Rev.
Fall
2019

MECE 3304
System Dynamics
Exam Booklet

DEPARTMENT OF MECHANICAL ENGINEERING

Booklet #

ELEMENT TYPES

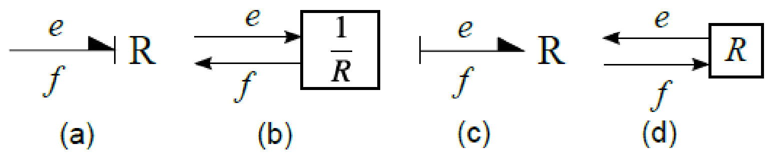


Figure 1: R-element

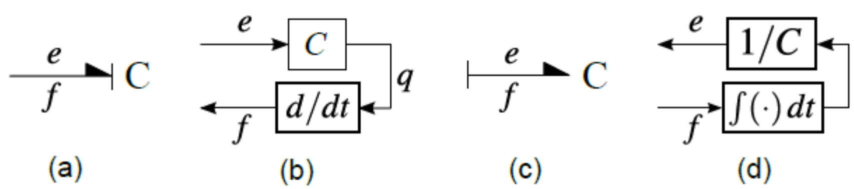


Figure 2: C-element

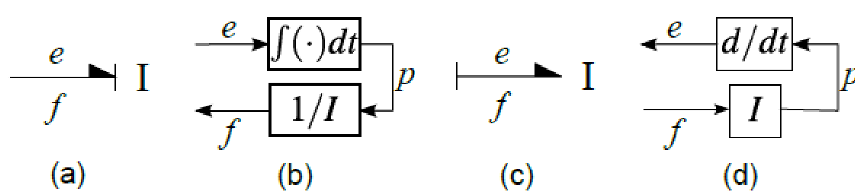


Figure 3: I-element

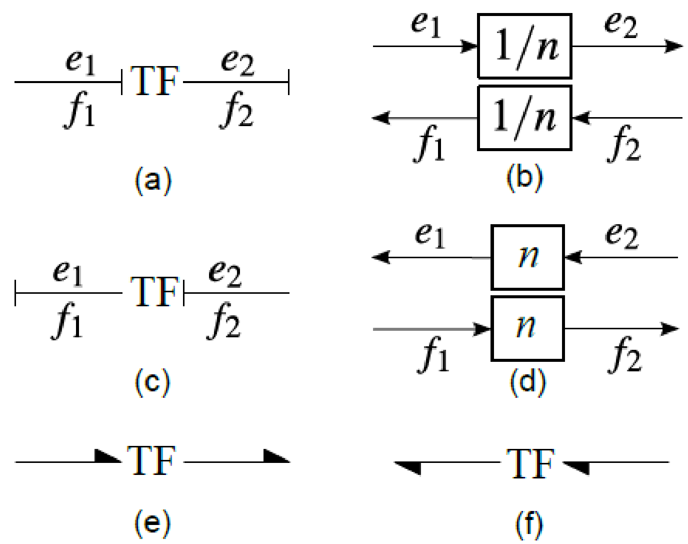


Figure 4: Transformer Element

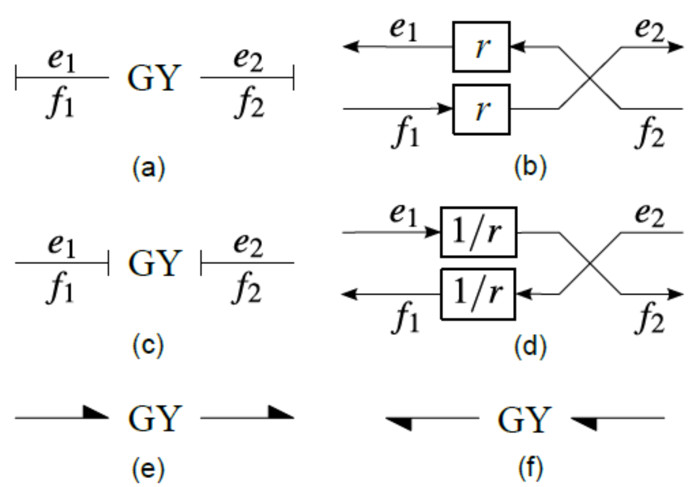


Figure 5: Gyrator Element

CONSTITUTIVE RELATIONS

Table 1: Constitutive relations for linear, 1-port, R-elements

DOMAIN	EFFORT-IN RELATION	FLOW-IN RELATION
GENERALIZED	$f = \frac{e}{R}$	$e = Rf$
TRANSLATIONAL	$v = \frac{F}{b}$	$F = bv$
ROTATIONAL	$\omega = \frac{\tau}{B}$	$\tau = B\omega$
ELECTRICAL	$i = \frac{e}{R}$	$e = Ri$
HYDRAULIC	$Q = \frac{P}{R_f}$	$P = R_f Q$

Table 2: Constitutive relations for linear, 1-port, C-elements.

DOMAIN	LINEAR RELATION	INTEGRAL RELATION	DERIVATIVE RELATION
GENERALIZED	$e = \frac{q}{C}$	$e = \frac{\int f\,dt}{C}$	$f = \frac{d}{dt}(Ce)$
TRANSLATIONAL	$F = kx$	$F = k\int v\,dt$	$v = \frac{d}{dt}\left(\frac{f}{k}\right)$
ROTATIONAL	$\tau = \kappa\theta$	$\tau = \kappa\int \omega\,dt$	$\omega = \frac{d}{dt}\left(\frac{\tau}{\kappa}\right)$
