Chapter 4: The State Space and Numerical Simulation

Samantha Ramirez

Preview Questions

- What mathematical formulation might we apply to the models we derived in the previous chapter in order to facilitate predicting and analyzing their dynamic responses?
- What are typical system inputs and outputs?
- How can we use a numerical language like MATLAB to simulate dynamic responses?
- What are typical responses for common types of inputs?

Objectives & Outcomes

- **Objectives:**
  - To understand how systems of first-order differential equations are converted to state-space representations,
  - To study and understand the use of mathematical functions utilized to model commonly occurring system inputs, and
  - To simulate dynamic systems represented by state-space equations using MATLAB.
- **Outcomes:** Upon completion, you should
  - be able to reformulate equations derived using bond graphs into state-space representations,
  - be able to model a variety of physical inputs using some basic mathematical functions, and
  - simulate dynamic responses for simple and moderately complex systems using the state-space formulation.

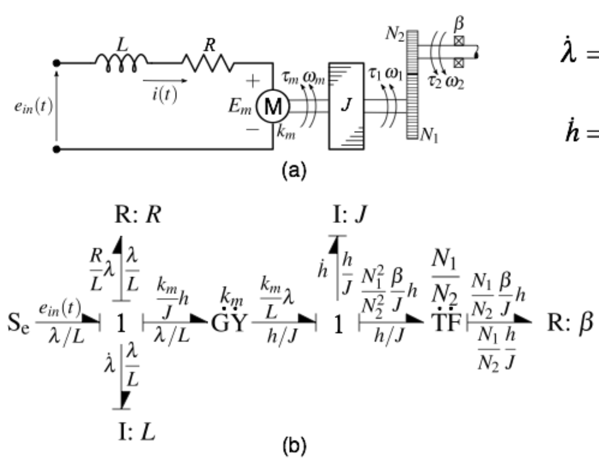
The State Space

- $\dot{x} = \boldsymbol{Ax} + \boldsymbol{Bu}$
  - **A** is an [n x n] matrix
  - **x** is an [n x 1] state vector
  - **B** is an [n x m] matrix
  - **u** is an [mx1] input vector
- $\boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du}$
  - **y** is an [r x 1] output vector
  - **C** is an [r x n] matrix
  - **D** is an [r x m] matrix

Composing State-Space Representations

- Identify the states, inputs, and outputs
- Rewrite each state equation as a linear combination of all the states and all the inputs
- Write the differential equations as a linear algebraic equation in terms of the state and input vectors
- Write each output as a linear combination of all the states and all the inputs
- Rewrite the linear system of output equations as a single linear algebraic equation in terms of the state and input vectors

Example 4.1



$$\dot{\lambda} = -\frac{R}{L}\lambda - \frac{k_m}{J}h + e_{in}(t)$$

$$\dot{h} = \frac{k_m}{L}\lambda - \left[\frac{N_1}{N_2}\right]^2 \frac{\beta}{J}h$$

$$y = \omega_{out} = \frac{N_1}{N_2}\frac{h}{J}.$$

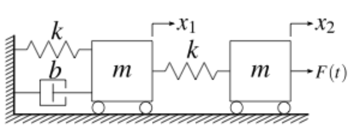
Example 4.1 Continued

$$\begin{aligned} \dot{\lambda} &= \left| \begin{array}{cc|c} -(R/L)\,\lambda & -(k_m/J)\,h & +1\,e_{in}(t) \\ (k_m/L)\,\lambda & -(N_1/N_2)^2(\beta/J)\,h & +0\,e_{in}(t) \end{array} \right| \\ \dot{h} &= \end{aligned}$$

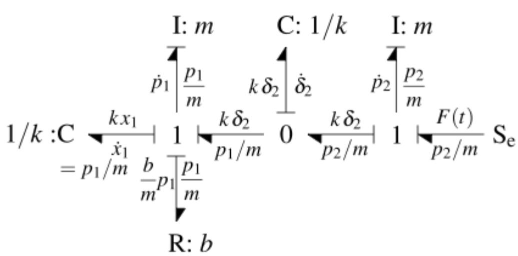
$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{\lambda} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} -R/L & -k_m/J \\ k_m/L & -(N_1/N_2)^2(\beta/J) \end{bmatrix} \begin{bmatrix} \lambda \\ h \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e_{in}(t) = \mathbf{A}\mathbf{x} + \mathbf{B}u\,.$$

$$\begin{aligned} y &= 0\lambda + \left(\frac{N_1}{N_2}\frac{1}{J}\right) h + 0\,e_{in}(t) \\ &= \begin{bmatrix} 0 & \dfrac{N_1}{N_2}\dfrac{1}{J} \end{bmatrix} \begin{bmatrix} \lambda \\ h \end{bmatrix} + 0\,e_{in}(t) \\ &= \mathbf{C}\mathbf{x} + \mathbf{D}u\,. \end{aligned}$$

