RIGID BODIES:
EQUIVALENT FORCE SYSTEMS

Rigid Bodies

- A large number of particles occupying fixed positions with respect to one another
- A body which does not deform

- Nothing is completely rigid, but assuming so is continually used in the applications of mechanics
- Forces that act on a rigid body are, in general, not concurrent.
- The condition $\vec{R} = \Sigma\vec{F} = 0$ is still a requirement, but it is not sufficient to guarantee static equilibrium of a rigid body.

Particle

Concurrent Forces

Rigid Body

Not Concurrent Forces

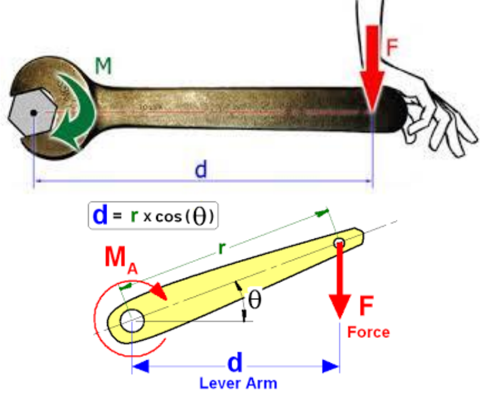
https://www.internationalcranes.media/Images/480xany/20220412-094711-photo-8.jpg
http://mobileimages.lowes.com/productimages/be8e19f4-4b8e-415a-93a8-5ec424888aa1/15122123.jpg
https://m.media-amazon.com/images/I/31w-1OaNC7L._AC_.jpg

MOMENTS

Moment of a Force

Sometimes called a torque

- The tendency of a force to rotate a body about a point or axis



http://www.mem50212.com/MDME/MEMmods/MEM30005A/moments/Moments.html

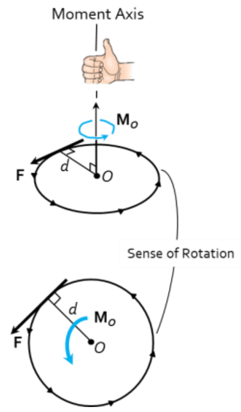
Moment of a Force – Scalar Formulation

- In 2-D, the magnitude of the moment is

$$M_o = Fd$$

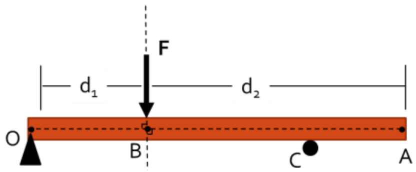
- d is the perpendicular distance form point *O* to the line of action of the force.

- In 2-D the direction of M_O is either clockwise (CW) or counter-clockwise (CCW), depending on the tendency for rotation.
- Units of Nm or lb-ft

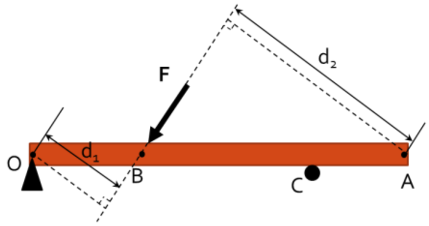


Moment of a Force

- Moment of **F** with respect to (w.r.t.) point O
  - $\overrightarrow{M_{F/O}} = F \cdot d_1$ CW
- Moment of **F** w.r.t. point A
  - $\overrightarrow{M_{F/A}} = F \cdot d_2$ CCW



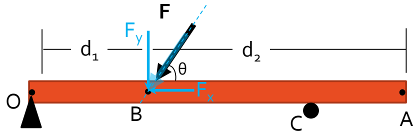
- Moment of **F** with respect to (w.r.t.) point O
  - $\overrightarrow{M_{F/O}} = F \cdot d_1$ CW
- Moment of **F** w.r.t. point A
  - $\overrightarrow{M_{F/A}} = F \cdot d_2$ CCW



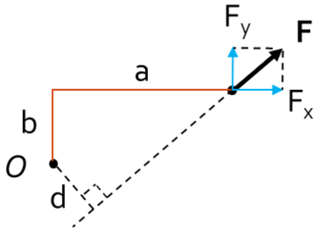
Moment of a Force – Using Components

Often, it is easier to determine M_O by using the components of **F** as shown.

- Moment of **F** with respect to (w.r.t.) point O
  - $\overrightarrow{M_{F/O}} = F_y \cdot d_1$ CW
- Moment of **F** w.r.t. point A
  - $\overrightarrow{M_{F/A}} = F_y \cdot d_2$ CCW

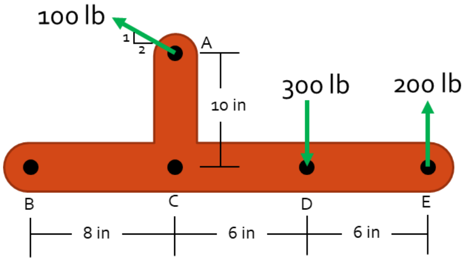


- Moment of **F** w.r.t. point *O*
  - $\vec{M} = F \cdot d$
  - $\vec{M} = (F_y \cdot a) - (F_x \cdot b)$



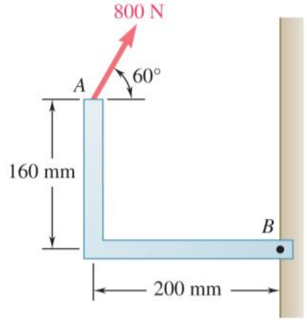
Example 1

- Three forces are applied to the bar as shown. Determine the following:
  - The moment of force F_A about point E,
  - The moment of force F_E about point A,
  - The moment of force F_D about point B, and
  - Challenge: The total moment about point C.