ELECTRICAL	$e = \frac{q}{C}$	$e = \frac{\int i\,dt}{C}$	$i = \frac{d}{dt}(Ce)$
HYDRAULIC	$P = \frac{V}{C_f}$	$P = \frac{\int Q\,dt}{C_f}$	$Q = \frac{d}{dt}(C_f P)$

Table 3: Constitutive relations for linear, 1-port, I-elements

DOMAIN	LINEAR RELATION	INTEGRAL RELATION	DERIVATIVE RELATION
GENERALIZED	$f = \frac{p}{I}$	$f = \frac{\int e\,dt}{I}$	$e = \frac{d}{dt}(If)$
TRANSLATIONAL	$v = \frac{p}{m}$	$v = \frac{\int F\,dt}{m}$	$F = \frac{d}{dt}(mv)$
ROTATIONAL	$\omega = \frac{h}{J}$	$\omega = \frac{\int \tau\,dt}{J}$	$\tau = \frac{d}{dt}(J\omega)$
ELECTRICAL	$i = \frac{\lambda}{L}$	$i = \frac{\int e\,dt}{L}$	$e = \frac{d}{dt}(\lambda i)$
HYDRAULIC	$Q = \frac{\Gamma}{I_f}$	$Q = \frac{\int P\,dt}{I_f}$	$P = \frac{d}{dt}(I_f Q)$

JUNCTIONS & GUIDELINES

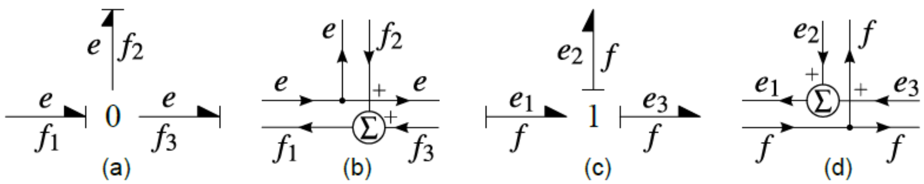


Figure 6: Junctions

Table 4: Summary of Junction Conditions

JUNCTION TYPE	PRIMARY CONDITION	SECONDARY CONDITION	BOND DIRECTION INDICATES
0-JUNCTION	Common effort $e_1 = e_2 = \cdots = e$	Sum of flows $\Sigma f_i = 0$	Flow sign convention (sign of the flows)
1-JUNCTION	Common flow $f_1 = f_2 = \cdots = f$	Sum of efforts $\Sigma e_i = 0$	Effort sign convention (sign of the efforts)

