Chapter 8:
Frequency Domain Analysis

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Preview Questions

1. What is the steady-state response of a linear system excited by a cyclic or oscillatory input?
2. How does one characterize the response at steady-state when the system is exposed to a consistent oscillatory input?
3. Is the time domain still appropriate for conducting our analyses of such systems?
4. What tools are useful for examining such dynamics?

Objectives and Outcomes

Objectives

1. To analyze mechanical vibration systems including transmission and modal analysis,
2. To be able to analyze basic AC circuits, and
3. To conduct frequency response analysis.

Outcomes: You will be able to

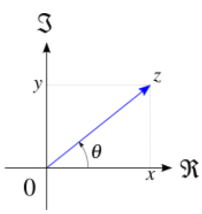
1. determine the steady-state response of a linear time-invariant system to a sinusoidal input,
2. calculate the force or motion transmitted by a vibration isolation system,
3. conduct basic modal analysis of free vibration systems,
4. conduct basic analyses of AC circuits,
5. identify the characteristics for frequency responses of first- and second-order systems, and
6. compose Bode plots that visualize the frequency response of an oscillatory system.

5.2.1 Complex Numbers

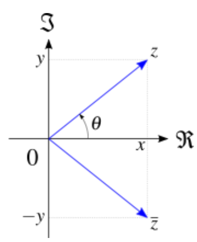
$z = x + jy$

$|z| = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\frac{y}{x}$



$\bar{z} = x - jy$



$$
\begin{aligned}
z &= x + jy \\
&= |z|\cos\theta + j|z|\sin\theta \\
&= |z|(\cos\theta + j\sin\theta) \\
&= |z|\underline{/\theta} \\
&= |z|e^{j\theta}.
\end{aligned}
$$

$$z = z + jy = |z|(\cos\theta + j\sin\theta) = |z|\underline{/\theta}$$
$$\bar{z} = x - jy = |z|(\cos\theta - j\sin\theta) = |z|\underline{/-\theta}.$$

$$z\bar{z} = (x + jy)(x - jy) = x^2 - jxy + jxy + y^2 = x^2 + y^2,$$

$$\frac{1}{z} = \frac{1}{z}\frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{x^2 + y^2}.$$

5.2.2 Euler’s Theorem

Refer to the textbook (§5.2.2 for derivation of theorem and identities)

Euler’s Theorem	$e^{-j\theta} = \cos\theta - j\sin\theta$
Cosine Identity	$\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$
Sine Identity	$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

5.2.3 Complex Algebra

$z = x + jy$ and $\mathrm{w = u + jv}$

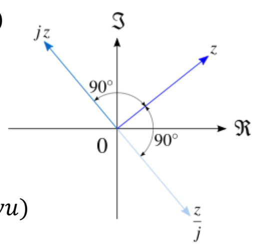
$z + w = (x + u) + j(y + v)$

$z - w = (x - u) + j(y - v)$

$az = ax + jay$

$jz = -y + jx = |z|\angle(0 + 90°)$

$\dfrac{z}{j} = y - jx = |z|\angle(\theta - 90°)$



$zw = (xu - yv) + j(xv + yu)$

$zw = |z||w|\angle(\theta + \phi)$

$\dfrac{z}{w} = \dfrac{|z|}{|w|}\angle(\theta - \phi) = \dfrac{xu + yv}{u^2 + y^2} + j\dfrac{yu - xv}{u^2 + y^2}$