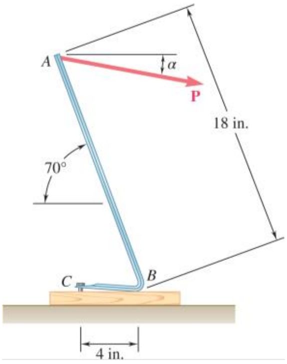
Example 2

- Find the moment at point B caused by the force applied at point A.



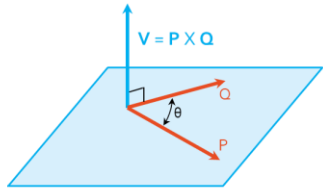
Example 3

- A normal force of 30 lb is required at point C to remove the nail. What force P will be the smallest force sufficient to remove the nail if angle α is 25°.

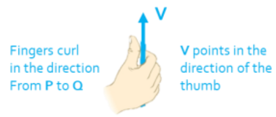


Vector (Cross) Product

$$\vec{V} = \vec{P} \times \vec{Q}$$



- The vector product of two vectors **P** and **Q** is defined as the vector **V** that satisfies the following conditions:
  1. The line of action of **V** is perpendicular to the plan containing **P** and **Q**
  2. $V = PQsin\theta$ where θ≤180°
  3. The direction of **V** is obtained from the right-hand rule.



- Vector product is not commutative: $\vec{P} \times \vec{Q} = -(\vec{Q} \times \vec{P})$
- Vector product is distributive: $\vec{P} \times (\overrightarrow{Q_1} + \overrightarrow{Q_2}) = (\vec{P} \times \overrightarrow{Q_1}) + (\vec{P} \times \overrightarrow{Q_2})$
- Vector product is not associative: $(\vec{P} \times \vec{Q}) \times \vec{S} \neq \vec{P} \times (\vec{Q} \times \vec{S})$

Vector (Cross) Product

- Rectangular Components of a Vector Product (Determinant Form)

$$\vec{V} = \vec{P} \times \vec{Q} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$

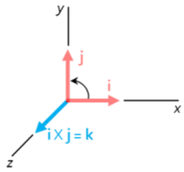
- Rectangular Components of a Vector Product

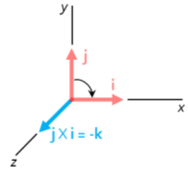
$$V_x = \hat{\imath}(P_y Q_z - P_z Q_y)$$

$$V_y = -\hat{\jmath}(P_x Q_z - P_z Q_x)$$

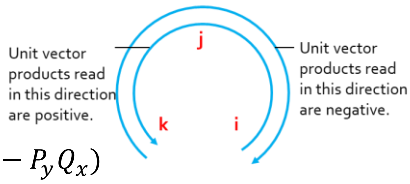
$$V_z = \hat{k}(P_x Q_y - P_y Q_x)$$

$$\vec{P} \times \vec{Q} = \hat{\imath}(P_y Q_z - P_z Q_y) - \hat{\jmath}(P_x Q_z - P_z Q_x) + \hat{k}(P_x Q_y - P_y Q_x)$$





$$\hat{\imath} \times \hat{\imath} = 0$$
$$\hat{\imath} \times \hat{\jmath} = \hat{k}$$
$$\hat{\imath} \times \hat{k} = -\hat{\jmath}$$
$$\hat{\jmath} \times \hat{\imath} = -\hat{k}$$
$$\hat{\jmath} \times \hat{\jmath} = 0$$
$$\hat{\jmath} \times \hat{k} = \hat{\imath}$$
$$\hat{k} \times \hat{\imath} = \hat{\jmath}$$
$$\hat{k} \times \hat{\jmath} = -\hat{\imath}$$
$$\hat{k} \times \hat{k} = 0$$