Example 4.2



(a)



(b)

$$\dot{x}_1 = \frac{1}{m} p_1$$

$$\dot{p}_1 = -kx_1 - \frac{b}{m} p_1 + k\delta_2$$

$$\dot{\delta}_2 = -\frac{1}{m} p_1 + \frac{1}{m} p_2$$

$$\dot{p}_2 = -k\delta_2 + F(t)$$

Example 4.2 Continued

$$
\begin{aligned}
\dot{x}_1 &= 0x_1 + (1/m)p_1 + 0\delta_2 + 0p_2 + 0F(t) \\
\dot{p}_1 &= -kx_1 - (b/m)p_1 + k\delta_2 + 0p_2 + 0F(t) \\
\dot{\delta}_2 &= 0x_1 - (1/m)p_1 + 0\delta_2 + (1/m)p_2 + 0F(t) \\
\dot{p}_2 &= 0x_1 + 0p_1 - k\delta_2 + 0p_2 + 1F(t)
\end{aligned}
$$

$$
\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{p}_1 \\ \dot{\delta}_2 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/m & 0 & 0 \\ -k & -b/m & k & 0 \\ 0 & -1/m & 0 & 1/m \\ 0 & 0 & -k & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ p_1 \\ \delta_2 \\ p_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} F(t) = \mathbf{Ax} + \mathbf{B}u .
$$

$$
\begin{aligned}
x_1 &= 1x_1 + 0p_1 + 0\delta_2 + 0p_2 + 0F(t) \\
x_2 &= 1x_1 + 0p_1 + 1\delta_2 + 0p_2 + 0F(t)
\end{aligned} .
$$

$$
\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ p_1 \\ \delta_2 \\ p_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} F(t) = \mathbf{Cx} + \mathbf{D}u .
$$

Example 4.3

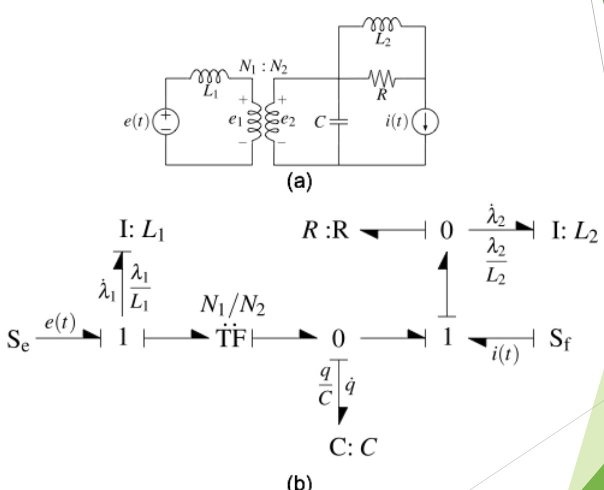


Figure 4.1

Example 4.3 Continued

$$\dot{\lambda}_1 = e(t) - \frac{N_1}{N_2}\frac{q}{C}, \;\; \dot{q} = \frac{N_1}{N_2}\frac{\lambda}{L_1} - i(t) \;\; \text{, and } \dot{\lambda}_2 = \left[i(t) - \frac{\lambda_2}{L_2} \right] R \,.$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -\frac{N_1}{N_2}\frac{1}{C} & 0 \\ \frac{N_1}{N_2}\frac{1}{L_1} & 0 & 0 \\ 0 & 0 & -\frac{R}{L_2} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ q \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & R \end{bmatrix} \begin{bmatrix} e(t) \\ i(t) \end{bmatrix} = \mathbf{Ax} + \mathbf{Bu} \,.$$