Table 5: Summary of energy-domain specific junction secondary conditions

DOMAIN	1-JUNCTION	0-JUNCTION
MECHANICAL TRANSLATION	Newton's Second Law $\Sigma F = \dot{p}$	Relative velocity e.g. $\Delta v = v_2 - v_1$
MECHANICAL ROTATION	Summation of moments $\Sigma \tau = \dot{h}$	Relative angular velocity e.g. $\Delta\omega = \omega_2 - \omega_1$
ELECTRIC CIRCUITS	Kirchhoff's Voltage Law $\Sigma e_j = 0$	Kirchhoff's Current Law $\Sigma i_j = 0$
HYDRAULIC CIRCUITS	Pressure drops around a loop $\Sigma \Delta P = 0$	Sum of flow into a junction $\Sigma Q_j = 0$

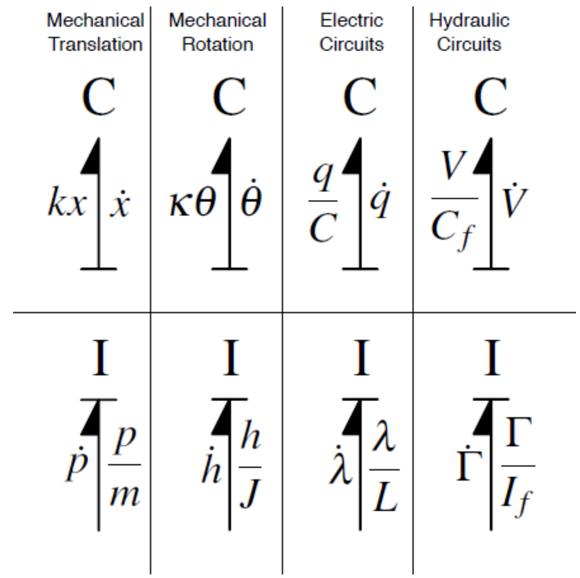


Figure 7 : Efforts and flows on C- and I-elements in the various domains

Guidelines for Mechanical Systems:

1. Identify distinct (angular) velocities by establishing 1-junctions.
2. Insert force-generating (torque-generating) 1-ports and energy-converting 2-ports.
3. Assign power directions.
4. Eliminate zero velocities & insert equivalencies.
5. Simplify.
6. Assign causality.

Guidelines for Circuits:

1. Identify distinct voltages (pressures) by establishing 0-junctions.
2. Insert 1 port circuit elements and energy-converting 2-ports.
3. Assign power directions.
4. Eliminate zero voltage/atmospheric pressure & insert equivalencies.
5. Simplify.
6. Assign causality

