Chapter 7: Time Domain Analysis

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Preview Questions

How do the system parameters affect the response?

How are the parameters linked to the system poles or eigenvalues?

How can Laplace transforms and transfer functions be used to analyze the time domain response of a system?

How can linear algebra and the state-space model be used to analyze the time domain response?

Objective and Outcomes

Objectives

- To understand first-order responses
- To understand second-order responses
- To understand how higher-order responses are composed from first- and second-order terms, and
- To understand the relation between the system poles or eigenvalues and the overall system response.

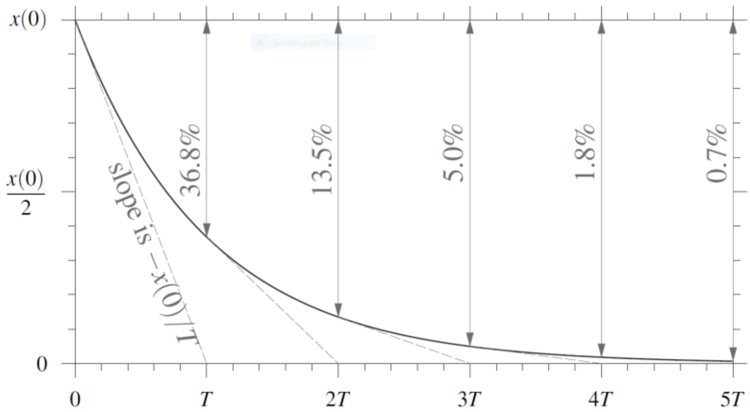
Outcomes

- Identify the characteristics of first-order responses
- Identify the characteristics of second-order responses
- Identify the dominant poles of higher-order systems,
- Use the transfer function or state-space representation to determine a system's characteristic roots, and
- Predict overall response based on pole placement

Transient Response of First-Order Systems

The Natural Response

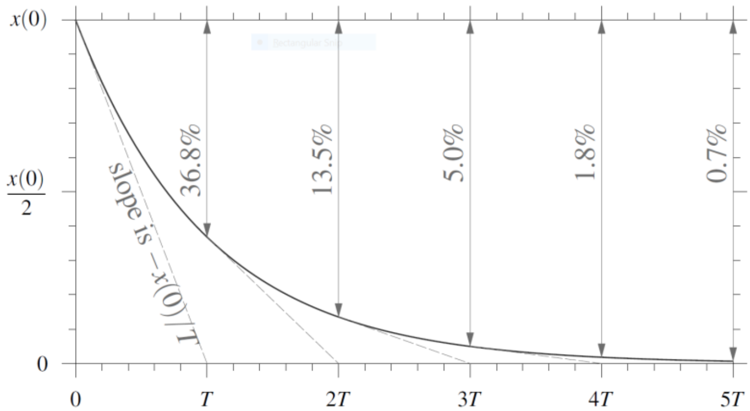
$$T\frac{dx}{dt} + x = 0 \Rightarrow x(t) = x(0)e^{-t/T}$$



The Impulse Response

$$T\frac{dx}{dt} + x = A\tilde{\delta}(t)$$

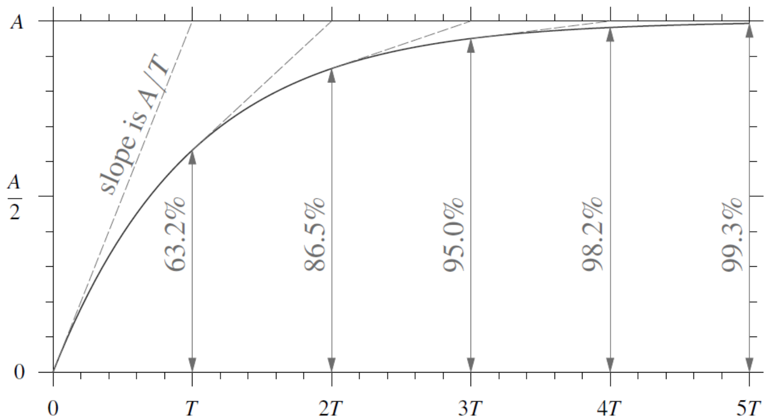
$$x(t) = \frac{A}{T}e^{-t/T}$$



The Step Response

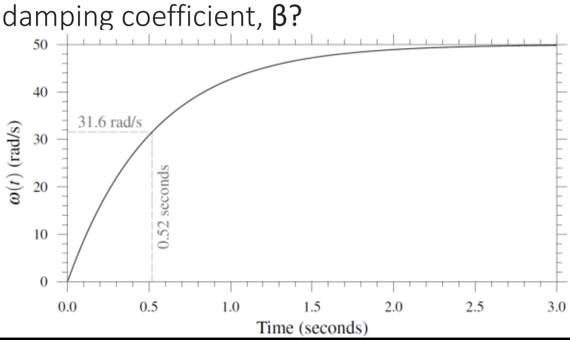
$$Tsx(s) + x(s) = (Ts + 1)x(s) = \frac{A}{x}$$

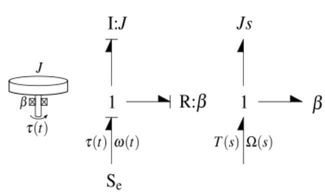
$$x(t) = A(1 - e^{-t/T})$$



Example 7.1

A simple torsion system is depicted. It is a disk mounted on a bearing and is excited by an input torque. The rotational inertia can be readily attained by measuring the mass and diameter of the disk. The bearing damping coefficient, on the other hand, is not something that is commonly advertised or supplied. However, this can be readily determined using experimental data and a model of the system. Given the measured step response plotted in the following figure, what are the rotational inertia, J, and damping coefficient, β?

