CHAPTER 5: LAPLACE TRANSFORMS

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PREVIEW QUESTIONS

- What are some commonly recurring functions in dynamic systems and their Laplace transforms?
- How can Laplace transforms be used to solve for a dynamic response in the time domain?
- What purpose does the Laplace transform play in analyzing the time domain response?
- What are the relationships between the time domain and s-domain?

OBJECTIVES & OUTCOMES

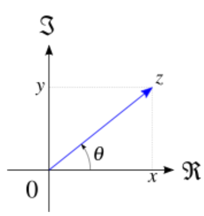
- Objectives
  - To review the transforms of commonly occurring functions used for modeling dynamic systems,
  - To review some of the more often used theorems that aid in the analysis and solution of dynamic systems, and
  - To understand algebraic analysis of Laplace transforms.
- Outcomes: Upon completion, you should
  - Be able to transform and inverse transform common functions,
  - Be able to conduct partial fraction expansions to find inverse Laplace transforms,
  - Be able to use MATLAB to analyze Laplace transforms, and
  - Be able to solve linear differential equations using Laplace transforms.

5.2.1 COMPLEX NUMBERS

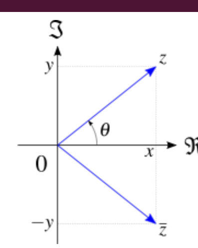
$z = x + jy$

$|z| = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\dfrac{y}{x}$



$\bar{z} = x - jy$



$$\begin{aligned} z &= x + jy \\ &= |z|\cos\theta + j|z|\sin\theta \\ &= |z|(\cos\theta + j\sin\theta) \\ &= |z|\underline{/\theta} \\ &= |z|e^{j\theta}. \end{aligned}$$

$$z = z + jy = |z|(\cos\theta + j\sin\theta) = |z|\underline{/\theta}$$
$$\bar{z} = x - jy = |z|(\cos\theta - j\sin\theta) = |z|\underline{/-\theta}.$$

$$z\bar{z} = (x + jy)(x - jy) = x^2 - jxy + jxy + y^2 = x^2 + y^2,$$

$$\frac{1}{z} = \frac{1}{z}\frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{x^2 + y^2}.$$

5.2.2 EULER'S THEOREM

- Refer to the textbook (§5.2.2 for derivation of theorem and identities)
- Euler's Theorem

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

- Cosine Identity

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

- Sine Identity

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

5.2.3 COMPLEX ALGEBRA

$z = x + jy$ and $w = u + jv$

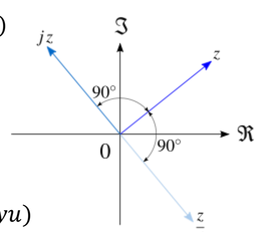
$$z + w = (x + u) + j(y + v)$$

$$z - w = (x - u) + j(y - v)$$

$$zw = (xu - yv) + j(xv + yu)$$

$$zw = |z||w|\angle(\theta + \phi)$$

$$\frac{z}{w} = \frac{|z|}{|w|}\angle(\theta - \phi) = \frac{xu + yv}{u^2 + y^2} + j\frac{yu - xv}{u^2 + y^2}$$



$$az = ax + jay$$

$$jz = -y + jx = |z|\angle(0 + 90°)$$

$$\frac{z}{j} = y - jx = |z|\angle(\theta - 90°)$$

$$z^n = (|z|\angle\theta)^n = |z|^n\angle(n\theta)$$

$$z^{1/n} = (|z|\angle\theta)^{1/n} = |z|^{1/n}\angle(\theta/n)$$

COMPLEX VARIABLES AND FUNCTIONS

$$s = \sigma + j\omega$$

$$G(s) = \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)}$$

- Transfer function
  - Ratio of polynomials in the s-domain
- Zeroes
  - Roots of the numerator
- Poles
  - Roots of the denominator

5.3 THE LAPLACE TRANSFORM

- Laplace transform of a function

$$\mathcal{L}[g(t)] = \mathrm{g(s)} = \int_0^\infty e^{-st} g(t)dt$$

- Inverse Laplace transform of a function

$$\mathcal{L}^{-1}[\mathrm{g(s)}] = g(t)$$

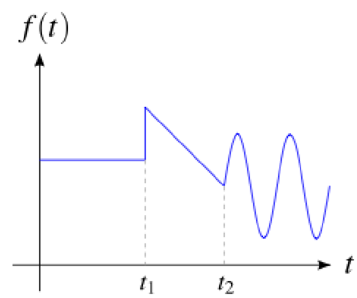
- The Laplace transform is a linear operation

$$\begin{aligned}\mathcal{L}[ag_1(t) + bg_2(t) + cg_3(t)] &= a\mathcal{L}[g_1(t)] + b\mathcal{L}[g_2(t)] + c\mathcal{L}[g_3(t)] \\ &= a\mathrm{g}_1(\mathrm{s}) + b\mathrm{g}_2(\mathrm{s}) + c\mathrm{g}_3(\mathrm{s})\end{aligned}$$