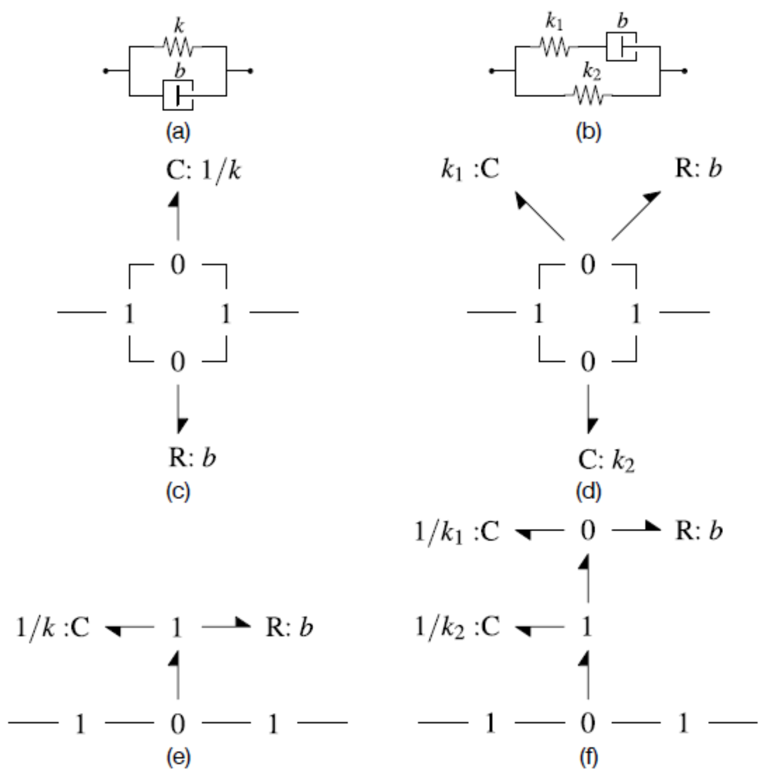


Figure 8: Mass-spring-damper equivalencies

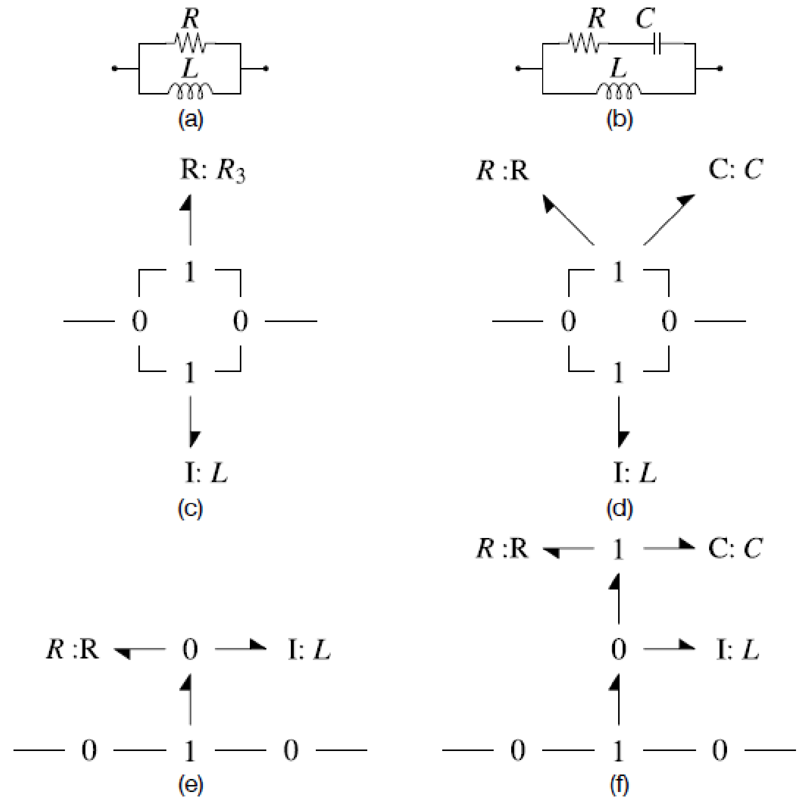


Figure 9: Electric circuit equivalencies

EQUIVALENCIES

IMPEDANCE BOND GRAPHS

Table 6: Mechanical Translation

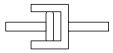
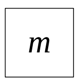
Element Type	Component	Force-Velocity	Impedance Z(s)=F(s)/V(s)
R-Element	Damper  b	$f(t) = bv(t)$ $F(s) = bV(s)$	b
C-Element	Spring k 	$f(t) = k\int v(t)dt$ $F(s) = \frac{k}{s}V(s)$	$\frac{k}{s}$
I-Element	Mass 	$f(t) = m\frac{dv(t)}{dt}$ $F(s) = msV(s)$	ms

Table 7: Mechanical Rotation

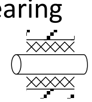
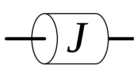
Element Type	Component	Torque-Velocity	Impedance Z(s)=τ(s)/ω(s)
R-Element	Bearing  B	$\tau(t) = B\omega(t)$ $\tau(s) = B\Omega(s)$	B
C-Element	Spring K 	$\tau(t) = \kappa\int \omega(t)dt$ $T(s) = \frac{\kappa}{s}\Omega(s)$	$\frac{\kappa}{s}$
I-Element	Inertia 	$\tau(t) = J\frac{d\omega(t)}{dt}$ $T(s) = Js\Omega(s)$	Js