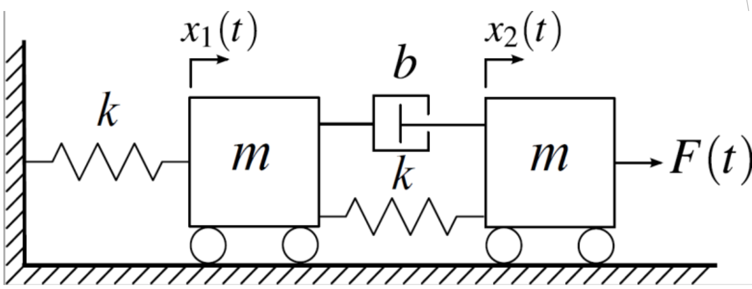
$$\mathbf{y} = \begin{bmatrix} i_{L_1} \\ v_C \\ i_{L_2} \end{bmatrix} = \begin{bmatrix} \lambda_1 / L_1 \\ q/C \\ \lambda_2 / L_2 \end{bmatrix} .$$

$$\mathbf{y} = \begin{bmatrix} \frac{1}{L_1} & 0 & 0 \\ 0 & \frac{1}{C} & 0 \\ 0 & 0 & \frac{1}{L_2} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ q \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ i(t) \end{bmatrix} = \mathbf{Cx} + \mathbf{Du} \,.$$

Review Problems

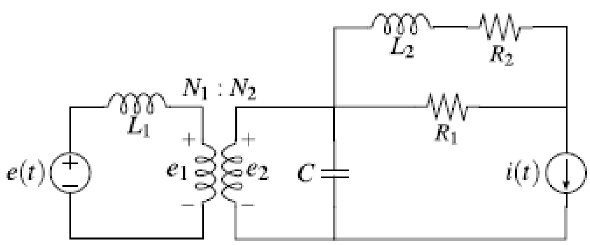
- Put the differential equations in state-space format for the following problems.

Review Problem 1



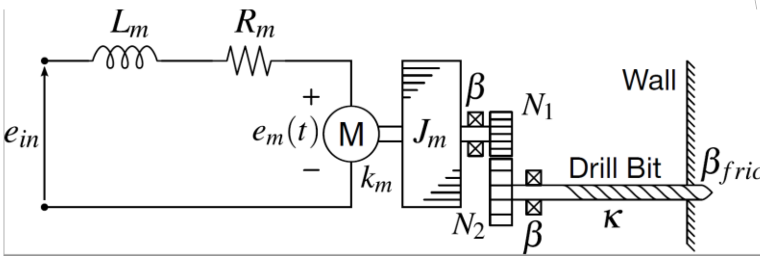
- $\dot{p_1} = \frac{b}{m}(p_2 - p_1) + k\delta_2 - kx_1$
- $\dot{x_1} = \frac{p_1}{m}$
- $\dot{p_2} = -\frac{b}{m}(p_2 - p_1) - k\delta_2 + F(t)$
- $\dot{\delta_1} = \frac{p_2}{m} - \frac{p_1}{m}$

Review Problem 2



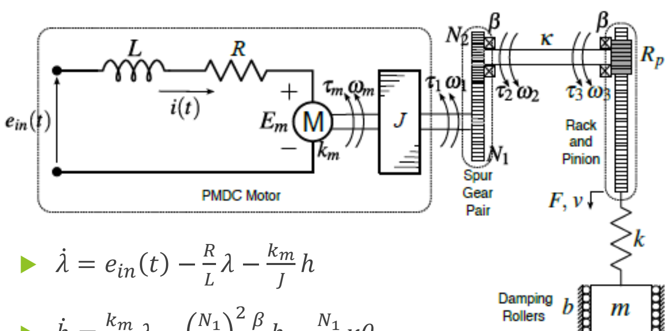
- $\dot{\lambda_1} = e(t) - \frac{N_1}{N_2}\frac{q}{C}$
- $\dot{q} = \frac{N_1}{N_2}\frac{\lambda_1}{L_1} - i(t)$
- $\dot{\lambda_2} = R_1 i(t) - \frac{R_1}{L_2}\lambda_2 - \frac{R_2}{L_2}\lambda_2$