EXISTENCE OF THE LAPLACE TRANSFORM

- Requirements
  1. *g(t)* is a piecewise continuous function on the interval 0<t<∞.
  2. *g(t)* is of exponential order. That is to say there exist real-valued positive constants A and t such that $|g(t)| \leq Ae^{at}$ for all t≥T.

- Does the Laplace transform exist for this function?

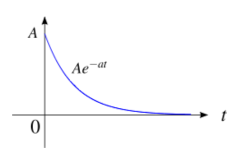


TRANSFORMS OF COMMON FUNCTIONS

Exponential Functions

$$g(t) = \begin{cases} 0, & t < 0 \\ Ae^{-at}, & t \geq 0 \end{cases}$$

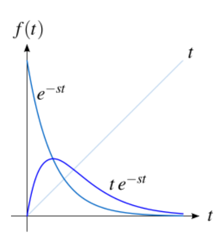
$$\mathcal{L}[Ae^{-at}] = A\frac{1}{s+a}$$



Step Functions

$$A1(t) = \begin{cases} 0, & t < 0 \\ A, & t \geq 0 \end{cases}$$

$$\mathcal{L}[A1(t)] = \frac{A}{s}$$



Ramp Functions

$$g(t) = \begin{cases} 0, & t < 0 \\ At, & t \geq 0 \end{cases}$$

$$\mathcal{L}[At] = \frac{A}{s^2}$$

Sinusoidal Functions

$$g(t) = \begin{cases} 0, & t < 0 \\ A\sin\omega t, & t \geq 0 \end{cases}$$

$$\mathcal{L}[A\sin\omega t] = A\frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}[A\cos\omega t] = A\frac{s}{s^2+\omega^2}$$

MULTIPLICATION BY EXPONENTIAL

$$\mathcal{L}[e^{-at}g(t)] = \mathrm{g(s + a)}$$

$$\mathcal{L}[e^{-at}\sin\omega t] = \frac{\omega}{(s+a)^2+\omega^2}$$

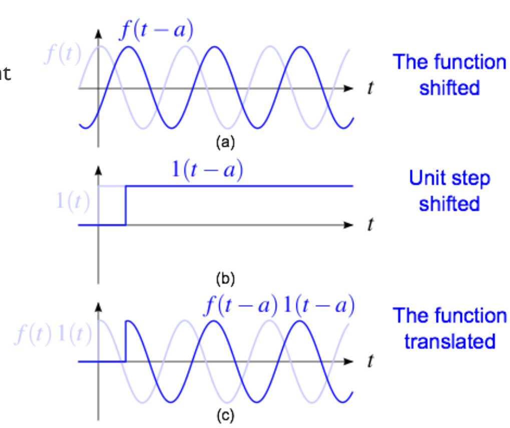
$$\mathcal{L}[e^{-at}\cos\omega t] = \frac{s+a}{(s+a)^2+\omega^2}$$

- These will require completing the square.

SHIFTING VS TRANSLATION OF A FUNCTION

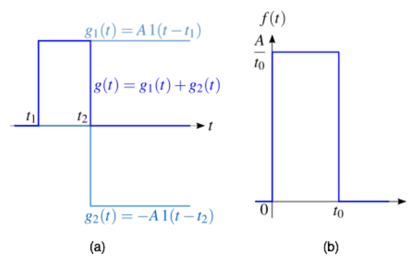
$$\mathcal{L}[g(t-a)1(t-a)] = e^{-as}g(s)$$

- Evaluating a mathematical function at *t-a* shifts the function to the right along *t* an amount *a*.
- However, when deriving Laplace transforms, we have assumed that the functions are "truncated" such that they are 0 for t<0.
- A translated function is truncated and shifted.



A RECTANGULAR PULSE

- Two step functions are summed to generate a rectangular pulse.