$z^n = (|z|\angle\theta)^n = |z|^n\angle(n\theta)$

$z^{1/n} = (|z|\angle\theta)^{1/n} = |z|^{1/n}\angle(\theta/n)$

Complex Variables and Functions

$$s = \sigma + j\omega$$

$$G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

Transfer function
  Ratio of polynomials in the s-domain

Zeroes
  Roots of the numerator

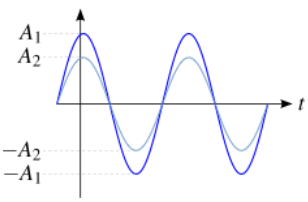
Poles
  Roots of the denominator

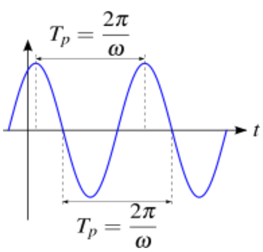
8.2 Properties of Sinusoids

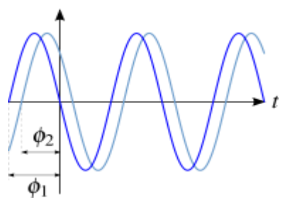
Sinusoids of different amplitude

Period of oscillation

Sinusoids of different phase angle







The Sinusoidal Transfer Function

$$\frac{y(s)}{u(s)} = G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

Partial fraction of sinusoidal response

$$y(s) = G(s)\frac{A\omega}{s^2+\omega^2} = \frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega} + \frac{b_1}{s+p_1} + \frac{b_2}{s+p_2} + \ldots + \frac{b_n}{s+p_n}$$

If the response is stable, these terms lead to decaying exponentials and decaying sinusoids.

$$y_{ss}(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t}$$

$$a = -\frac{A}{2j}G(-j\omega)$$

$$\bar{a} = \frac{A}{2j}G(j\omega)$$

See textbook for derivation of these coefficients

$$y_{ss}(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} = -\frac{A}{2j}|G(j\omega)|e^{-j\phi}e^{-j\omega t} + \frac{A}{2j}|G(j\omega)|e^{j\phi}e^{j\omega t}$$

$$= |G(j\omega)|A\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j} = |G(j\omega)|A\sin(\omega t+\phi)$$

$$= Y\sin(\omega t+\phi)$$

We use Euler's Theorem

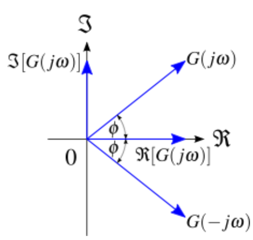
Magnitude and Phase Angle

$$y_{ss}(t) = |G(j\omega)|A\sin(\omega t+\phi)$$

$$= Y\sin(\omega t+\phi)$$

$\boldsymbol{Y = |G(j\omega)|A}$ and $\boldsymbol{\phi = \angle G(j\omega)}$

$$|G(j\omega)| = \left|\frac{y(j\omega)}{u(j\omega)}\right| = output\ to\ input\ amplitude\ ratio$$



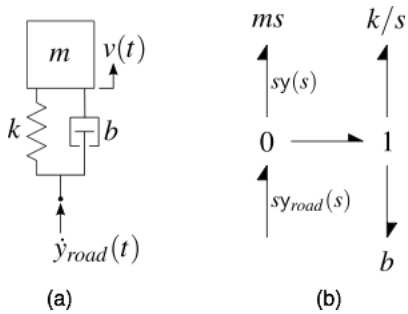
$$\angle G(j\omega) = \angle\frac{y(j\omega)}{u(j\omega)} = \tan^{-1}\left\{\frac{\Im[G(j\omega)]}{\Re[G(j\omega)]}\right\}$$

$$= \tan^{-1}\left\{\frac{\Im[y(j\omega)]}{\Re[y(j\omega)]}\right\} - \tan^{-1}\left\{\frac{\Im[u(j\omega)]}{\Re[u(j\omega)]}\right\}$$

$$= phase\ shift\ of\ output\ with\ respect\ to\ input$$

Example 8.1

Find the steady-state response to a sinusoidal input displacement of the form $y_{road} = (A \sin \omega t)\ m/s$ where A=0.03 m and ω=10 rad/s. The system parameters for m, b, and k are 500 kg, 8000 N-s/m, and 34,000 N/m, respectively. The transfer function is given below.



$$\frac{y(s)}{y_{road}(s)} = G(s) = \frac{bs + k}{ms^2 + bs + k}$$

Complex Operations in MATLAB

Function	Description
abs(X)	Returns magnitude(s) of complex element(s) in X
angle(X)	Returns phase angle(s) of complex element(s) in X
conj(X)	Returns complex conjugate(s) of complex element(s) in X
imag(X)	Returns imaginary part(s) of complex element(s) in X
real(X)	Returns real part(s) of complex element(s) in X