Review Problem 3



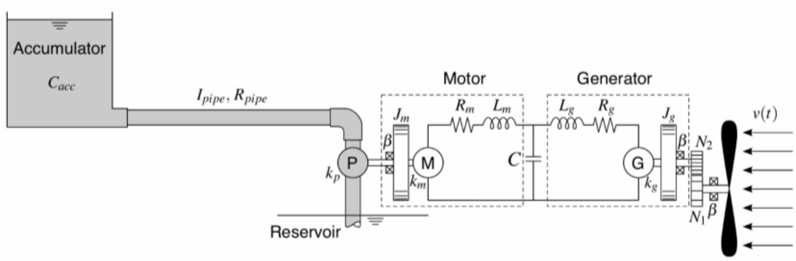
- $\dot{\lambda} = e_{in} - \frac{R_m}{L_m}\lambda - \frac{k_m}{J_m}h$
- $\dot{h} = \frac{k_m}{L_m}\lambda - \frac{\beta}{J_m}h - \left(\frac{N_1}{N_2}\right)^2 \frac{\beta}{J_m}h - \frac{N_1}{N_2}\kappa\theta$
- $\dot{\theta} = \frac{N_1}{N_2}\frac{1}{J_m}h - \frac{\kappa}{\beta_{fric}}\theta$

Review Problem 4



- $\dot{\lambda} = e_{in}(t) - \frac{R}{L}\lambda - \frac{k_m}{J}h$
- $\dot{h} = \frac{k_m}{L}\lambda - \left(\frac{N_1}{N_2}\right)^2 \frac{\beta}{J}h - \frac{N_1}{N_2}\kappa\theta$
- $\dot{\theta} = \frac{N_1}{N_2}\frac{h}{J} - \frac{\kappa}{\beta}\theta + \frac{R_p\kappa}{\beta}\delta$
- $\dot{\delta} = \frac{R_p\kappa}{\beta}\theta - \frac{R_p^2\kappa}{\beta}\delta - \frac{p}{m}$
- $\dot{p} = \kappa\delta - \frac{b}{m}p$

Challenge Problem



- $\dot{\Gamma} = \frac{k_p}{J_m} h_m - \frac{R_{pipe}}{I_{pipe}} \Gamma - \frac{V}{C_{acc}}$
- $\dot{V} = \frac{\Gamma}{I_{pipe}}$
- $\dot{h_m} = \frac{k_m}{L_m} \lambda_m - \frac{k_p}{I_{pipe}} \Gamma - \frac{\beta_3}{J_m} h_m$
- $\dot{\lambda_m} = \frac{q}{C} - \frac{R_m}{L_m} \lambda_m - \frac{k_m}{J_m} h_m$
- $\dot{q} = \frac{\lambda_g}{L_g} - \frac{\lambda_m}{L_m}$
- $\dot{\lambda_g} = \frac{k_g}{J_g} h_g - \frac{R_g}{L_g} \lambda_g - \frac{q}{C}$
- $\dot{h_g} = \frac{N_2}{N_1} k_{fan} v(t) - \left(\frac{N_2}{N_1}\right)^2 \frac{\beta_1}{J_g} h_g - \frac{\beta_2}{J_g} h_g - \frac{k_g}{L_g} \lambda_g$

Basic Transient Responses

The Unit Impulse Response

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

- Also known as the Dirac delta function
- Infinitely high
- Infinitesimal width
- Total area under the curve of 1
- Sometimes used to approximate inputs of large magnitude that operate over short periods of time (e.g. an impact force)

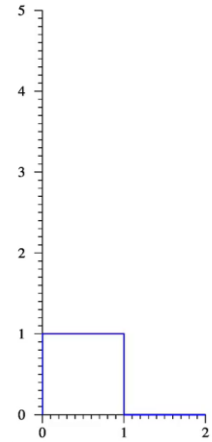


Figure 4.2

The Unit Step Response

$$1(t) = \int_{-\infty}^{t} \delta(t)\, dt = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

- Also known as the Heaviside Function
- The unit step is the integral of the unit impulse
- Often used to represent constant inputs that are "switched on" at initial time

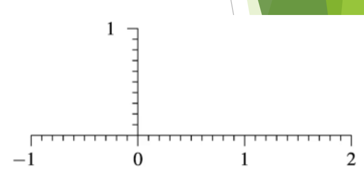


Figure 4.3

The Unit Ramp Response

$$\int_{-\infty}^{t} 1(t)\,dt = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$$

- The unit ramp response is the integral of the unit step response
- Used to represent inputs that vary proportionally with time

