CHAPTER 5:
LAPLACE TRANSFORMS
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PREVIEW QUESTIONS

- What are some commonly recurring functions in dynamic systems and their Laplace transforms?
- How can Laplace transforms be used to solve for a dynamic response in the time domain?
- What purpose does the Laplace transform play in analyzing the time domain response?
- What are the relationships between the time domain and s-domain?
OBJECTIVES & OUTCOMES

- Objectives
  - To review the transforms of commonly occurring functions used for modeling dynamic systems,
  - To review some of the more often used theorems that aid in the analysis and solution of dynamic systems, and
  - To understand algebraic analysis of Laplace transforms.
- Outcomes: Upon completion, you should
  - Be able to transform and inverse transform common functions,
  - Be able to conduct partial fraction expansions to find inverse Laplace transforms,
  - Be able to use MATLAB to analyze Laplace transforms, and
  - Be able to solve linear differential equations using Laplace transforms.
5.3 THE LAPLACE TRANSFORM

- Laplace transform of a function
  \[ \mathcal{L}[g(t)] = g(s) = \int_0^\infty e^{-st}g(t)dt \]

- Inverse Laplace transform of a function
  \[ \mathcal{L}^{-1}[g(s)] = g(t) \]

- The Laplace transform is a linear operation
  \[ \mathcal{L}[a g_1(t) + b g_2(t) + c g_3(t)] = a \mathcal{L}[g_1(t)] + b \mathcal{L}[g_2(t)] + c \mathcal{L}[g_3(t)] \]
  \[ = a g_1(s) + b g_2(s) + c g_3(s) \]
EXISTENCE OF THE LAPLACE TRANSFORM

- Requirements
  1. $g(t)$ is a piecewise continuous function on the interval $0 < t < \infty$.
  2. $g(t)$ is of exponential order. That is to say there exist real-valued positive constants $A$ and $t$ such that $|g(t)| \leq A e^{at}$ for all $t \geq T$.

- Does the Laplace transform exist for this function?

\[ f(t) \]

\[ t_1 \quad t_2 \]
**Transforms of Common Functions**

**Exponential Functions**
\[ g(t) = \begin{cases} 
0, & t < 0 \\
Ae^{-at}, & t \geq 0 
\end{cases} \]
\[ \mathcal{L}[Ae^{-at}] = A \frac{1}{s + a} \]

**Step Functions**
\[ A1(t) = \begin{cases} 
0, & t < 0 \\
A, & t \geq 0 
\end{cases} \]
\[ \mathcal{L}[A1(t)] = \frac{A}{s} \]

**Ramp Functions**
\[ g(t) = \begin{cases} 
0, & t < 0 \\
At, & t \geq 0 
\end{cases} \]
\[ \mathcal{L}[At] = \frac{A}{s^2} \]

**Sinusoidal Functions**
\[ g(t) = \begin{cases} 
0, & t < 0 \\
A \sin \omega t, & t \geq 0 
\end{cases} \]
\[ \mathcal{L}[A \sin \omega t] = A \frac{\omega}{s^2 + \omega^2} \]
\[ \mathcal{L}[A \cos \omega t] = \frac{A}{s^2 + \omega^2} \]
MULTIPLICATION BY EXPONENTIAL

\[ \mathcal{L}[e^{-at}g(t)] = g(s + a) \]

\[ \mathcal{L}[e^{-at}\sin(\omega t)] = \frac{\omega}{(s + a)^2 + \omega^2} \]

\[ \mathcal{L}[e^{-at}\cos(\omega t)] = \frac{s + a}{(s + a)^2 + \omega^2} \]

- These will require completing the square.
SHIFTING VS TRANSLATION OF A FUNCTION

\[ \mathcal{L}[g(t-a)1(t-a)] = e^{-as}g(s) \]

- Evaluating a mathematical function at \( t-a \) shifts the function to the right along \( t \) an amount \( a \).
- However, when deriving Laplace transforms, we have assumed that the functions are “truncated” such that they are 0 for \( t<0 \).
- A translated function is truncated and shifted.