Table 8: Electrical Circuit

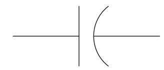
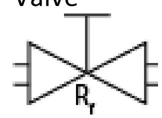
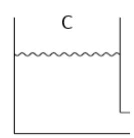
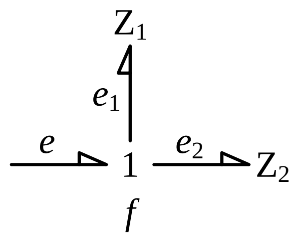
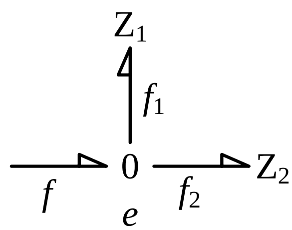
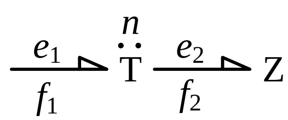
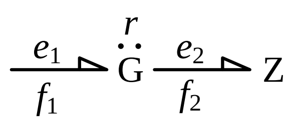
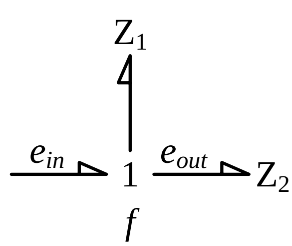
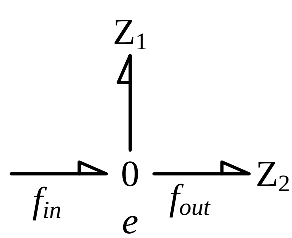
Element Type	Component	Voltage-Current	Impedance Z(s)=e(s)/i(s)
R-Element	Resistor  R	$e(t) = Ri(t)$ $E(s) = RI(s)$	R
C-Element	Capacitor  C	$e(t) = \frac{1}{C}\int i(t)dt$ $E(s) = \frac{1}{Cs}I(s)$	$\frac{1}{Cs}$
I-Element	Inductor L 	$e(t) = L\frac{di(t)}{dt}$ $E(s) = LsI(s)$	Ls

Table 9: Hydraulic Circuit

Element Type	Component	Pressure-Flow Rate	Impedance Z(s)=P(s)/Q(s)
R-Element	Valve 	$P(t) = R_f Q(t)$ $P(s) = R_f Q(s)$	R_f
C-Element	Accumulator 	$P(t) = \frac{1}{C_f}\int Q(t)dt$ $P(s) = \frac{1}{C_f s}Q(s)$	$\frac{1}{C_f s}$
I-Element	Inertia 	$P(t) = I_f\frac{dQ(t)}{dt}$ $P(s) = I_f sQ(s)$	$I_f s$

IMPEDANCE BOND GRAPHS

Table 10: Junctions, Transformers, Gyrators

	Impedance Bond Graph	Mathematical (Label) Relations	Equivalent Impedance, Z_{eq}
1-Junction		$Z_1 = \frac{e_1}{f_1} = \frac{e_1}{f}$ $Z_2 = \frac{e_2}{f_2} = \frac{e_2}{f}$ $e = e_1 + e_2$ $= Z_1 f + Z_2 f$ $= (Z_1 + Z_2) f$	$Z_{eq} = \frac{e(s)}{f(s)} = Z_1 + Z_2$
0-Junction		$Z_1 = \frac{e_1}{f_1} = \frac{e}{f_1}$ $Z_2 = \frac{e_2}{f_2} = \frac{e}{f_2}$ $f = f_1 + f_2$ $= \frac{1}{Z_1} e + \frac{1}{Z_2} e$ $= \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) e$	$Z_{eq} = \frac{e(s)}{f(s)} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}}$ $= \frac{Z_1 Z_2}{Z_1 + Z_2}$
Transformer		$e_1 = n e_2 = n Z f_2 = n^2 Z f_1$	$Z_{eq} = \frac{e_1(s)}{f_1(s)} = n^2 Z$
Gyrator		$e_1 = r f_2 = r \frac{e_2}{Z} = \frac{r^2}{Z} f_1$	$Z_{eq} = \frac{e_1(s)}{f_1(s)} = \frac{r^2}{Z}$
Effort Divider		$Z_2 = \frac{e_{out}}{f}$ $e_{out} = Z_2 f = Z_2 \frac{e_{in}}{Z_1 + Z_2}$ $\frac{e_{out}}{e_{in}} = \frac{Z_2}{Z_1 + Z_2}$	N/A
Flow Divider		$Z_2 = \frac{e}{f_{out}}$ $f_{out} = \frac{1}{Z_2} e = \frac{1}{Z_2} \frac{Z_1 Z_2}{Z_1 + Z_2} f_{in}$ $\frac{f_{out}}{f_{in}} = \frac{Z_1}{Z_1 + Z_2}$	N/A