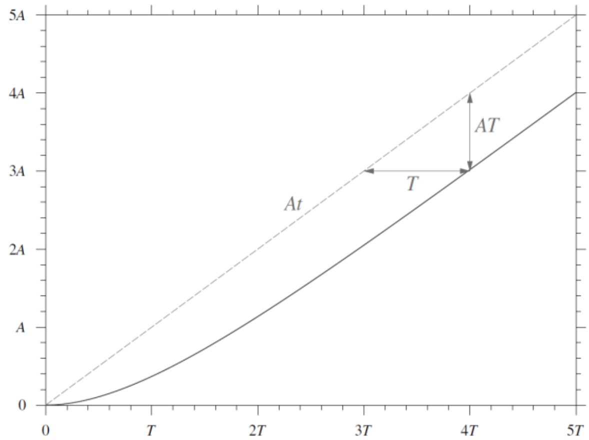




The Ramp Response

$$T\frac{dx}{dt} + x = At$$

$$x(t) = A(t - T - Te^{-\frac{t}{T}})$$



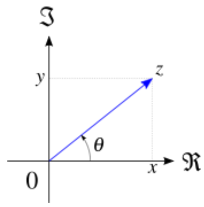
Complex Numbers

5.2.1 Complex Numbers

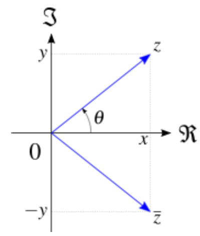
$z = x + jy$

$|z| = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\dfrac{y}{x}$



$\bar{z} = x - jy$



$$\begin{aligned} z &= x + jy \\ &= |z|\cos\theta + j|z|\sin\theta \\ &= |z|(\cos\theta + j\sin\theta) \\ &= |z|\underline{/\theta} \\ &= |z|e^{j\theta}. \end{aligned}$$

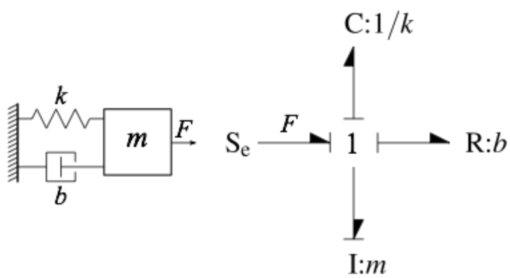
$$z = z + jy = |z|(\cos\theta + j\sin\theta) = |z|\underline{/\theta}$$
$$\bar{z} = x - jy = |z|(\cos\theta - j\sin\theta) = |z|\underline{/-\theta}.$$

$$z\bar{z} = (x + jy)(x - jy) = x^2 - jxy + jxy + y^2 = x^2 + y^2,$$

$$\frac{1}{z} = \frac{1}{z}\frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{x^2 + y^2}.$$

Transient Responses of Second-Order Systems

Prototypical Second-Order System



$$\dot{x} = \frac{p}{m}$$

$$\dot{p} = -kx - \frac{b}{m}p + F(t)$$

The Natural Response

Prototypical second-order differential equation with no input

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

Associated characteristic equation in the s-domain

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Characteristic roots

$$s_{1,2} = \frac{-2\zeta\omega_n \pm \sqrt{4(\zeta\omega_n)^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Undamped Natural Response (ζ=0)

Differential Equation

$$\ddot{x} + \omega_n^2 x = 0$$

Characteristic Equation

$$s^2 + \omega_n^2 = 0$$

Characteristic Roots

$$s_{1,2} = \pm j\omega_n$$

← Purely Imaginary Roots

Response

$$x(t) = x(0) cos\omega t$$

Underdamped Natural Response (ζ<1)

Roots

$$\begin{aligned} s_{1,2} &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\ &= -\zeta\omega_n \pm \omega_n\sqrt{-1}\sqrt{1-\zeta^2} \\ &= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \end{aligned}$$

← Complex Conjugate Roots

Response

$$\begin{aligned} x(t) &= x(0)\left[e^{-\zeta\omega_n t}\cos\omega_n\sqrt{1-\zeta^2}t + \frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}t\right] \\ &= x(0)\left[e^{-\zeta\omega_n t}\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t\right] \\ &= x(0)e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right] \end{aligned}$$

where

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

← Damping Frequency

Critically Damped Natural Response (ζ=1)

Characteristic Roots

$$\begin{aligned} s_{1,2} &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\ &= -\zeta\omega_n \pm \omega_n\sqrt{1^2 - 1} \\ &= -\omega_n \end{aligned}$$

← Repeated Negative Real Roots

Response

$$x(t) = x(0)(1 + \omega_n t)e^{-\omega_n t}$$

Overdamped Natural Response (ζ>1)

Characteristic Roots

$$\begin{aligned} s_{1,2} &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\ &= -\omega_n(\zeta \mp \sqrt{\zeta^2 - 1} \end{aligned}$$

← Distinct Negative Real Roots

Response

$$x(t) = ae^{-\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right)t} + be^{-\omega_n\left(\zeta-\sqrt{\zeta^2-1}\right)t}$$