Example 8.2

```
>> m=500; b=8000; k=34000; w=10; A=0.03;

>> G=(k+j*b*w)/((k-m*w^2)+j*b*w)

G =   0.8798 - 0.6010i

>> abs(G)

ans =    1.0655

>> angle(G)

ans =   -0.5993
```

```
>> num=k+j*b*w;
>> den=(k-m*w^2)+j*b*w

den =  -1.6000e+04 + 8.0000e+04i

>> G=(num*conj(den))/(den*conj(den))

G =   0.8798 - 0.6010i

>> magnitude=abs(num)/abs(den)

magnitude =    1.0655

>> phase=angle(num)-angle(den)

phase =   -0.5993
```

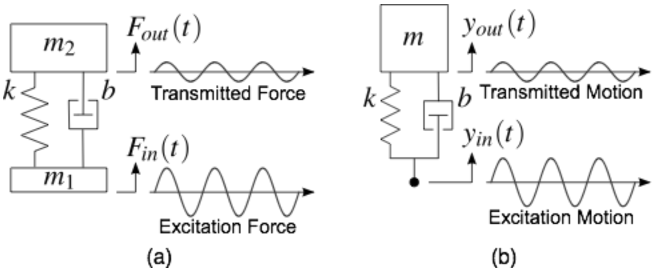
```
>> magnitude=sqrt(real(G)^2+imag(G)^2)

magnitude =    1.0655

>> phase=atan(imag(G)/real(G))

phase =   -0.5993
```

Mechanical Vibration

Transmissibility

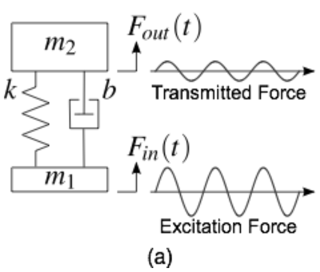
In vibration isolation systems, transmissibility is the amplitude ratio of the transmitted force (displacement) to the excitation force (displacement).

$$TR = \left|\frac{F_{out}(j\omega)}{F_{in}(j\omega)}\right|$$



Example 8.3

Find the transmissibility if the foundation is forced by an excitation $F_{in}(t) = (5sin2t)N$. The first mass, damping constant, spring stiffness, and second mass are 2 kg, 2 N-s/m, 5 N/m, and 1 kg, respectively.



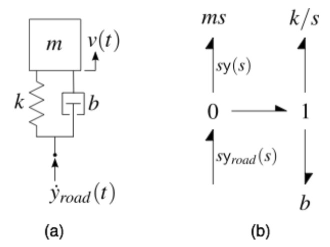
Motion Transmissibility Example 8.1

Input

$$y_{road} = (0.03 \sin 10t)\ m/s$$

Output Response

$$y_{ss}(t) = 0.031965 sin(10t - 0.59927)\ m/s$$



Amplitude ratio of output to input

$$|G(j\omega)| = \frac{0.031965}{0.03} = 1.0655$$

Motion Transmissibility

$$TR = \frac{|y(j\omega)|}{|y_{road}(j\omega)|} = 1.0655$$

$$\frac{y(s)}{y_{road}(s)} = G(s) = \frac{bs + k}{ms^2 + bs + k}$$

$$G(j\omega) = \frac{y(j\omega)}{y_{road}(j\omega)} = \frac{k + bj\omega}{m(j\omega)^2 + bj\omega + k}$$

Resonant Frequency

Occurs when a system's natural frequency is equal to the input frequency.

Recall, from the prototypical second-order system:

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{b}{2\sqrt{km}}$$

Normalized frequency is the input sinusoidal frequency divided by the natural frequency

$$\frac{\omega}{\omega_n}$$

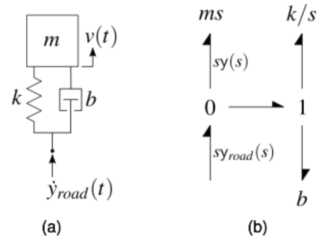
Resonant frequency for a prototypical second-order system

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$$

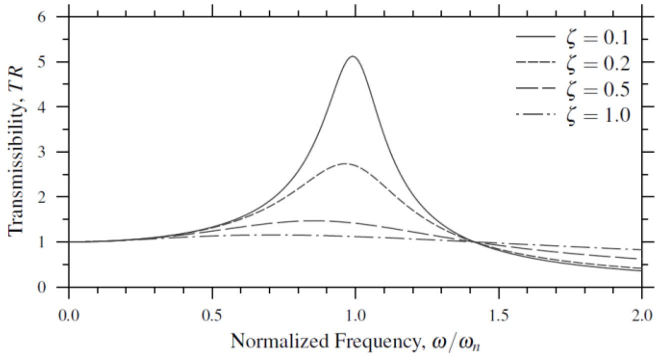
The Quarter-Car Suspension

Rewrite the motion transmissibility in terms of the damping ratio and normalized frequency.