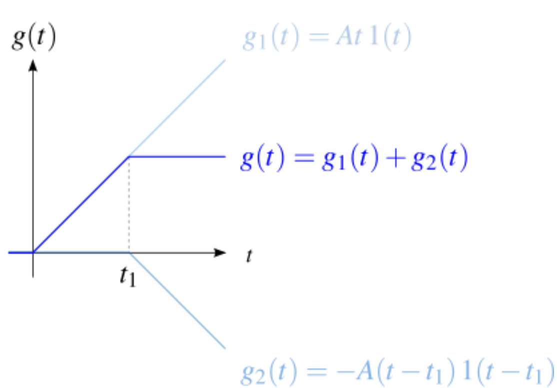


$$g(t) = A1(t - t_1) - A1(t - t_2) = A[1(t - t_1) - 1(t - t_2)]$$

$$\mathcal{L}[g(t)] = A\left[\frac{e^{-t_1 s}}{s} - \frac{e^{-t_2 s}}{s}\right] = \frac{A}{s}[e^{-t_1 s} - e^{-t_2 s}]$$

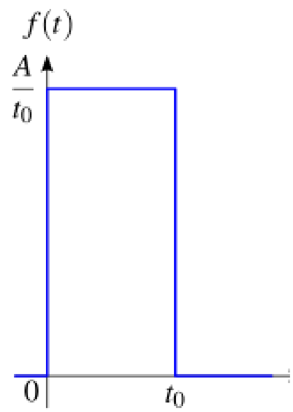
EXAMPLE 5.1

- Find the Laplace transform for the piecewise continuous function in the figure shown.



IMPULSE FUNCTIONS

$$\mathcal{L}[A\tilde{\delta}(t)] = \lim_{t_o \to 0} \frac{A}{t_o s}[1 - e^{-t_o s}]$$
$$= \lim_{t_o \to 0} \frac{\frac{d}{dt_o} A[1 - e^{-t_o s}]}{\frac{d}{dt_o} t_o s}$$
$$= \lim_{t_o \to 0} \frac{Ase^{-t_o s}}{s}$$
$$= \frac{As}{s}$$
$$= A$$



(b)

DIFFERENTIATION THEOREM

- Laplace transform of the first derivative

$$\mathcal{L}[\dot{g}(t)] = s\mathrm{g}(\mathrm{s}) - g(0)$$

- Laplace transform of the second derivative

$$\mathcal{L}[\ddot{g}(t)] = s^2\mathrm{g}(\mathrm{s}) - sg(0) - \dot{g}(0)$$

- Laplace transform of the nth derivative

$$\mathcal{L}[g^{(n)}(t)] = s^n\mathrm{g}(\mathrm{s}) - s^{n-1}g(0) - s^{n-2}\dot{g}(0) - \cdots - g^{(n-1)}(0)$$

EXAMPLE 5.2

- Using the differentiation theorem, derive the Laplace transforms of the untranslated unit step and impulse functions from the unit ram function.

INTEGRATION, FINAL VALUE, AND INITIAL VALUE THEOREMS

- Integration Theorem

$$\mathcal{L}\left[\int_0^t g(t)dt\right] = \frac{\mathrm{g(s)}}{s}$$

- Final Value Theorem

$$\lim_{t\to\infty} g(t) = \lim_{s\to 0} s\mathrm{g(s)}$$

- Initial Value Theorem

$$g(0+) = \lim_{s\to 0} s\mathrm{g(s)}$$

EXAMPLE 5.3

- Using the integration theorem, beginning with the impulse function, derive the Laplace transforms of the unit step and ramp functions.

PARTIAL FRACTION EXPANSION WITH DISTINCT POLES

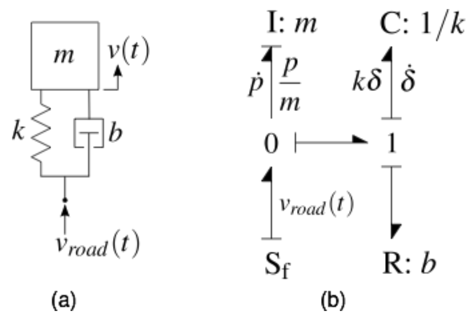
$$G(s) = \frac{B(s)}{A(s)} = \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)} = \frac{r_1}{(s+p_1)} + \frac{r_2}{(s+p_2)} + \cdots + \frac{r_n}{(s+p_n)}$$

$$\left[(s+p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k} = \left[\frac{r_1}{(s+p_1)}(s+p_{k)} + \frac{r_2}{(s+p_2)}(s+p_k) + \cdots \right.$$
$$\left. + \frac{r_k}{(s+p_k)}(s+p_k) + \cdots + \frac{r_n}{(s+p_n)}(s+p_k)\right]_{s=-p_k} = r_k$$

$$r_k = \left[(s+p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k}$$

EXAMPLE 5.4

- Recall the quarter-car suspension simulated in Example 4.4. Assume for this particular example that the mass, damping constant, and spring rate are 500 kg, 8,000 N-s/m, and 30,000 N/m, respectively. Derive the response of the system to a unit impulse displacement.