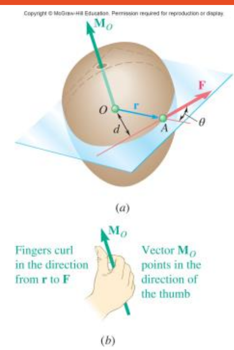


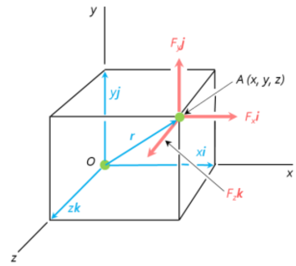
Moment of a Force in Space

- Moment of a force about a point *O* is equal to the vector product of **r** and **F**

$$\overrightarrow{M_o} = \vec{r} \times \vec{F}$$

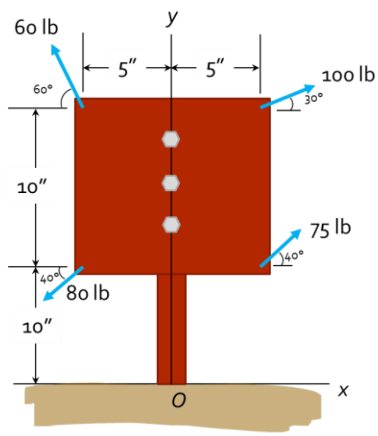
- $\vec{r}$ is the position vector relative to a reference point (in vector cartesian form)
- $\vec{F}$ is the force expressed in vector cartesian form
- M_O must be perpendicular to the plan containing *O* and force **F**
- The Right-Hand Rule can facilitate determining the sense of the moment





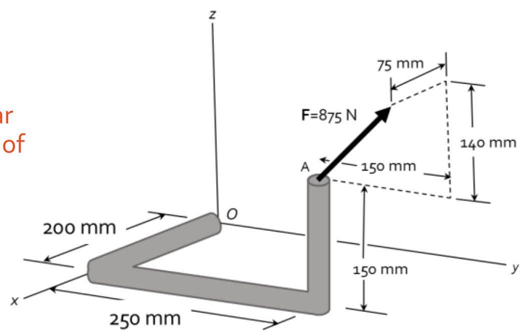
Example 4

- Four forces are applied to a square plate as shown. Determine the moments produced by each of the forces about the origin *O*.



Example 5

- A bar is bent and loaded as shown. Determine the moment of force **F** about point *O*, and the perpendicular distance, d, from point *O* to the line of action of the force.

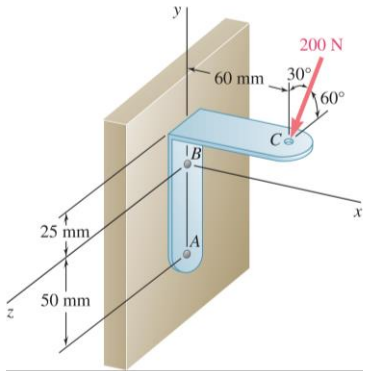


Example 6

- Determine the moment about the origin *O* of the following
  - $\vec{F} = 4\hat{\imath} + 10\hat{\jmath} + 6\hat{k}$ that acts at point A ($\vec{r} = 6\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$)
  - $\vec{F} = 5\hat{\imath} - 9\hat{\jmath} + 7\hat{k}$ that acts at point B ($\vec{r} = -2\hat{\imath} - 6\hat{\jmath} - 3\hat{k}$)

Example 7

- Find the moment at B caused by the 200 N force applied at C.
- If you were asked to find the moment at A instead of C, how would the computation change?

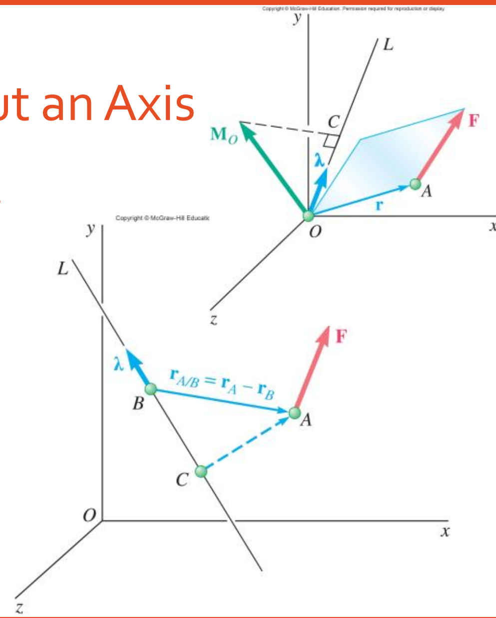


Moment of a Force about an Axis

- The moment, M_O, of a force about a given axis

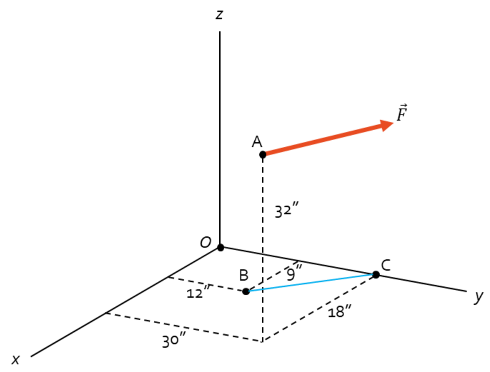
$$M_{BL} = \vec{\lambda} \cdot \vec{M} = \vec{\lambda} \cdot (\vec{r} \times \vec{F})$$

$$M_{BL} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x_{A/B} & y_{A/B} & z_{A/B} \\ F_x & F_y & F_z \end{vmatrix}$$



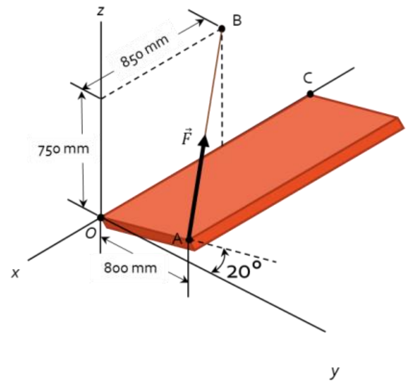
Example 8

- Determine the moment of force F about line BC if $\vec{F} = 60\hat{\imath} + 100\hat{\jmath} + 120\hat{k}$ lb.



