Math 6382 Exam 1
assigned on 10/21/2016
to be done in 48 hours

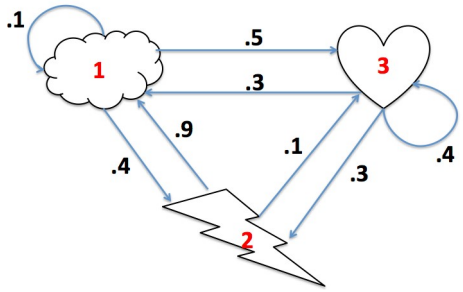
Show all work. It is open book and open time so you have 48 hour to finish and submit the exam through blackboard. Please send a legible copy. Remember there is always more than one way to reach a point but one of them is easier than the others.

1. (15 pts) Consider the hierarchical model (mixture)

$$[Y|N,\lambda] \sim \text{Gamma}(N,\lambda)$$
$$[N|r,p] \sim \text{Negative binomial}(r,p)$$

   Find analytically (mathematically) the resulting distribution of the mixture; that is, find the probability density function of $[Y|r,p,\lambda]$. Then write up an R code to prove the first part computationally using figures (histograms, Q-Q plots, density functions) and the appropriate statistical analyses.

2. (10 pts) The following transition graph shows the daily modes of Romeo and Juliets relationship and their transition probabilities every day, where 1: neutral, 2: hate, and 3: love.



   Find the stationary distribution computationally. Can you show if the Markov Chain has converged, how?

3. (15 pts) Consider a random sample $X_1,\ldots,X_{n_1}$ of size $n_1 = 10$ from a population $N(\mu_1 = 5, \sigma_1 = 1)$ show computationally that

   (a) $\frac{(n_1-1)S_1^2}{\sigma_1^2} \sim \chi^2(n_1-1)$

   (b) $\frac{\bar{X}-\mu_1}{S_1/\sqrt{n_1}} \sim$ T-dist(n_1-1)

   (c) Consider another independent sample $Y_1,\ldots,Y_{n_2}$ of size $n_2 = 12$ from a population $N(\mu_2 = 8, \sigma_2 = 2)$ show computationally that

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F(n_1-1, n_2-1)$$

4. (20 pts) Let

$$\theta = \int_0^1 e^{-x^3} dx$$

Find an estimate of the integral θ, the standard error of the estimate, the amount of reduction in the variance in comparison to the Monte Carlo method, and 95% confidence intervals and put them in a nice table in R, using

(i) Monte Carlo method,

(ii) antithetic variable,

(iii) control variate using the simplest but appropriate control function (think about a function that is either linear, parabolic, cubic, etc.),

(iv) importance sampling: using beta(.5,.5) distribution, using beta(1.5,.5) distribution, using beta(.5,1.5) distribution, and using a simple density function supported on (0,1),

(v) and stratified sampling from 2 sub-intervals and from 4 sub-intervals.

Which one is the most efficient method?

5. (10 pts) Generate $n = 100$ random numbers from the probability density function

$$f_X(x) = \frac{1}{\alpha} e^{-x^3}$$

where $0 < x < 1$ and α is the normalization constant. (Hint: consider the previous problem with taking uncertainty into account and use acceptance-rejection method.)

Best wishes!