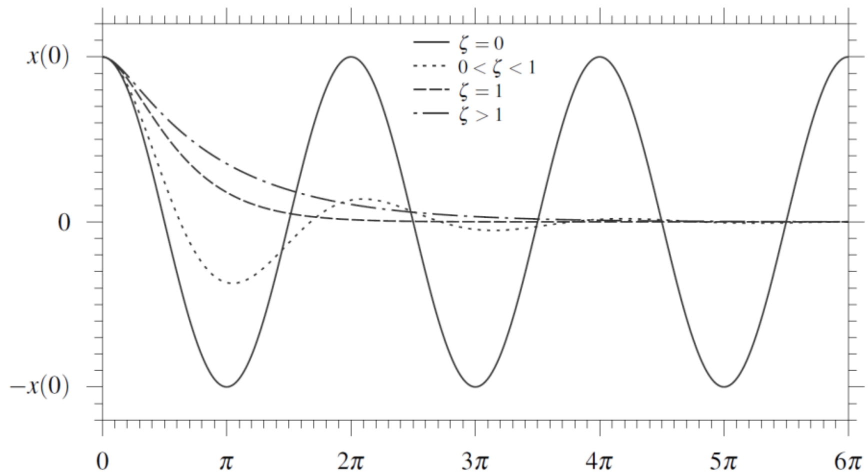
where

$$a = \frac{-\zeta + \sqrt{\zeta^2 - 1}}{2\sqrt{\zeta^2 - 1}}x(0)$$

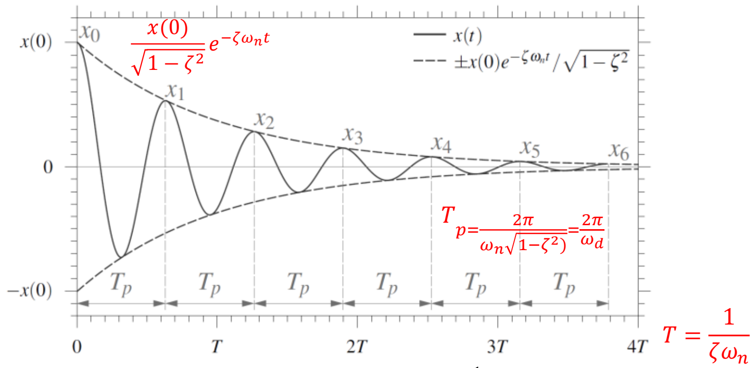
and

$$b = \frac{\zeta + \sqrt{\zeta^2 - 1}}{2\sqrt{\zeta^2 - 1}}x(0)$$

Second-Order Natural Responses



Second-Order Natural, Underdamped Response

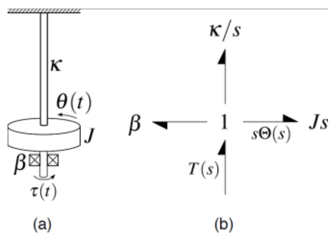


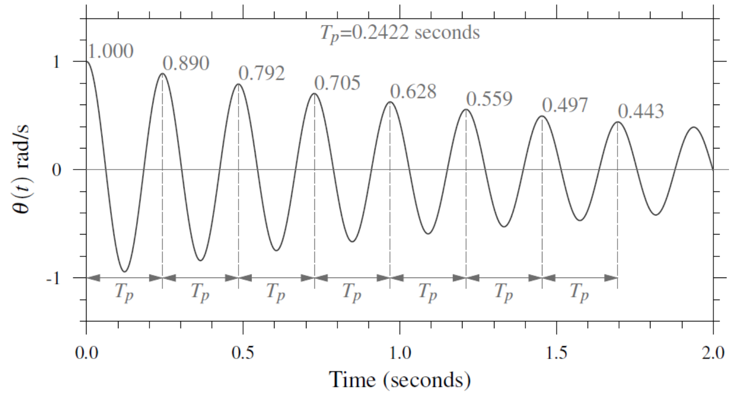
From Logarithmic Decrement

$$\zeta = \frac{\frac{1}{n-1} ln \frac{x_1}{x_n}}{\sqrt{4\pi^2 + \left(\frac{1}{n-1} ln \frac{x_1}{x_n}\right)^2}}$$

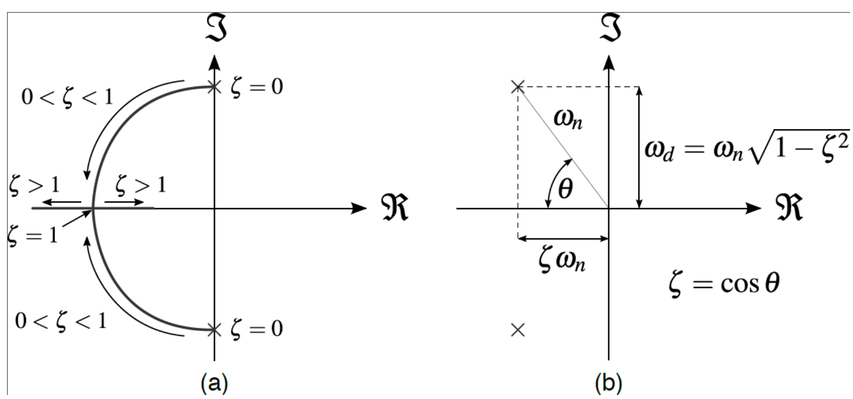
Example 7.2

Given the natural response, how could you determine the damping coefficient, β, and the shaft rigidity, κ? The disk is the same as from Example 1.