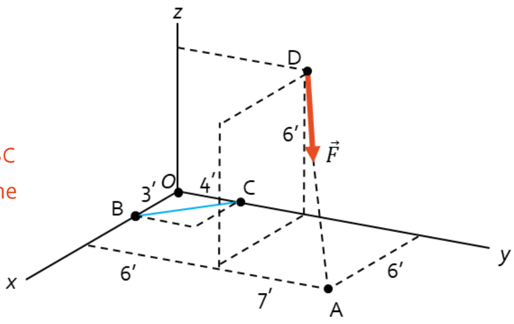
Example 9

- A door is secured to a wall (the x-z plane) via hinges and is held open at A by a rope secured to the wall at B. The magnitude of force $\vec{F}$ in the rope is 500 N. Determine the scalar component of the moment at point O about line OC.



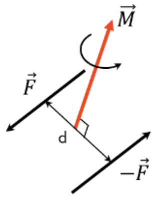
Example 10

- The force, $\vec{F}$, has a magnitude of 440 lb. Determine
  - The moment $\overrightarrow{M_B}$ of the force about point B,
  - The component of moment $\overrightarrow{M_B}$ parallel to line BC,
    - Hint: This is the projection of moment $\overrightarrow{M_B}$ along line BC
  - The component of moment $\overrightarrow{M_B}$ perpendicular to line BC, and
    - Hint: The sum of the parallel and perpendicular components of $\overrightarrow{M_B}$ is equal to $\overrightarrow{M_B}$.
  - The unit vector associated with the component of moment $\overrightarrow{M_B}$ perpendicular to line BC.



Couples

- Two forces that have the same magnitude, parallel lines of action, and opposite sense are said to form a couple.



- Using scalar analysis

$$M_O = Fd$$

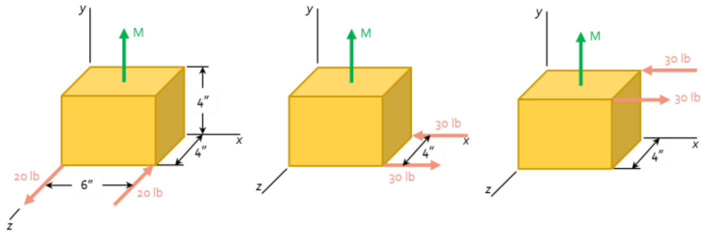
- Using vector analysis

$$\overrightarrow{M_O} = \vec{r} \times \vec{F}$$

https://di-uploads-development.dealerinspire.com/toyotaofnorthcharlotte/uploads/2018/05/Car-Tires-in-NC.png

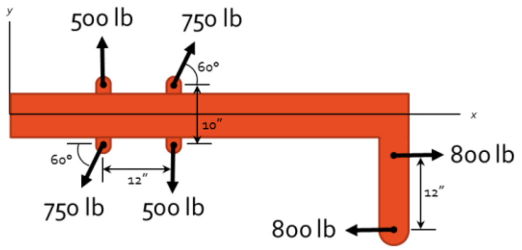
Equivalent Couples

- The net external effect of a couple is that the net force equals zero and the magnitude of the net moment equals F*d.
- Since the moment of a couple depends only on the distance between the forces, it can be moved anywhere on the body and have the same external effect (b & c).
- Moments due to couples can be added together using the same rules as adding any vectors.

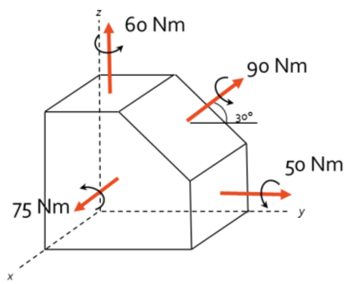


Example 11

- Express the resultant of the force system in Cartesian vector form in the beam is loaded with a system of forces as shown.



Example 12

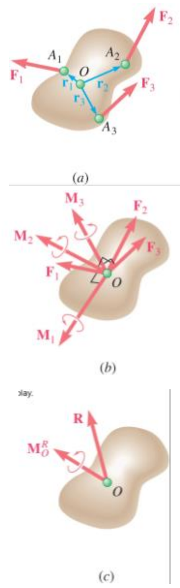


- The magnitudes of the four couples applied to the block are shown. Determine the magnitude of the resultant couple $\vec{C}$ and the direction angles associated with the unit vector used to describe the normal to the plane of the resultant couple $\vec{C}$.
- Hint: The components of the resultant couple are the sums of the components of the individual couples.