\( f(t) \), \( f(t-a) \) (a) The function shifted

\( 1(t-a) \) (b) Unit step shifted

\( f(t-a)1(t-a) \) (c) The function translated
A RECTANGULAR PULSE

- Two step functions are summed to generate a rectangular pulse.

\[ g(t) = A_1(t - t_1) - A_1(t - t_2) = A[1(t - t_1) - 1(t - t_2)] \]

\[ \mathcal{L}[g(t)] = A \left[ \frac{e^{-t_1 s}}{s} - \frac{e^{-t_2 s}}{s} \right] = \frac{A}{s} \left[ e^{-t_1 s} - e^{-t_2 s} \right] \]
EXAMPLE 5.1

- Find the Laplace transform for the piecewise continuous function in the figure shown.

\[ g_1(t) = At \cdot 1(t) \]

\[ g(t) = g_1(t) + g_2(t) \]

\[ g_2(t) = -A(t - t_1) \cdot 1(t - t_1) \]
\[ \mathcal{L}[A\delta(t)] = \lim_{t_0 \to 0} \frac{A}{t_0} [1 - e^{-t_0 s}] \]

\[ = \lim_{t_0 \to 0} \frac{d}{dt_0} \left[ A[t_0 - e^{-t_0 s}] \right] \]

\[ = \lim_{t_0 \to 0} \frac{d}{dt_0} t_0 s \]

\[ = \lim_{t_0 \to 0} \frac{Ase^{-t_0 s}}{s} \]

\[ = \frac{As}{s} \]

\[ = A \]
**DIFFERENTIATION THEOREM**

- Laplace transform of the first derivative
  \[ \mathcal{L}[\dot{g}(t)] = s g(s) - g(0) \]

- Laplace transform of the second derivative
  \[ \mathcal{L}[\ddot{g}(t)] = s^2 g(s) - sg(0) - \dot{g}(0) \]

- Laplace transform of the nth derivative
  \[ \mathcal{L}[g^{(n)}(t)] = s^n g(s) - s^{n-1}g(0) - s^{n-2}\dot{g}(0) - \cdots - g^{(n-1)}(0) \]
EXAMPLE 5.2

- Using the differentiation theorem, derive the Laplace transforms of the untranslated unit step and impulse functions from the unit ramp function.
INTEGRATION, FINAL VALUE, AND INITIAL VALUE THEOREMS

- Integration Theorem
  \[ L \left[ \int_0^t g(t)dt \right] = \frac{g(s)}{s} \]

- Final Value Theorem
  \[ \lim_{t \to \infty} g(t) = \lim_{s \to 0} sg(s) \]

- Initial Value Theorem
  \[ g(0+) = \lim_{s \to 0} sg(s) \]
EXAMPLE 5.3

- Using the integration theorem, beginning with the impulse function, derive the Laplace transforms of the unit step and ramp functions.
PARTIAL FRACTION EXPANSION
WITH DISTINCT POLES

\[
G(s) = \frac{B(s)}{A(s)} = \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)} = \frac{r_1}{s + p_1} + \frac{r_2}{s + p_2} + \cdots + \frac{r_n}{s + p_n}
\]

\[
\left( s + p_k \right) \frac{B(s)}{A(s)} \bigg|_{s=-p_k} = \left[ \frac{r_1}{s + p_1} + \frac{r_2}{s + p_2} + \cdots + \frac{r_k}{s + p_k} + \cdots + \frac{r_n}{s + p_n} \right]_{s=-p_k} = r_k
\]

\[
r_k = \left( s + p_k \right) \frac{B(s)}{A(s)} \bigg|_{s=-p_k}
\]
Recall the quarter-car suspension simulated in Example 4.4. Assume for this particular example that the mass, damping constant, and spring rate are 500 kg, 8,000 N-s/m, and 30,000 N/m, respectively. Derive the response of the system to a unit impulse displacement.