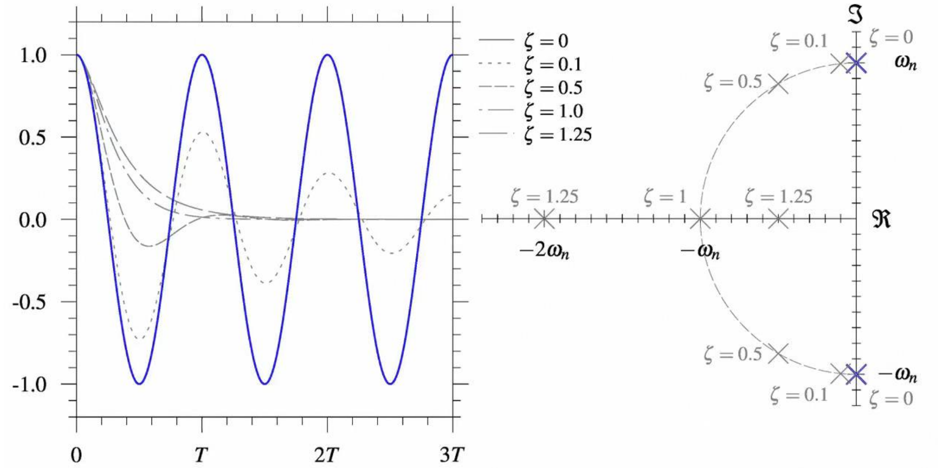




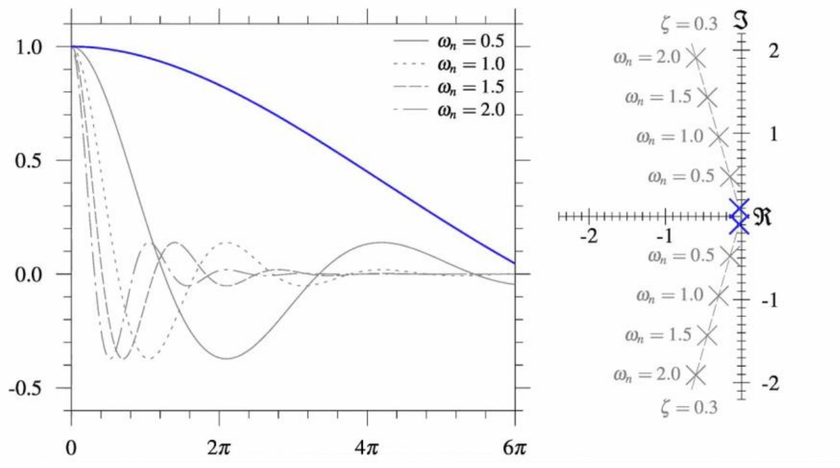
The Natural Frequency, Damping Ratio, and Pole Placement



Second-Order Response of Varying Damping Ratio



Second-Order Response of Varying Natural Frequency

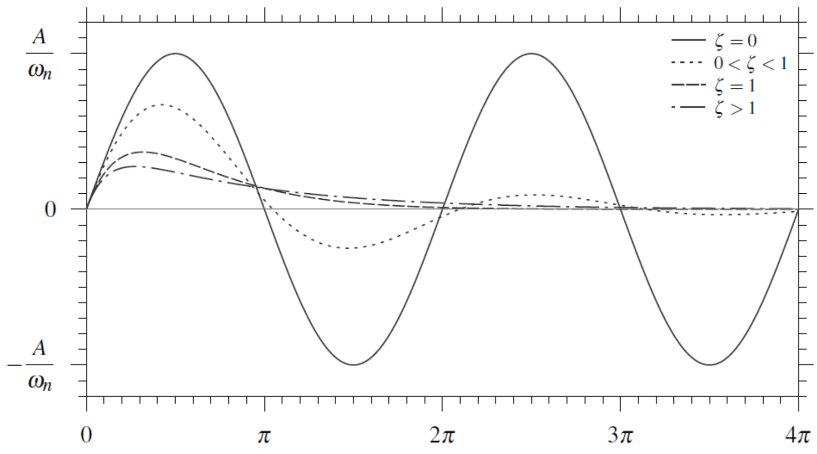


The Impulse Response

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = A\tilde{\delta}(t)$$

Undamped	$x(t) = \frac{A}{\omega_n} sin\omega_n t$	
Underdamped	$x(t) = \frac{A}{\omega_n} e^{-\zeta\omega_n t} sin\omega_d t$	
Critically Damped	$x(t) = Ate^{-\omega_n t}$	
Overdamped	$x(t) = \frac{A}{2\omega_n\sqrt{\zeta^2 - 1}}\left[-e^{-\omega_n\zeta_1 t} + e^{-\omega_n\zeta_2 t}\right]$	
	where	$\zeta_1 = \zeta + \sqrt{\zeta^2 - 1}$
	and	$\zeta_2 = \zeta - \sqrt{\zeta^2 - 1}$

Second-Order Impulse Response

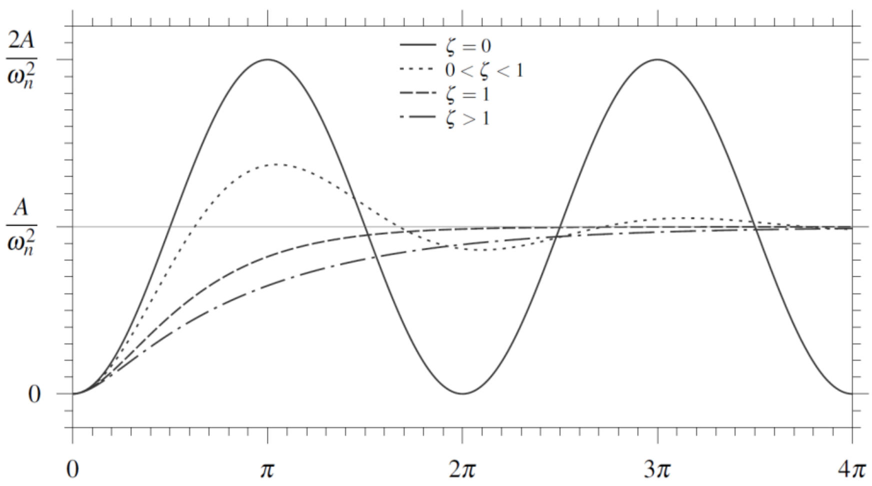


The Step Response

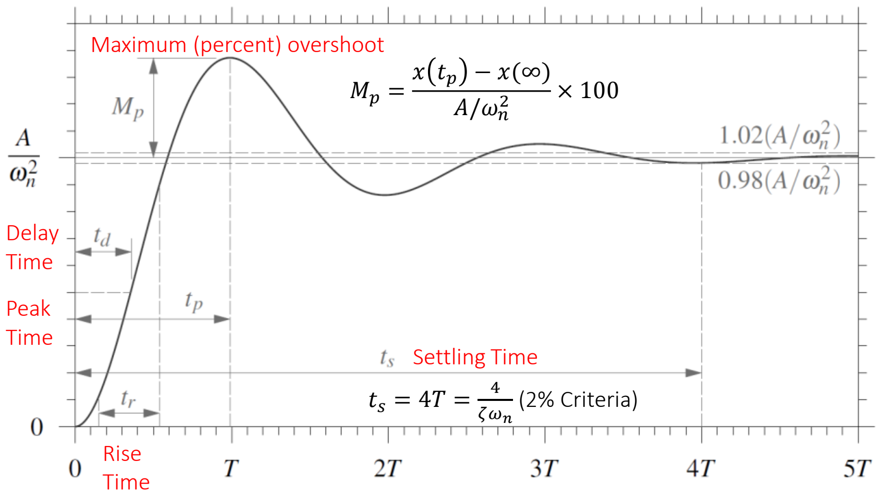
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = A1(t)$$