Simplifying Systems of Forces

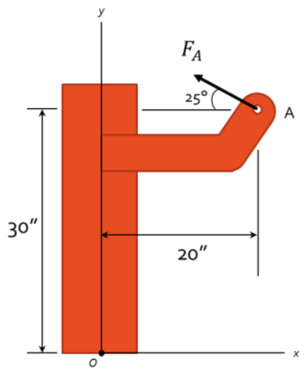
- When a number of forces and couple moments are acting on a body, it is easier to understand their overall effect on the body if they are combined into a single force and couple moment having the same external effect.
- The two force and couple systems are called equivalent systems since they have the same external effect on the body.
- When several forces and couple moments act on a body, you can move each force and its associated couple moment to a common point O.
- Now you can add all the force and couple moments together to find one resultant force-couple moment pair.

$$\vec{R} = \Sigma\vec{F} \qquad \overrightarrow{M_O^R} = \Sigma\overrightarrow{M_O} = \Sigma(\vec{r} \times \vec{F})$$



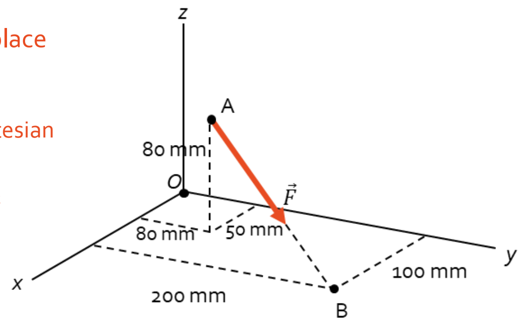
Example 13

- A 300-lb force $\mathbf{F_A}$ is applied to a bracket at point A as shown. Replace the force $\mathbf{F_A}$ by a for $\mathbf{F_O}$ and a couple $\mathbf{C}$ at point O.
- Hint: Any force can be moved to a parallel position through an arbitrary point if a couple is added that has a moment equal to the moment of the original force about the arbitrary point.



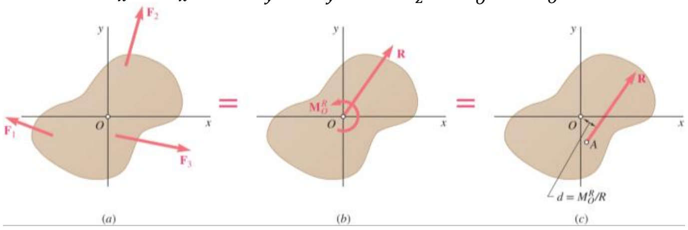
Example 14

- The force $\vec{F}$ has a magnitude of 763 N. Replace the force $\vec{F}$ by a force $\mathbf{F_O}$ at point O and a couple **C**.
  - Express the force $\mathbf{F_O}$ and the couple **C** in Cartesian vector form.
  - Determine the direction angles θ_x, θ_y, and θ_z associated with the couple vector.



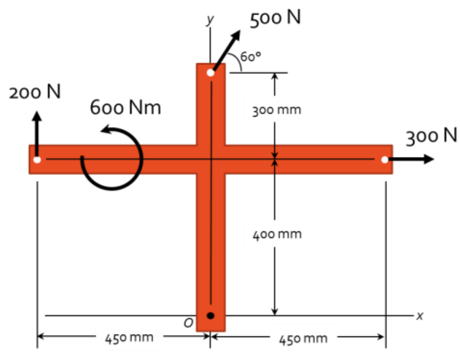
Additional Simplification of a System of Forces

- Concurrent Forces
  - Act at the same point and can be added together to obtain their resultant
- Coplanar Forces
  - Act in the same plane
  - The components of the resultant of any coplanar forces are the sum of the components of each force. $R_x = \Sigma F_x$ $R_y = \Sigma F_y$ $M_z^R = M_O^R = \Sigma M_O$



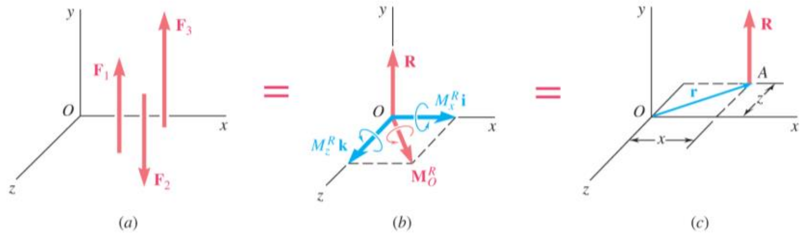
Example 15

- Three forces and a couple are applied to a bracket as shown. Determine:
  - The magnitude and direction of the resultant
  - The perpendicular distance from point O to the line of action of the resultant
  - The distance from point O to the intercept of the line of action of the resultant with the x-axis



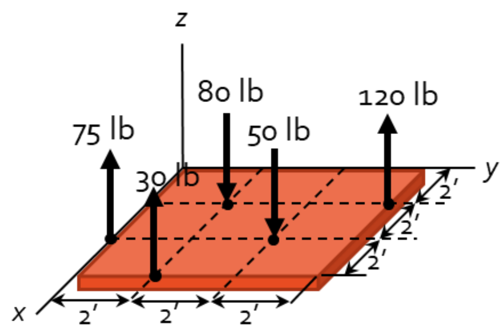
Additional Simplification of a System of Forces