\[ \dot{p} = k \delta + b \left[ v_{\text{road}}(t) - \frac{p}{m} \right] \]
\[ \delta = v_{\text{road}}(t) - \frac{p}{m} \]
\[ \dot{y} = \frac{p}{m} \]
**PARTIAL FRACTION EXPANSION WITH REPEATED POLES**

\[
G(s) = \frac{B(s)}{A(s)} = \frac{K(s + z_1)(s + z_2) \ldots (s + z_m)}{(s + p_1)(s + p_2) \ldots (s + p_n)} = \frac{r_1}{(s + p_1)} + \frac{r_2}{(s + p_2)} + \ldots + \frac{r_n}{(s + p_n)}
\]

\[
A(s) \frac{B(s)}{A(s)} = (s + p_1)(s + p_2)^2 \ldots (s + p_n) \frac{K(s + z_2)(s + z_2) \ldots (s + z_m)}{(s + p_1)(s + p_2) \ldots (s + p_n)}
\]

\[
K(s + z_1)(s + z_2) \ldots (s + z_m) = (s + p_2)^2 \ldots (s + p_n)r_1 + (s + p_1)(s + p_2) \ldots (s + p_n)r_2 + (s + p_1)(s + p_2) \ldots (s + p_{n-1})r_n
\]
EXAMPLE 5.5

Find the partial fraction expansion of \( G(s) = \frac{s + 4}{(s + 1)(s + 2)^2(s + 3)} \)
EXAMPLE 5.6

- Find the unit impulse response of the quarter-car suspension from Example 5.4 if the mass, damping constant, and spring rate are 500 kg, 8,000 N-s/m, and 32,000 N/m, respectively. Recall that the transform we derived was

\[ y(s) = \frac{bs + k}{ms^2 + bs + k} y_{\text{road}}(s), \]
\[ G(s) = \frac{B(s)}{A(s)} = \frac{K(s + z_1)(s + z_2)}{(s + p_1)(s + p_2)((s + a)^2 + \omega^2)} \]

\[ = \frac{r_1}{(s + p_1)} + \frac{r_2}{(s + p_2)} + \frac{r_3s + r_4}{(s + a)^2 + \omega^2} \]

\[ K(s + z_1)(s + z_2) = (s + p_2)((s + a)^2 + \omega^2) r_1 \]
\[ + (s + p_1)((s + a)^2 + \omega^2) r_2 \]
\[ + (s + p_1)(s + p_2)(r_3s + r_4) \]
EXAMPLE 5.7

- Find the partial fraction expansion of \[ G(s) = \frac{s + 4}{(s + 1)(s^2 + 2s + 5)} \]
EXAMPLE 5.8

- Find the unit impulse response of the quarter-car suspension from Example 5.4 if the mass, damping constant, and spring rate are 500 kg, 8,000 N-s/m, and 34,000 N/m, respectively. Recall that the transform we derived was

\[ y(s) = \frac{bs + k}{ms^2 + bs + k} y_{road}(s) \]
## POLES, ZEROS, PARTIAL FRACTION EXPANSION, AND MATLAB

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>tf(num,den)</td>
<td>defines transfer function using polynomials</td>
</tr>
<tr>
<td>pole(sys)</td>
<td>computes poles of a LTI system object</td>
</tr>
<tr>
<td>zero(sys)</td>
<td>computes zeros of a LTI system object</td>
</tr>
<tr>
<td>zpk(Z,P,K)</td>
<td>defines transfer function with zeros, poles, and gain</td>
</tr>
<tr>
<td>conv(A,B)</td>
<td>calculates polynomial multiplication</td>
</tr>
<tr>
<td>residue(num,den)</td>
<td>computes partial fraction expansion</td>
</tr>
</tbody>
</table>
EXAMPLE 5.9

- Using MATLAB, find the poles, zeros, and partial fraction expansion of

\[ G(s) = \frac{s^2 + 9s + 20}{s^4 + 4s^3 + 10s^2 + 12s + 5} \]
SUMMARY

- Laplace transforms are used to convert differential equations in the time domain to algebraic equations in the s-domain.

- Linear differential equations become polynomials in the s-domain.

- Sinusoids can be represented using complex exponential functions. As such, they can be manipulated using basic algebraic principles.

- A Laplace transform can be decomposed through partial fraction expansions into terms that can be readily inverse Laplace transformed using Laplace transform primitives.

- Laplace transforms lead to transfer function models. A transfer function is an algebraic construct that represents the output/input relation in the s-domain.

- The zeros are defined as the roots of the polynomial in the numerator of a transfer function.

- The poles are defined as the roots of the polynomial in the denominator of a transfer function.

- MATLAB includes a series of functions to represent transfer functions, compute poles and zeros, conduct polynomial multiplication, and compute partial fraction expansions.