$$TR = \frac{\sqrt{k^2 + (b\omega)^2}}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}}$$



Resonance for the Quarter-Car Suspension

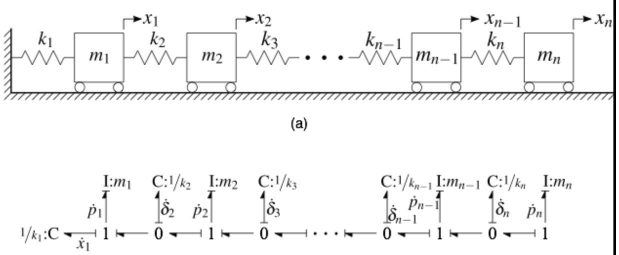


$$\zeta = \frac{b}{2\sqrt{km}} = 0.97$$

$$\omega_n = \sqrt{\frac{k}{m}} = 8.25\ rad/s$$

Modal Analysis of Free Vibration

$$\dot{x}_1 = \frac{p_1}{m_1}$$
$$\dot{p}_1 = -k_1 x_1 + k_2 \delta_2$$
$$\dot{\delta}_2 = -\frac{p_1}{m_1} + \frac{p_2}{m_2}$$
$$\dot{p}_2 = -k_2 \delta_2 + k_3 \delta_3$$
$$\vdots$$
$$\dot{\delta}_{n-1} = -\frac{p_{n-2}}{m_{n-2}} + \frac{p_{n-1}}{m_{n-1}}$$
$$\dot{p}_{n-1} = -k_{n-1}\delta_{n-1} + k_n \delta_n$$
$$\dot{\delta}_n = -\frac{p_{n-1}}{m_{n-1}} + \frac{p_n}{m_n}$$
$$\dot{p}_n = -k_n \delta_n .$$



(a)

(b)

Set & Tensor Notation

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$ Vector

$$\{a\} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix}$$ Vector in set form

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = [A]$$ Matrix referenced as tensor

Set of second-order, free, undamped vibration equations

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

where $[M]$ and $[K]$ are the mass and stiffness tensors and $\{x\}$ and $\{\ddot{x}\}$ are the displacement and acceleration sets

Converting System of First-Order D.E. to System of Second-Order D.E.

Recognizing that

$$\delta_k = x_k - x_{k-1} \quad (k = 1, \ldots, n)$$

and

$$p_k = mv_k = m\dot{x}_k \quad \Rightarrow \quad \dot{p}_k = m\ddot{x}_k$$

we can reformulate the system of first-order differential equations as

$$\begin{aligned}
m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 &= 0 \\
m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 &= 0 \\
&\vdots \\
m_{n-1}\ddot{x}_{n-1} - k_{n-1}x_{n-2} + (k_{n-1} + k_n)x_{n-1} - k_nx_n &= 0 \\
m_n\ddot{x}_n - k_nx_{n-1} + k_nx_n &= 0\,.
\end{aligned}$$

Matrix Form of the Vibration Equations

$$[M]\{\ddot{x}\} + [K]\{x\} = \begin{bmatrix} m_1 & 0 & \cdots & 0 & 0 \\ 0 & m_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & m_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & m_n \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_{n-1} \\ \ddot{x}_n \end{Bmatrix} +$$

$$\begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 & \cdots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 & \cdots & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -k_{n-1} & k_{n-1} + k_n & -k_n \\ 0 & 0 & \cdots & 0 & -k_n & k_n \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{Bmatrix}.$$

Frequencies and Modes of Vibration

In free vibration we assume

$$\begin{cases} x_k = A_k \sin \omega t \\ \dot{x}_k = -A_k \omega \sin \omega t \\ \ddot{x}_k = -A_k \omega^2 \sin \omega t \end{cases}$$