LAPLACE TRANSFORMS

Table 11: Commonly used Laplace transforms

	g(t)	g(s)
1	$\tilde{\delta}(t)$	1
2	$1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	e^{-at}	$\frac{1}{s+a}$
5	te^{-at}	$\frac{1}{(s+a)^2}$
6	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
7	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
9	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
10	$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
11	$\mathrm{e}^{-\mathrm{at}}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
12	$\dot{g}(t)$	$sg(s)-g(0)$
13	$\ddot{g}(t)$	$s^2g(s)-sg(0)-\dot{g}(0)$
14	$\int g(t)dt$	$\frac{g(s)}{s}+\frac{g^{-1}(0)}{s}$
15	$e^{-at}g(t)$	$g(s+a)$
16	$g(t-a)1(t-a)$	$e^{-as}g(s)$
17	$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}[F(s)]$

Prototypical second-order system:

$$\frac{x(s)}{F(s)} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ (Roots of characteristic equation or poles)

Table 12 Characteristics of an ***underdamped (ζ<1)*** *second- order system*

$\omega_d = \omega_n\sqrt{1-\zeta^2}$	Damped frequency of oscillation
$T_p = \frac{2\pi}{\omega_d}$	Period of oscillation
$T = \frac{1}{\zeta\omega_n}$	Time constant
$\zeta = \frac{\frac{1}{n-1} ln \frac{x_1}{x_n}}{\sqrt{4\pi^2 + \left(\frac{1}{n-1} ln \frac{x_1}{x_n}\right)^2}}$	Logarithmic decrement, damping ratio

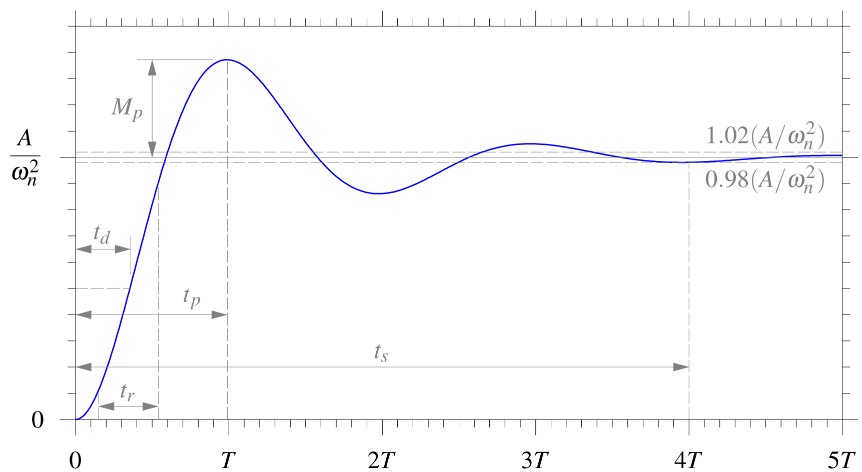


Figure 10: **Step response** characteristics for an **underdamped** second-order system