- Parallel Forces
  - Have parallel lines of action
  - The resultant of a number of non-coplanar parallel forces is the sum of the individual forces
  - The moment of the resultant about any axis equals the sum of moments of the individual forces about the same axis



Example 16

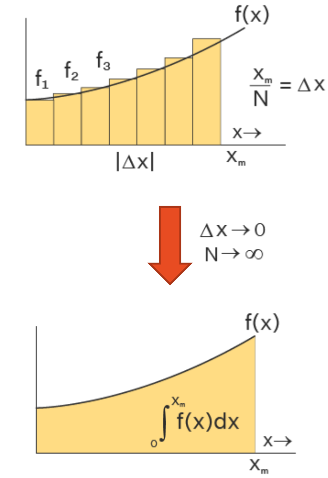
- Determine the resultant of the parallel force system shown and locate the intersection of the line of action of the resultant with the xy-plane.



CENTROIDS & MOMENTS OF INERTIA

What is an Integral?

- Integral is the representation of the area of a region under a curve.
- We approximate the actual value of an integral by drawing rectangles under the curve. The area of each rectangle is summed to calculate the total area under the curve. This is called a Riemann Sum.



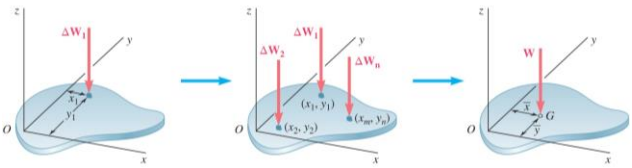
$$\text{Area} = \int_0^{x_m} f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f_i(x)\Delta x$$

https://www.cuemath.com/calculus/integral/

Center of Gravity (CG)

- The center of gravity (CG) is a point, often shown a G, which locates the resultant weight of a system of particles or a solid body.
- Consider the flat plate below made of many particles. The net or resultant weight of all particles is given as $W = \sum W_n = \int dW$.
- The center of gravity (G) is determined using the following equations:

$$\bar{x} = \frac{\sum \bar{x}_i W_i}{\sum W_i} \qquad \bar{y} = \frac{\sum \bar{y}_i W_i}{\sum W_i} \qquad \bar{z} = \frac{\sum \bar{z}_i W_i}{\sum W_i}$$



(a) Single element of the plate (b) Multiple elements of the plate (c) Center of gravity


https://static.boredpanda.com/blog/wp-content/uploads/2014/10/gravity-stone-balancing-michael-grab-4.jpg

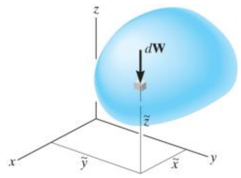
Center of Gravity (CG)

- A rigid body can be considered as made up of an infinite number of particles. Hence, using the same principles as with the center of mass, we get the coordinates of G by simply replacing the discrete summation sign (Σ) by the continuous summation sign (∫) and W by dW.

$$\bar{x} = \frac{\sum \bar{x}_i W_i}{\sum W_i} \qquad \bar{y} = \frac{\sum \bar{y}_i W_i}{\sum W_i} \qquad \bar{z} = \frac{\sum \bar{z}_i W_i}{\sum W_i}$$

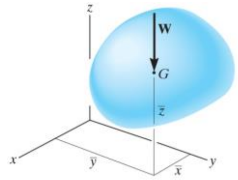
$$\Sigma = \int \qquad W = dW$$

$$\bar{x} = \frac{\int \tilde{x} dW}{\int dW} \qquad \bar{y} = \frac{\int \tilde{y} dW}{\int dW} \qquad \bar{z} = \frac{\int \tilde{z} dW}{\int dW}$$



(a)

Figure: 09_001a

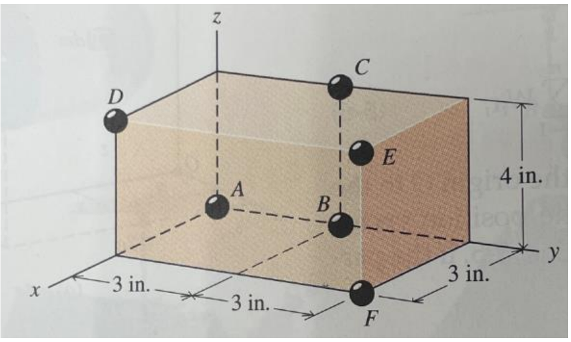


(b)

Figure: 09_001b

Example 1

- Locate the center of gravity for the six particles shown if W_A=50 lb, W_B=25 lb, W_C=30 lb, W_D=35 lb, W_E=20 lb, and W_F=40 lb.