Plug into $\rightarrow$ $[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$

$$[M]\{A_k\}(-\omega^2 \sin \omega t) + [K]\{A_k\} \sin \omega t = \{0\}$$

$$\left[-\omega^2[M] + [K]\right] \{A_k\} \sin \omega t = \{0\}$$

$$\left[-\omega^2[M]^{-1}[M] + [M]^{-1}[K]\right] \{A_k\} \sin \omega t = \{0\}$$

$$\left[-\omega^2[I] + [M]^{-1}[K]\right] \{A_k\} \sin \omega t = \{0\}$$

$$\left[\omega^2[I] - [M]^{-1}[K]\right] \{A_k\} \sin \omega t = \{0\}$$

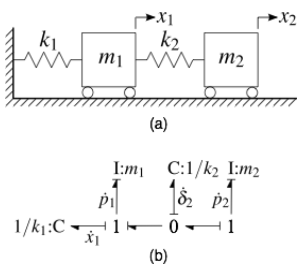
$$\left[\omega^2[I] - [M]^{-1}[K]\right] \{A_k\} = 0$$

$$\left[\lambda[I] - [M]^{-1}[K]\right] \{A_k\} = 0$$

$$\left[\lambda[I] - [A]\right] \{A_k\} = 0$$

The eigenvalues are the squares of the frequencies of vibration and the eigenvectors are the mode shapes.

Example 8.4



Find the natural frequencies of vibration and the respective mode shapes for the two DOF structure model and an axially vibrating beam where $k_1 = 2k_2$ and $m_1 = 2m_2$. For simplicity, $k_2 = k$ and $m_2 = m$.

$$\dot{x}_1 = \frac{p_1}{m_1} = \frac{p_1}{2m},$$

$$\dot{p}_1 = -k_1 x_1 + k_2 \delta_2 = -2kx_1 + k\delta_2,$$

$$\dot{\delta}_2 = -\frac{p_1}{m_1} + \frac{p_2}{m_2} = -\frac{p_1}{2m} + \frac{p_2}{m}, \text{ and}$$

$$\dot{p}_2 = -k_2 \delta_2 = -k\delta_2.$$

$\rightarrow$

$$2m\ddot{x}_1 + 3kx_1 - kx_2 = 0$$

$$m\ddot{x}_2 - kx_1 + kx_2 = 0$$

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Example 8.4

$$det\left[\omega^2[I] - [M]^{-1}[K]\right] = 0$$

$$\det\left\{\begin{bmatrix}\omega^2 & 0\\ 0 & \omega^2\end{bmatrix} - \begin{bmatrix}2m & 0\\ 0 & m\end{bmatrix}^{-1}\begin{bmatrix}3k & -k\\ -k & k\end{bmatrix}\right\} = 0$$

$$\omega^4 - \frac{5k}{2m}\omega^2 + \frac{k^2}{2m^2} = 0$$

$$\left(\omega^2 - \frac{2k}{m}\right)\left(\omega^2 - \frac{k}{2m}\right) = 0$$

Eigenvalues

$$\omega_1^2 = \frac{2k}{m} \quad \text{or} \quad \omega_1 = \sqrt{\frac{2k}{m}}$$

$$\omega_2^2 = \frac{k}{2m} \quad \text{or} \quad \omega_2 = \sqrt{\frac{k}{2m}}$$

$$\begin{bmatrix}\frac{2k}{m} - \frac{3k}{2m} & \frac{k}{2m}\\ \frac{k}{m} & \frac{2k}{m} - \frac{k}{m}\end{bmatrix}\begin{Bmatrix}A_1\\ A_2\end{Bmatrix} = 0$$

$$\frac{k}{2m}\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}\begin{Bmatrix}A_1\\ A_2\end{Bmatrix} = 0$$

Eigenvector 1 $\begin{Bmatrix}1\\ -1\end{Bmatrix}$

$$\begin{bmatrix}\frac{k}{2m} - \frac{3k}{2m} & \frac{k}{2m}\\ \frac{k}{m} & \frac{k}{2m} - \frac{k}{m}\end{bmatrix}\begin{Bmatrix}A_1\\ A_2\end{Bmatrix} = 0$$

$$\frac{k}{2m}\begin{bmatrix}-2 & 1\\ 2 & -1\end{bmatrix}\begin{Bmatrix}A_1\\ A_2\end{Bmatrix} = 0$$

Eigenvector 2 $\begin{Bmatrix}1\\ 2\end{Bmatrix}$