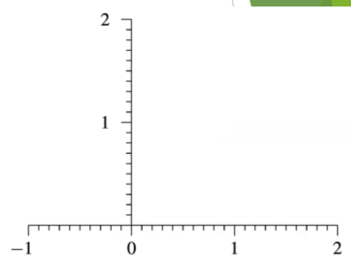


Figure 4.4

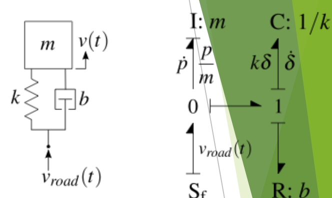
State-Space Simulations Using MATLAB

- `sys = ss(A,B,C,D)`, state-space representation
- `step(sys)`, unit step response
- `impulse(sys)`, unit impulse response
- `lsim(sys)`, general input response
- `initial(sys,x0)`, simulate the initial condition response

Applications

Example 4.4

- This is a quarter-car suspension model. Typical vehicle mass, damping constant, and spring rate for a family sedan are 1700 kg, 750 Ns/m, and 30 kN/m, respectively. If the vehicle drives over a curb, what is the approximate response?



(a) (b)

$$\dot{p} = k\delta + b\left[v_{road}(t) - \frac{p}{m}\right] \text{ and}$$

$$\dot{\delta} = v_{road}(t) - \frac{p}{m}$$

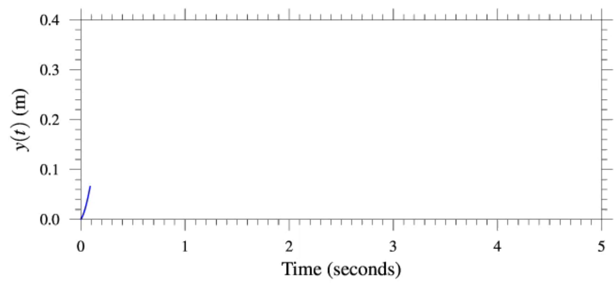
$$\dot{y} = \frac{p}{m}$$

$$\begin{bmatrix} \dot{p} \\ \dot{\delta} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -b/m & k & 0 \\ -1/m & 0 & 0 \\ 1/m & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \delta \\ y \end{bmatrix} + \begin{bmatrix} b \\ 1 \\ 0 \end{bmatrix} v_{road}(t) = \mathbf{Ax} + \mathbf{B}u.$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ \delta \\ y \end{bmatrix} + 0v_{road}(t) = \mathbf{Cx} + \mathbf{D}u.$$

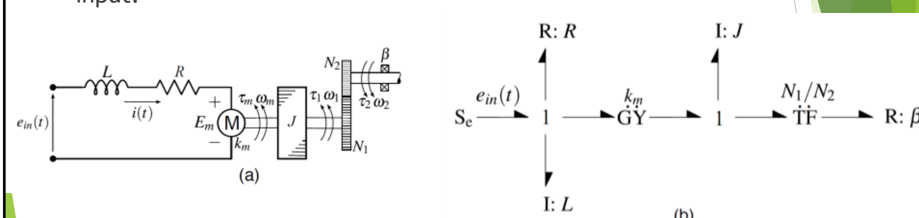
Example 4.4

```
>> m = 0.30*1700;
>> b = 750; k = 30e3;
>> A = [-b/m,k,0;-1/m,0,0;1/m,0,0];
>> B = [b;1;0];C = [0,0,1]; D = 0;
>> sys = ss(A,B,C,D);
>> [Y,T] = impulse(sys);
>> plot(T,0.20*Y);
>> xlabel('Time (seconds)')
>> ylabel('y(t) (m)')
```



Example 4.5

- A PMDC motor operates at a constant input voltage. When the voltage is initially applied, the input instantaneously jumps from zero to a constant value just like a step input. A transient response ensues. Plot the response of the system to a 24-volt input.



$$\begin{bmatrix}\dot{\lambda}\\ \dot{h}\end{bmatrix} = \begin{bmatrix} -R/L & -k_m/J \\ k_m/L & -(N_1/N_2)^2(\beta/J)\end{bmatrix}\begin{bmatrix}\lambda \\ h\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} e_{in}(t)$$

$$y = \omega_{out} = \begin{bmatrix} 0 & \frac{N_1}{N_2}\frac{1}{J}\end{bmatrix}\begin{bmatrix}\lambda \\ h\end{bmatrix} + 0\, e_{in}(t)$$