Prototypical first-order system:

$$\frac{x(s)}{F(s)} = \frac{1}{T}\left(\frac{1}{s + \frac{1}{T}}\right)$$

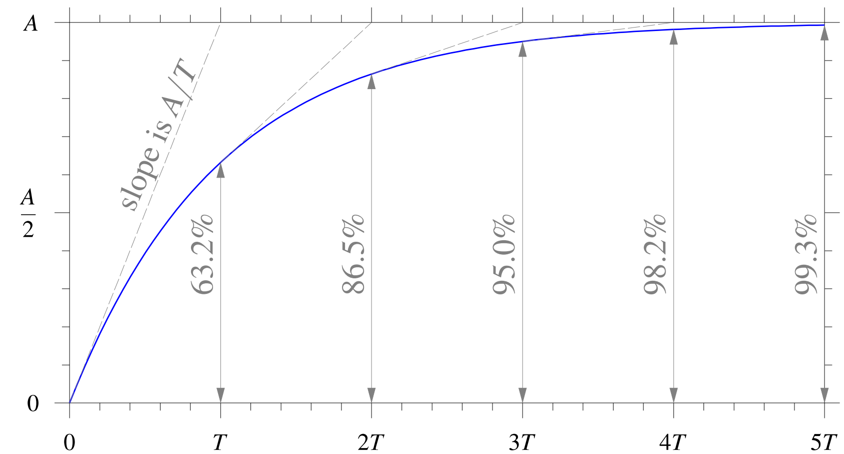


Figure 11: ***Step response*** *of prototypical first-order system*

Pole-Zero Analysis:

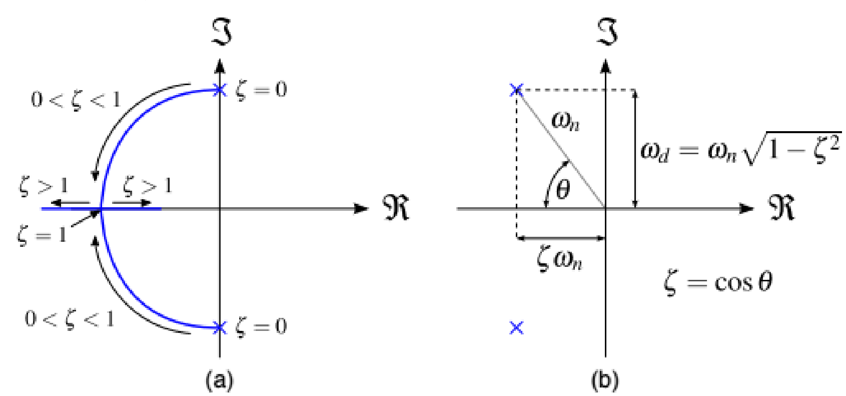


Figure 12: (a) damping as it relates to pole placement and (b) daming ratio and natural frequency as they relate to pole placement

Steady-State Response for a Sinusoidal Input:

$$y_{ss}(t) = Y sin(\omega t + \phi)$$

$$Y = |G(j\omega)|A = \left|\frac{y(j\omega)}{u(j\omega)}\right| A$$

$$\phi = \angle G(j\omega) = \angle \frac{y(j\omega)}{u(j\omega)} = \tan^{-1}\left\{\frac{\Im[G(j\omega)]}{\Re[G(j\omega)]}\right\}$$

Transmissibility:

$$TR = \left|\frac{F_{out}(j\omega)}{F_{in}(j\omega)}\right|$$

$$TR = \left|\frac{y_{out}(j\omega)}{y_{in}(j\omega)}\right|$$

Free Vibration Modal Analysis:

$$[\boldsymbol{M}]\{\ddot{x}\} + [\boldsymbol{K}]\{x\} = 0$$

$$\big[\lambda[\boldsymbol{I}] - [\boldsymbol{A}]\big]\{A_k\} = 0$$

$$[\boldsymbol{A}] = [\boldsymbol{M}]^{-1}[\boldsymbol{K}]$$

State-Space Representation:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

State-Space to Transfer Function:

$$\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$$

Inverse Operations for a Matrix:

2x2

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$|\boldsymbol{A}| = ad - cb$$

3x3

$$\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|}\begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & \begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & \begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & \begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

$$|\boldsymbol{A}| = a(ei - hf) - b(di - gf) + c(dh - ge)$$