Undamped	$x(t) = \frac{A}{\omega_n^2}(1 - cos\omega_n t)$
Underdamped	$x(t) = \frac{A}{\omega_n^2}\left[1 - e^{-\zeta\omega_n t}\left(\frac{\zeta}{\sqrt{1-\zeta^2}} sin\omega_d t + cos\omega_d t\right)\right]$
Critically Damped	$x(t) = \frac{A}{\omega_n^2}[1 - e^{-\omega_n t} - \omega_n e^{-\omega_n t} t]$
Overdamped	$x(t) = \frac{A}{2\omega_n^2}\left[\frac{\zeta}{\sqrt{\zeta^2 - 1}}\left(e^{-\omega_n\zeta_1 t} - e^{-\omega_n\zeta_2 t}\right) - e^{-\omega_n\zeta_1 t} - e^{-\omega_n\zeta_2 t} + 2\right]$

Second-Order Step Response



Characteristics of an Underdamped Step Response

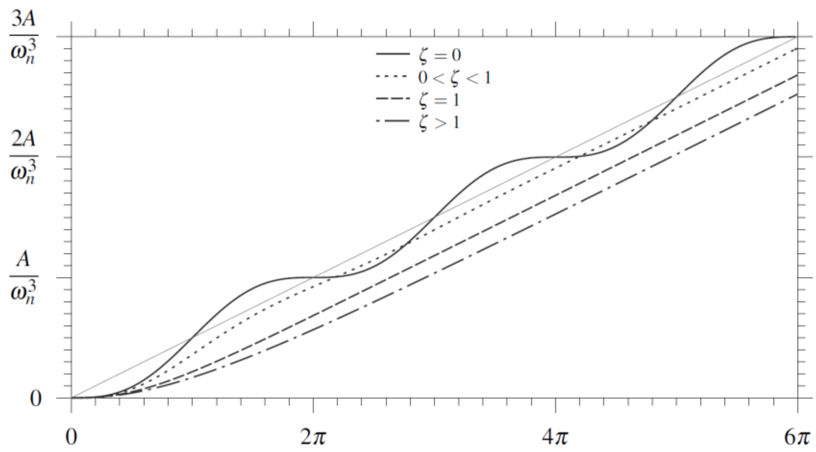


The Ramp Response

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = At$$

Undamped	$x(t) = \frac{A}{\omega_n^3}(\omega_n t - sin\omega_n t)$
Underdamped	$x(t) = \frac{A}{\omega_n^3}\left\{\frac{e^{-\zeta\omega_n t}}{\omega_n}\left[\frac{\zeta}{\sqrt{1-\zeta^2}}(\omega_d cos\omega_d t + \zeta\omega_n sin\omega_d t) + \zeta\omega_n cos\omega_d t - \omega_d sin\omega_d t\right] + \omega_n t - 2\zeta\right\}$
Critically Damped	$x(t) = \frac{A}{\omega_n^3}[\omega_n t + \omega_n e^{-\omega_n t}t - 2e^{-\omega_n t} - 2]$
Overdamped	$x(t) = \frac{A}{2\omega_n^3}\left[\frac{\zeta}{\sqrt{\zeta^2-1}}\left(-\zeta_2 e^{-\omega_n\zeta_1 t} + \zeta_1 e^{-\omega_n\zeta_2 t} - 2\sqrt{\zeta^2-1}\right) + \zeta_2 e^{-\omega_n\zeta_1 t} + \zeta_1 e^{-\omega_n\zeta_2 t} - 2\zeta + 2\omega_n t\right]$

Second-Order Ramp Response



Transient Responses of Higher-Order Systems

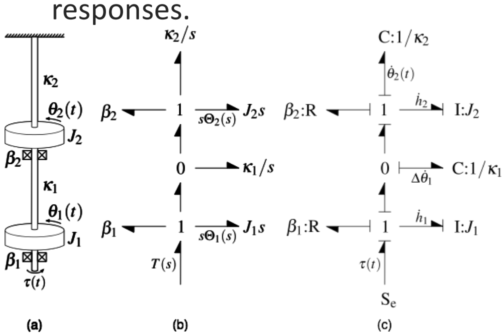
$$\frac{\mathsf{x}(s)}{\mathsf{u}(s)} = \frac{\hat{b}_0 s^m + \hat{b}_1 s^{m-1} + \cdots + \hat{b}_{n-1} s + \hat{b}_n}{s^n + \hat{a}_1 s^{n-1} + \cdots + \hat{a}_{n-1} s + \hat{a}_n}$$

$$\mathsf{x}(s) = \frac{a}{s} + \sum_{j=1}^{q} \frac{a_j}{s+p_j} + \sum_{k=1}^{r} \frac{b_k(s+\zeta_k\omega_k) + c_k\omega_k\sqrt{1-\zeta_k^2}}{s^2 + 2\zeta_k\omega_k s + \omega_k^2}$$

$$x(t) = a + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} b_k e^{-\zeta_k\omega_k t}\cos\omega_k\sqrt{1-\zeta_k^2}\,t + \sum_{k=1}^{r} c_k e^{-\zeta_k\omega_k t}\sin\omega_k\sqrt{1-\zeta_k^2}\,t$$