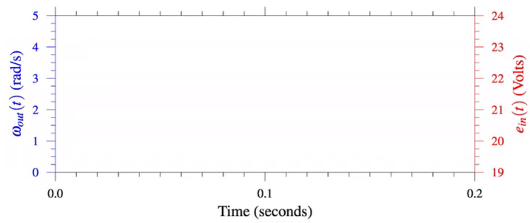
Example 4.5 Continued

- Recall PMDC motor from Example 4.1

```
>> ein_max = 24; L = 8.62e-3; R = 17.2; km = 2.73e-2;
>> J = 9.2e-7; n = 1/187.7; beta = 5.9e-7;
>> A = [-R/L,-km/J;km/L,-n^2*beta/J];
>> B = [1;0]; C = [0,n/J]; >> D = 0;
>> sys = ss(A,B,C,D);
>> t = [0:0.001:0.2]';
>> [Y,T] = step(sys,t);
>> wout = 24*Y; % scale the output response
>> u = ein_max*ones(length(t),1); % array of ein_max
>> [AX,H1,H2] = plotyy(T,wout,t,u);
>> xlabel('Time (seconds)');
>> set(get(AX(1),'Ylabel'),'String','\omega_{out}(t) (rad/s)')
>> set(get(AX(2),'Ylabel'),'String','e_{in}(t) (volts)')
```

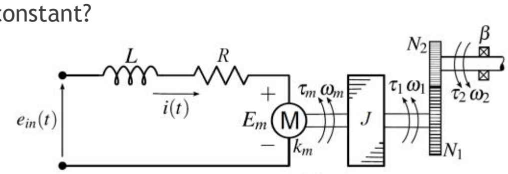
$$\begin{bmatrix} \dot{\lambda} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} -R/L & -k_m/J \\ k_m/L & -(N_1/N_2)^2(\beta/J) \end{bmatrix} \begin{bmatrix} \lambda \\ h \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e_{in}(t)$$

$$y = \omega_{out} = \begin{bmatrix} 0 & \frac{N_1}{N_2}\frac{1}{J} \end{bmatrix} \begin{bmatrix} \lambda \\ h \end{bmatrix} + 0\, e_{in}(t)$$

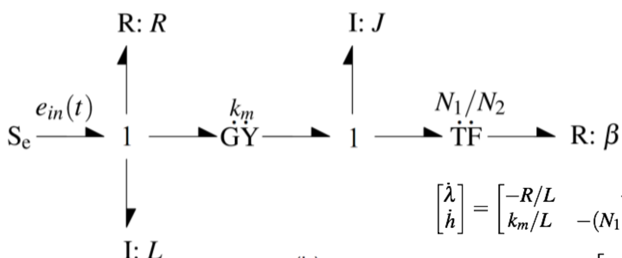


Example 4.6

- What would a response of the PMDC motor be if the voltage is ramped from 0 to 24 V in 0.2 seconds and then maintained constant?



(a)



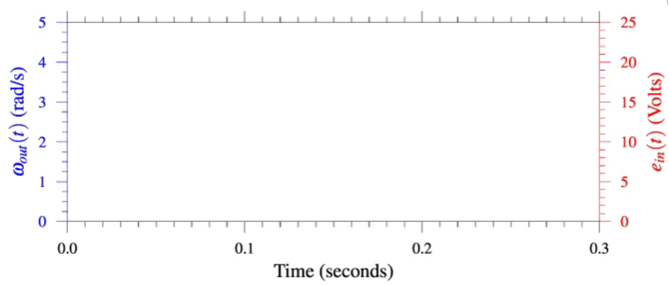
(b)

$$\begin{bmatrix} \dot{\lambda} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} -R/L & -k_m/J \\ k_m/L & -(N_1/N_2)^2(\beta/J) \end{bmatrix} \begin{bmatrix} \lambda \\ h \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e_{in}(t)$$

$$y = \omega_{out} = \begin{bmatrix} 0 & \frac{N_1}{N_2}\frac{1}{J} \end{bmatrix} \begin{bmatrix} \lambda \\ h \end{bmatrix} + 0\, e_{in}(t)$$