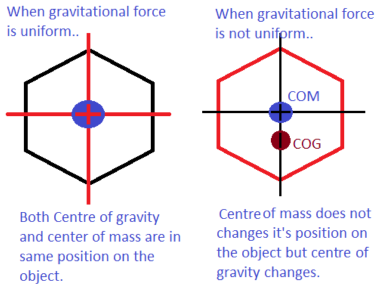
Center of Mass (CM)

- Since W=mg, by replacing the weights W with the masses m in the equations, the coordinates of the center of mass can be found.

$$\bar{x} = \frac{\sum \bar{x}_i W_i}{\sum W_i} \qquad \bar{y} = \frac{\sum \bar{y}_i W_i}{\sum W_i} \qquad \bar{z} = \frac{\sum \bar{z}_i W_i}{\sum W_i}$$

$$W = mg \Rightarrow$$

$$\bar{x} = \frac{\sum \bar{x}_i m_i}{\sum m_i} \qquad \bar{y} = \frac{\sum \bar{y}_i m_i}{\sum m_i} \qquad \bar{z} = \frac{\sum \bar{z}_i m_i}{\sum m_i}$$

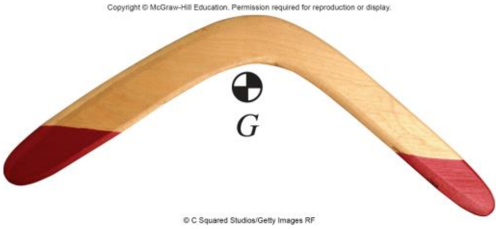


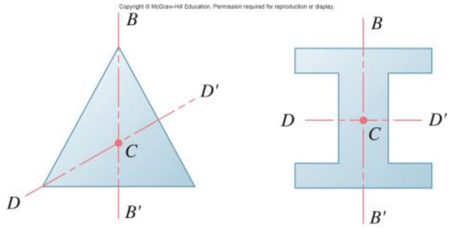
$\bar{x}W$ is the first moment about an axis or plane.

https://www.mechanicaleducation.com/wp-content/uploads/2019/05/Centre-of-gravity-mass.png

Centroid of an Area

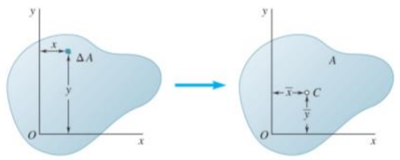
- The centroid is the geometric center of a body
  - This coincides with the center of mass and the center of gravity only if the material is uniform or homogenous (density and specific weight is constant throughout the body).
- The centroid may be located at a point that is not on the object.
- If there is a plane of symmetry, the centroid will lie upon that plane.





Centroids and First Moments of

AREAS



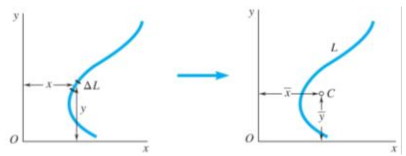
- Centroid

$$\bar{x} = \frac{\int x dA}{A} \qquad \bar{y} = \frac{\int y dA}{A}$$

- First Moment of Area

$$Q_x = \int y dA \qquad Q_y = \int x dA$$

LINES



- Centroid

$$\bar{x} = \frac{\int x dL}{L} \qquad \bar{y} = \frac{\int y dL}{L}$$

- First Moment of Line

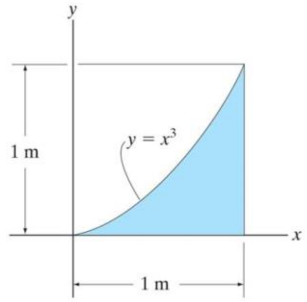
$$Q_x = \int y dL \qquad Q_y = \int x dL$$

Centroids of Areas via Integration

1. Sketch the body approximately to scale.
2. Establish a coordinate system.
    - Rectangular coordinates for flat planes for boundaries
    - Polar coordinates for circular boundaries
  - When a body has an axis of symmetry, the centroid is located on that axis
    - What happens if a body has two axes of symmetry?
3. Choose an appropriate differential element dA at a general point.
  - Generally, if y is easily express in terms of x (e.g. $y=x^2+1$), use a vertical rectangular element. If the opposite it true (e.g. $x=\sqrt{(y-1)}$), then use a horizontal element.
4. Express dA in terms of the differentiating elements dx (or dy).
5. Determine the coordinates $(\bar{x}, \bar{y})$ of the centroid of the rectangular element in terms of the general point (x,y).
6. Express all variables and integral limits in the formula using either x or y depending on whether the differential element is in terms of dx or dy, respectively, and integrate.

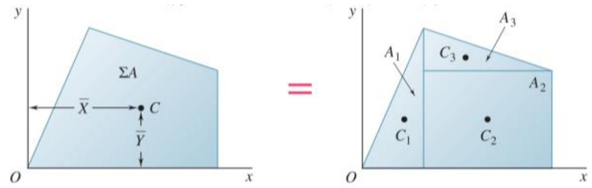
Example 2

- Locate the centroid of the area shown.