Example 7.3

A two-disk system is depicted, and the system parameters are provided in Table 7.1 in your textbook. Determine the lower-order components that contribute to the overall responses $\Theta_1(t)$ and $\Theta_2(t)$. Plot and compare the individual contributions to the overall responses.



$$\frac{\theta_1(s)}{T(s)} = \frac{J_2 s^2 + \beta_2 s + \kappa_2 + \kappa_1}{a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

$$\frac{\theta_2(s)}{T(s)} = \frac{\kappa_1}{a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

$$a_0 = J_1 J_2$$
$$a_1 = J_1\beta_2 + J_2\beta_1$$
$$a_2 = J_1\kappa_2 + J_1\kappa_1 + J_2\kappa_1 + \beta_1\beta_2$$
$$a_3 = \beta_1\kappa_2 + \beta_1\kappa_1 + \beta_2\kappa_1$$
$$a_4 = \kappa_1\kappa_2$$

Example 7.3

Use MATLAB to find poles

```
>> J1 = 0.0104;
>> J2 = J1;
>> K1 = 7;
>> K2 = 2;
>> B1 = 0.01;
>> B2 = 0.02;
>> num1 = [J2 B2 (K1+K2)];
>> num2 = K1;
>> den = [J1*J2 (J1*B2+J2*B1) ...
     (J1*(K1+K2)+J2*K1+B1*B2) ...
     (B1*(K1+K2)+B2*K1) K1*K2];
>> G1 = tf(num1,den);
>> G2 = tf(num2,den);
>> pole(G1)

ans =

  -0.7550 +38.0560i
  -0.7550 -38.0560i
  -0.6872 + 9.4265i
  -0.6872 - 9.4265i
```

Responses are combinations of two second-order underdamped responses

$$\Theta_1(s) = \frac{-0.03333s + 41.26}{s^2 + 1.51s + 1449} + \frac{0.03333s + 54.88}{s^2 + 1.374s + 89.33} = \Theta_{1a}(s) + \Theta_{1b}(s)$$

and

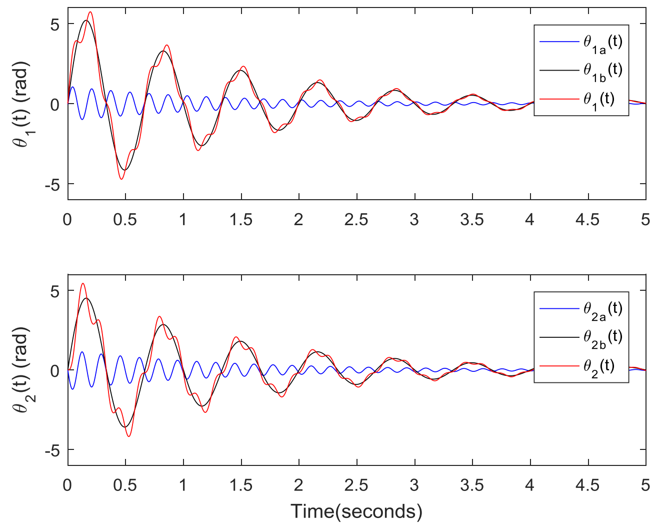
$$\Theta_2(s) = \frac{0.004746s - 47.6}{s^2 + 1.51s + 1449} + \frac{-0.004746s + 47.6}{s^2 + 1.374s + 89.33} = \Theta_{2a}(s) + \Theta_{2b}(s),$$

$$\begin{aligned}\theta_1(t) &= \theta_{1a}(t) + \theta_{1b}(t)\\ &= -0.03333e^{-0.755t}(\cos 38.05t - 32.54\sin 38.05t)\\ &\quad + 0.03333e^{-0.687t}(\cos 9.427t + 175\sin 9.427t)\end{aligned}$$

and

$$\begin{aligned}\theta_2(t) &= \theta_{2a}(t) + \theta_{2b}(t)\\ &= 0.0047e^{-0.755t}(\cos 38.05t - 263.6\sin 38.05t)\\ &\quad - 0.0047e^{-0.687t}(\cos 9.427t - 1064\sin 9.427t).\end{aligned}$$

Example 7.3 Response



An Introduction to Pole-Zero Analysis

$$G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

When a zero is in close proximity to a pole

$$G(s) = \frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)(s+p_3)} \approx \frac{(s+z_1)}{(s+p_1)(s+p_3)}$$

Except for oscillatory or undamped responses, stable responses will tend to attenuate

Real roots will result in terms like

$$Ae^{-at}$$

and recalling the roots of underdamped systems,

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2},$$

the response is bonded by a decaying exponential

$$e^{-\zeta\omega_n t}.$$

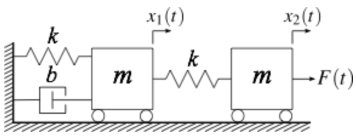
Remember that underdamped second-order responses will attenuate to within 2% in 4T

Pole-Zero Analysis using MATLAB

Function	Description
damp(sys)	Computes natural frequencies and damping ratios
pzmap(sys)	Plots pole-zero map of dynamic systems
sgrid	Generates s-plane grid lines for a pole-zero map
poly(A)	Computes characteristic polynomial of matrix

Example 7.4

Determine the dominant pole of the system.