Example 4.6

```
>> t = [0:1/1000:0.3]';
>> u = [0:24/200:24 24*ones(1,100)]';
>> [Y,T] = lsim(sys,u,t);
>> [AX,H1,H2] = plotyy(t,Y,t,u);
>> xlabel('Time (seconds)');
>> set(get(AX(1),'Ylabel'),'String','\omega_{out}(t) (rad/s)')
>> set(get(AX(2),'Ylabel'),'String','e_{in}(t) (Volts)')
```



Example 4.7

- This mass-spring-damper system has an external force that induces motion. Motion, however, can be induced by simply displacing the system from equilibrium. The first mass can be displaced a given amount and then released resulting in free vibration. Simulate the free vibration response for an initial displacement of the leftmost mass of a few centimeters. Assume that the masses, damping constants, and spring rates are 10 kg, 20 N-s/m, and 60 N/m, respectively.

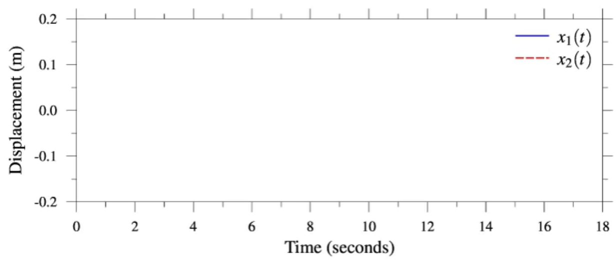
$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{p}_1 \\ \dot{\delta}_2 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/m & 0 & 0 \\ -k & -b/m & k & 0 \\ 0 & -1/m & 0 & 1/m \\ 0 & 0 & -k & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ p_1 \\ \delta_2 \\ p_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} F(t) = \mathbf{Ax} + \mathbf{Bu}.$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ p_1 \\ \delta_2 \\ p_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} F(t) = \mathbf{Cx} + \mathbf{Du}.$$

Example 4.7

- Recall Example 4.2

```
>> m = 10; b = 20; k = 60;
>> A = [0,1/m,0,0;-k,-b/m,k,0;0,-1/m,0,1/m;0,0,-k,0];
>> B = [0;0;0;1]; C = [1,0,0,0;1,0,1,0]; D = [0;0];
>> sys = ss(A,B,C,D);
>> [Y,T,X] = initial(sys,[0.20 0 0 0]);
>> plot(T,Y)
>> xlabel('Time (seconds)')
>> ylabel('Displacement (m)')
>> legend('x_1(t)','x_2(t)')
```



State Transformations

Take two different models of the same system given in state-space form as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

and

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\mathbf{u}$$
$$\mathbf{y} = \tilde{\mathbf{C}}\tilde{\mathbf{x}} + \mathbf{D}\mathbf{u}$$

where

$$\mathbf{x} = \mathbf{T}\tilde{\mathbf{x}} \quad \text{or} \quad \tilde{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{x}.$$

The state transformation can be substituted into the first model to arrive at the second,

$$\dot{\tilde{\mathbf{x}}} = \mathbf{T}^{-1}\dot{\mathbf{x}} = \mathbf{T}^{-1}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) = \mathbf{T}^{-1}\mathbf{A}\mathbf{x} + \mathbf{T}^{-1}\mathbf{B}\mathbf{u} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\tilde{\mathbf{x}} + \mathbf{T}^{-1}\mathbf{B}\mathbf{u}$$
$$= \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\mathbf{u}$$

and

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} = \mathbf{C}\mathbf{T}\tilde{\mathbf{x}} + \mathbf{D}\mathbf{u} = \tilde{\mathbf{C}}\tilde{\mathbf{x}} + \mathbf{D}\mathbf{u}.$$

State Transformations Continued

Thus the transformation matrix also relates the state-space representations,

$$\widetilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T},\ \widetilde{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B},\ \text{and}\ \widetilde{\mathbf{C}} = \mathbf{C}\mathbf{T}.$$

Similar matrices have the same eigenvalues. A matrix $\widetilde{\mathbf{A}}$ is said to be similar to matrix $\mathbf{A}$ if there exists a *similarity transformation* such that

$$\widetilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}.$$

Take two matrices $\widetilde{\mathbf{A}}$ and $\mathbf{A}$ with bases $\widetilde{\mathbf{x}}$ and $\mathbf{x}$ related through a transformation matrix, $\mathbf{x} = \mathbf{T}\widetilde{\mathbf{x}}$ (or $\widetilde{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{x}$). If the matrices are similar, it can be shown that

$$\widetilde{\mathbf{A}}\widetilde{\mathbf{x}} = \lambda\widetilde{\mathbf{x}}.$$

This is accomplished using the basis transformation,

$$\widetilde{\mathbf{A}}\widetilde{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\widetilde{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{x} = \mathbf{T}^{-1}\lambda\mathbf{x} = \lambda\mathbf{T}^{-1}\mathbf{x} = \lambda\widetilde{\mathbf{x}}.$$

Hence λ are the solutions for both eigenvalue problems and thus matrices $\widetilde{\mathbf{A}}$ and $\mathbf{A}$ are similar.