$$\dot{p} = k\delta + b\left[v_{road}(t) - \frac{p}{m}\right]$$

$$\dot{\delta} = v_{road}(t) - \frac{p}{m}$$

$$\dot{y} = \frac{p}{m}$$

PARTIAL FRACTION EXPANSION WITH REPEATED POLES

$$G(s) = \frac{B(s)}{A(s)} = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

$$= \frac{r_1}{(s+p_1)} + \frac{r_2}{(s+p_2)} + \cdots + \frac{r_n}{(s+p_n)}$$

$$A(s)\frac{B(s)}{A(s)} = (s+p_1)(s+p_2)^2\ldots(s+p_n)\frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

$$\begin{aligned} K(s+z_1)(s+z_2)\ldots(s+z_m) = {} & (s+p_2)^2\ldots(s+p_n)r_1 \\ & +(s+p_1)(s+p_2)\ldots(s+p_n)r_2 \\ & +(s+p_1)(s+p_2)\ldots(s+p_{n-1})r_n \end{aligned}$$

EXAMPLE 5.5

- Find the partial fraction expansion of $G(s) = \dfrac{s+4}{(s+1)(s+2)^2(s+3)}$

EXAMPLE 5.6

- Find the unit impulse response of the quarter-car suspension from Example 5.4 if the mass, damping constant, and spring rate are 500 kg, 8,000 N-s/m, and 32,000 N/m, respectively. Recall that the transform we derived was

$$y(s) = \frac{bs+k}{ms^2+bs+k} y_{road}(s).$$

PARTIAL FRACTION EXPANSION WITH COMPLEX POLES

$$G(s) = \frac{B(s)}{A(s)} = \frac{K(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)[(s+a)^2+\omega^2]}$$

$$= \frac{r_1}{(s+p_1)} + \frac{r_2}{(s+p_2)} + \frac{r_3 s + r_4}{(s+a)^2+\omega^2}$$

$$\begin{aligned} K(s+z_1)(s+z_2) = {} & (s+p_2)[(s+a)^2+\omega^2]\, r_1 \\ & +(s+p_1)[(s+a)^2+\omega^2] r_2 \\ & +(s+p_1)(s+p_2)(r_3 s + r_4) \end{aligned}$$

EXAMPLE 5.7

- Find the partial fraction expansion of $G(s) = \dfrac{s+4}{(s+1)(s^2+2s+5)}$

EXAMPLE 5.8

- Find the unit impulse response of the quarter-car suspension from Example 5.4 if the mass, damping constant, and spring rate are 500 kg, 8,000 N-s/m, and 34,000 N/m, respectively. Recall that the transform we derived was

$$y(s) = \frac{bs + k}{ms^2 + bs + k} y_{road}(s)$$

POLES, ZEROS, PARTIAL FRACTION EXPANSION, AND MATLAB

Function	Description
tf(num,den)	defines transfer function using polynomials
pole(sys)	computes poles of a LTI system object
zero(sys)	computes zeros of a LTI system object
zpk(Z,P,K)	defines transfer function with zeros, poles, and gain
conv(A,B)	calculates polynomial multiplication
residue(num,den)	computes partial fraction expansion

EXAMPLE 5.9

- Using MATLAB, find the poles, zeros, and partial fraction expansion of

$$G(s) = \frac{s^2 + 9s + 20}{s^4 + 4s^3 + 10s^2 + 12s + 5}$$

SUMMARY

- Laplace transforms are used to convert differential equations in the time domain to algebraic equations in the s-domain.
- Linear differential equations become polynomials in the s-domain.
- Sinusoids can be represented using complex exponential functions. As such, they can be manipulated using basic algebraic principles.
- A Laplace transform can be decomposed through partial fraction expansions into terms that can be readily inverse Laplace transformed using Laplace transform primitives.
- Laplace transforms lead to transfer function models. A transfer function is an algebraic construct that represents the output/input relation in the s-domain.
- The zeros are defined as the roots of the polynomial in the numerator of a transfer function.
- The poles are defined as the roots of the polynomial in the denominator of a transfer function.
- MATLAB includes a series of functions to represent transfer functions, compute poles and zeros, conduct polynomial multiplication, and compute partial fraction expansions.