The transfer function is

$$\frac{X_2(s)}{F(s)} = \frac{ms^2 + bs + 2k}{m^2s^4 + mbs^3 + 3mks^2 + bks + k^2}.$$

The zeros are

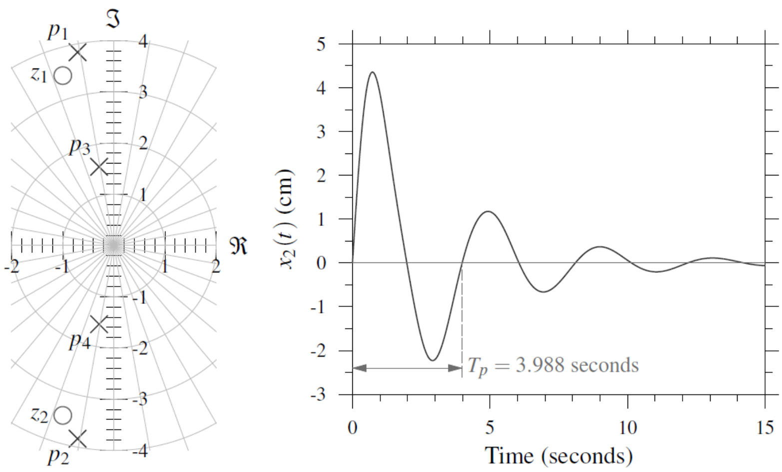
$$z_{1,2} = -1.0000 \pm 3.3166j$$

and the poles are

$$p_{1,2} = -0.7106 \pm 3.7722j \quad \text{and}$$
$$p_{3,4} = -0.2894 \pm 1.5361j.$$

```
>> m = 10; b = 20; k = 60;
>> sys2 = tf([m b 2*k],...
[m^2 m*b 3*m*k b*k k^2]);
>> Z2 = zero(sys2)
Z2 =
  -1.0000 + 3.3166i
  -1.0000 - 3.3166i
>> P2 = pole(sys2)
P2 =
  -0.7106 + 3.7722i
  -0.7106 - 3.7722i
  -0.2894 + 1.5361i
  -0.2894 - 1.5361i
>> pzmap(sys2); sgrid
```

Example 7.4 Response



Example 7.4

For poles $p_{1,2}$ the natural frequency is

$$\omega_{n1} = \sqrt{(-0.7106)^2 + (3.7722)^2} = 3.8385 \text{ rad/s},$$

and for poles $p_{3,4}$ the frequency is

$$\omega_{n2} = \sqrt{(-0.2894)^2 + (1.5361)^2} = 1.5631 \text{ rad/s}.$$

The damping ratio for poles $p_{1,2}$ is

$$\zeta_1 = \frac{0.7106}{3.8385} = 0.1851$$

and for poles $p_{3,4}$ is

$$\zeta_2 = \frac{0.2894}{1.5631} = 0.1851.$$

The contribution due to poles $p_{1,2}$ should attenuate in

$$t_{s1} = \frac{4}{\zeta_1 \omega_{n1}} = \frac{4}{0.7106 \text{ rad/s}} = 5.6289 \text{ seconds}.$$

For poles $p_{3,4}$, the settling time is

$$t_{s2} = \frac{4}{\zeta_2 \omega_{n_2}} = \frac{4}{0.2894 \text{ rad/s}} = 13.8228 \text{ seconds}.$$

```
>> damp(sys2)

         Eigenvalue              Damping      Frequency

 -2.89e-01 + 1.54e+00i          1.85e-01      1.56e+00
 -2.89e-01 - 1.54e+00i          1.85e-01      1.56e+00
 -7.11e-01 + 3.77e+00i          1.85e-01      3.84e+00
 -7.11e-01 - 3.77e+00i          1.85e-01      3.84e+00

(Frequencies expressed in rad/seconds)

>> pzmap(sys2)
>> sgrid(0:0.1:1,1:4)
>> impulse(sys2)
```

$$T_{p1} = \frac{2\pi}{\omega_{d1}} = \frac{2\pi}{\omega_{n1}\sqrt{1-\zeta_1^2}}$$

$$= \frac{2\pi}{(3.8385 \text{ rad/s})\sqrt{1-0.1851^2}} = 1.6657 \text{ s}$$

$$T_{p2} = \frac{2\pi}{\omega_{d2}} = \frac{2\pi}{\omega_{n2}\sqrt{1-\zeta_2^2}}$$

$$= \frac{2\pi}{(1.5631 \text{ rad/s})\sqrt{1-0.1851^2}} = 4.0904 \text{ s}.$$

Pole-Zero Analysis in the State-Space

The poles are the roots of the denominator

To find the poles, we seek the eigenvalues or the roots of the characteristic equation.

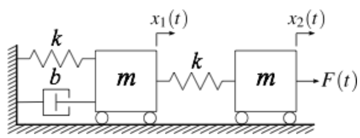
$$\det(s\boldsymbol{I} - \boldsymbol{A})\Big|_{s=p} = \det(p\boldsymbol{I} - \boldsymbol{A}) = 0$$

The zeros are the roots of the numerator

To find the zeros, we seek the roots of

$$det\begin{bmatrix} \boldsymbol{A} - z\boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} = 0$$

Example 7.5



Recall that the state-space model of the mass-spring-damper system in Example 7.4 is composed of the following matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 1/m & 0 & 0 \\ -k & -b/m & k & 0 \\ 0 & -1/m & 0 & 1/m \\ 0 & 0 & -k & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

For this problem, we only require the second displacement, $x_2(t)$, as an output making matrices $\mathbf{C}$ and D:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \text{ and } D = 0.$$

$$\det(p\mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} p & -1/m & 0 & 0 \\ k & p + b/m & -k & 0 \\ 0 & 1/m & p & -1/m \\ 0 & 0 & k & p \end{bmatrix} = \frac{m^2p^4 + mbp^3 + 3mkp^2 + bkp + k^2}{m^2} = 0$$

$$\det \begin{bmatrix} \mathbf{A} - z\mathbf{I} & \mathbf{B} \\ \mathbf{C} & D \end{bmatrix} = \det \left[\begin{array}{cccc|c} -z & 1/m & 0 & 0 & 0 \\ -k & -b/m - z & k & 0 & 0 \\ 0 & -1/m & -z & 1/m & 0 \\ 0 & 0 & -k & -z & 1 \\ \hline 1 & 0 & 1 & 0 & 0 \end{array}\right] = \frac{mz^2 + bz + 2k}{m^2} = 0.$$