Example 4.8

$$\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & m \end{bmatrix} \widetilde{\mathbf{x}} = \mathbf{T}\widetilde{\mathbf{x}} \quad \text{or} \quad \widetilde{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/m & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/m \end{bmatrix} \mathbf{x} = \mathbf{T}^{-1}\mathbf{x}.$$

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/m & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/m \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{p}_1 \\ \dot{\delta}_2 \\ \dot{p}_2 \end{bmatrix} = \mathbf{T}^{-1}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/m & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/m \end{bmatrix} \left\{ \begin{bmatrix} 0 & 1/m & 0 & 0 \\ -k & -b/m & k & 0 \\ 0 & -1/m & 0 & 1/m \\ 0 & 0 & -k & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ p_1 \\ \delta_2 \\ p_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} F(t) \right\} \\
&= \begin{bmatrix} 0 & 1/m & 0 & 0 \\ k/m & -b/m^2 & k/m & 0 \\ 0 & 0 & 0 & 1/m \\ 0 & 0 & -k/m & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ p_1 \\ \delta_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m \end{bmatrix} F(t) \\
&= \begin{bmatrix} 0 & 1/m & 0 & 0 \\ k/m & -b/m^2 & k/m & 0 \\ 0 & 0 & 0 & 1/m \\ 0 & 0 & -k/m & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & m \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m \end{bmatrix} F(t) \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2k/m & -b/m & k/m & 0 \\ 0 & 0 & 0 & 1 \\ k/m & 0 & -k/m & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m \end{bmatrix} F(t) = \widetilde{\mathbf{A}}\widetilde{\mathbf{x}} + \widetilde{\mathbf{B}}\mathbf{u}.
\end{aligned}$$

Example 4.8 Continued

```
>> m = 100; b = 25; k = 50;
>> A = [0 1/m 0 0; -k -b/m k 0; 0 -1/m 0 1/m; 0 0 -k 0];
>> T = [1 0 0 0; 0 m 0 0; -1 0 1 0; 0 0 0 m];
>> Aalt = inv(T)*A*T;
>> eig(A)

ans =

  -0.0902 + 1.1341i
  -0.0902 - 1.1341i
  -0.0348 + 0.4381i
  -0.0348 - 0.4381i

>> eig(Aalt)

ans =

  -0.0902 + 1.1341i
  -0.0902 - 1.1341i
  -0.0348 + 0.4381i
  -0.0348 - 0.4381i
```

Summary

- For linear systems, the differential equations and outputs can be written as a linear combination of the states and inputs using Linear Algebra. This type of formulation is called the state-space representation.
- State-space models are composed by identifying the state, input, and output vectors. The individual first-order differential equations and output equations are written as linear combinations of the states and inputs. This facilitates identifying and separating the coefficients, states, and inputs in each equation.
- At $t = 0$, the unit impulse function (or Dirac delta function), $\tilde{\delta}(t)$, has infinite height and infinitesimal width. The function is referred to as unit impulse because the integral under the curve is one.
- The unit step function (or Heaviside function), $1(t)$, is the integral of the unit impulse. The function is unity for all values of time greater than zero ($t > 0$).
- The unit ramp function is the integral of the unit impulse. For values of $t > 0$ the function increases at a constant rate of unity.
- Because the impulse, step, and ramp functions are related through integration and differentiation, so are the output responses to these inputs.
- MATLAB provides a variety of commands for defining and simulating the responses of state-space models, including commands to define a state-space object and to simulate responses to an impulse, a step, an arbitrary function, or an initial condition.
- State-space representations are not unique. Several models can be derived to represent the same system in terms of distinct sets of states. A state transformation can be used to transfer from one set of states to another. Regardless of the state vector chosen, the eigenvalues of the system are unique.