Steady-State Analysis in the State Space

Generally, at steady-state

$$\dot{\mathbf{x}}(t \to \infty) = \dot{\mathbf{x}}_{ss} = \mathbf{0}.$$

Thus, for the natural response

$$\mathbf{A}\mathbf{x}_{ss} = \mathbf{0} \quad \Rightarrow \quad \mathbf{x}_{ss} = \mathbf{0}.$$

For the impulse and step responses

$$\mathbf{A}\mathbf{x}_{ss} + \mathbf{B}u_{ss} = \mathbf{0}$$

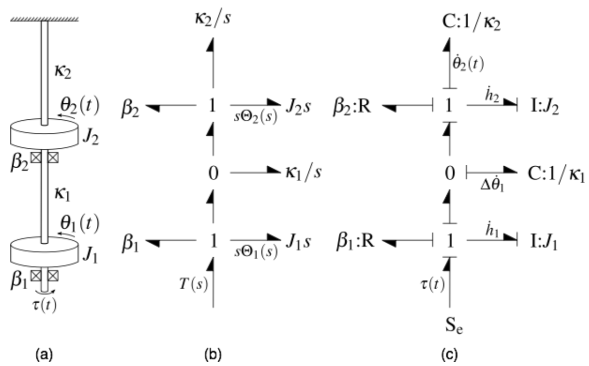
where $u_{ss} = u(t \to \infty)$. We can show

$$\begin{aligned} \mathbf{A}\mathbf{x}_{ss} &= -\mathbf{B}u_{ss} \\ \mathbf{x}_{ss} &= -\mathbf{A}^{-1}\mathbf{B}u_{ss}. \end{aligned}$$

Knowing the steady-state input and states, one can also find the steady-state outputs:

$$\begin{aligned} \mathbf{y}_{ss} &= \mathbf{C}\mathbf{x}_{ss} + \mathbf{D}u_{ss} \\ &= -\mathbf{C}\mathbf{A}^{-1}\mathbf{B}u_{ss} + \mathbf{D}u_{ss} \\ &= \left[-\mathbf{C}\mathbf{A}^{-1}\mathbf{B} + \mathbf{D}\right]u_{ss}. \end{aligned}$$

Example 7.6



$$\begin{bmatrix} \dot{h}_1 \\ \Delta\dot{\theta}_1 \\ \dot{h}_2 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -\beta_1/J_1 & -\kappa_1 & 0 & 0 \\ 1/J_1 & 0 & -1/J_2 & 0 \\ 0 & \kappa_1 & -\beta_2/J_2 & -\kappa_2 \\ 0 & 0 & 1/J_2 & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ \Delta\theta_1 \\ h_2 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tau(t) \quad \text{and} \quad \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ \Delta\theta_1 \\ h_2 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tau(t)$$

Summary

The natural response (also known as the unforced response) is the dynamic response that results when there are no inputs but the initial conditions are generally not zero.

Natural and impulse responses for first-order systems exponentially decay. The time constant, T, is the amount of time required for the response to decay to 36.8% of the initial value. The responses decay to under 2% in 4T.

The first-order step response asymptotically approaches the steady-state value. The time constant,T, is the time required to reach 63.2% of the steady-state value. Within 4T the step response is within 98% of the final value.

The first-order ramp response (increases at a constant rate that runs parallel to the input. At steady-state, there is a constant time delay between the input and response equal to the time constant, T. Also, at steady-state, the difference between the response and the input is AT, where A is the rate.

Second-order systems can be categorized based on the amount of damping:

Undamped, $\zeta = 0$

Underdamped, $0 < \zeta < 1$

Critically damped, $\zeta = 1$

Overdamped, $\zeta > 1$

Summary Continued

Undamped responses oscillate forever. Underdamped responses will oscillate, but attenuate to the steady-state value. Critically damped and overdamped responses do not oscillate, but rather asymptotically approach steady-state.

The roots of the characteristic equation are referred to as the poles. Poles are generally complex. The placement of second-order poles in the real-complex plane is correlated with the damping ratio (ζ), undamped natural frequency (ω_n), and the damped frequency of oscillation (ω_d):

Undamped, purely imaginary roots

Underdamped, complex conjugate roots

Critically damped, real repeated negative roots

Overdamped, real distinct negative roots

The second-order natural and impulse responses oscillate about, attenuate to, or asymptotically approach zero.

The unit step responses for second-order systems oscillate about, attenuate to, or asymptotically approach one.

The ramp responses of second-order systems oscillate about the ramp or settle to a steady-state response that runs parallel to the ramp with a constant error.

Summary Continued

Higher-order responses are just combinations of first- and second-order responses.

Transfer functions can be factored into first- and second-order terms. The second-order terms have complex poles that have associated damping ratios and natural frequencies. Generally, the poles closest to the imaginary axis have dynamics that take longer to settle. If not negated by the presence of zeros, the poles closest to the imaginary axis will tend to dominate the overall response.

MATLAB includes functions that facilitate transient analysis including commands to determine eigenvalues, damping ratios, and natural frequencies. It also includes functions for generating and annotating pole-zero maps.

Though poles and zeros are commonly determined by factoring the numerator and denominator of the transfer function representation, they can also be readily determined using the state-space model. The poles, for example, are the eigenvalues of the A matrix in the state-space representation.

For systems that reach a static condition at steady-state, the rate of change of the state vector, $\mathbf{x}$, is 0. Thus, the steady-state condition, $\mathbf{x}_{ss}$, can be determined algebraically from the steady-state input, $\mathbf{u}_{ss}$.