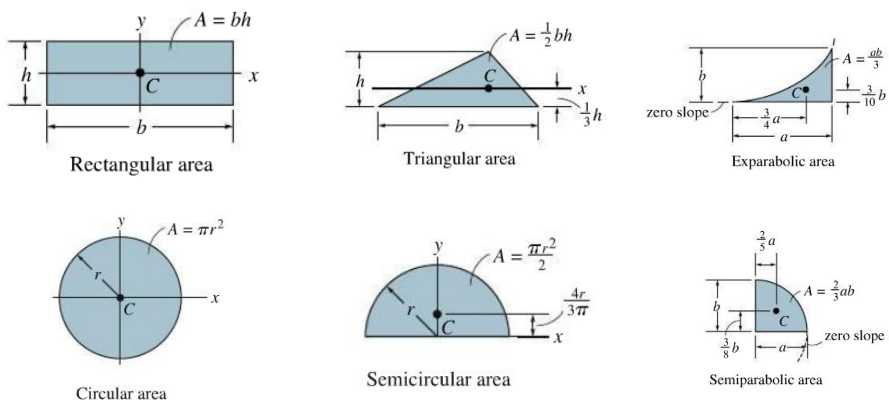


Composite Centroids and First Moments

- Some shapes can be broken up into simple shapes with known centroids and first moments.
- Take the quadrilateral shown below. It can be broken into two basic shapes (one rectangle and two triangles) with known centroids and first moments.
- The sum of the components of the individual centroids and first moments will yield the overall centroid and first moment of the quadrilateral

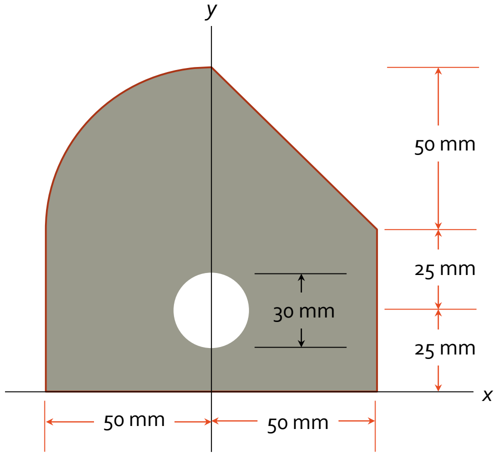


Centroids of Known Shapes



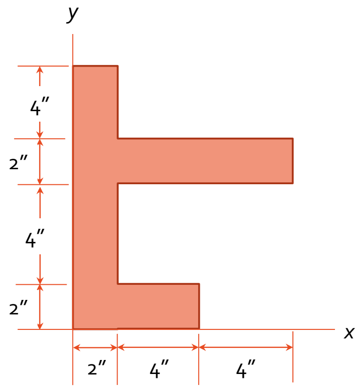
Example 3

- Locate the centroid of the composite area shown.



Example 4

- Locate the centroid of the composite area shown.



Centroid Project/Quiz

Project Parameters

- Groups of 3 of less
- Shape ≤ 1 ft^2
  - Has to be cut using 1 solid piece
- Minimum 3 shapes in your composite piece
  - Rectangles, Circles, Triangles, etc.
  - Must have negative space
  - Cannot be symmetric
- Use grid paper & establish a coordinate system
  - Label 1" increments
  - Mark centroid location
- Centroid location CANNOT be in empty space

What You Will Need

- Cardboard
  - CANNOT sag
- Scissors/Xacto Knife
- Ruler

You will bring in your shape and your calculated centroid of the shape (with all work shown) on the designated class day.
For your in-class quiz, I will test your centroid location. You will get a 100 if your shape balances, a 70 if it does not, and a 0 if you do not do the assignment.

Theorems of Pappus-Guldinus

- Surface Area
  - The area A of a surface of revolution equals the product of the length of the generating curve and the distance traveled by the centroid of the curve in generating the surface area.

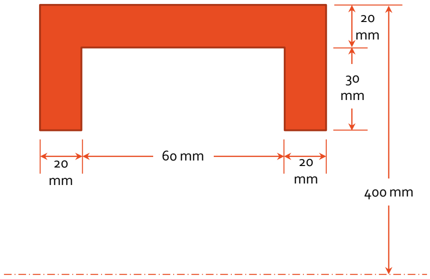
$$A = 2\pi\bar{y}L$$

- Volume
  - The volume V of a body of revolution equals the product of the generating area and the distance traveled by the centroid of the area in generating the volume.

$$V = 2\pi\bar{y}A$$

Example 5

- The outer diameter of a pulley is 0.8 m, and the cross-section of its rim is as shown. Knowing that the pully is made of steel which has a density of $7.85x10^3$ kg/m^3, determine the mass and weight of the rim.

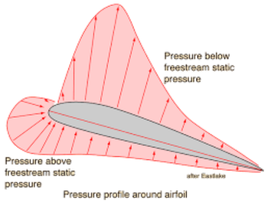


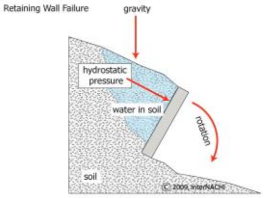
Centroid of a General Distributed Loading

- A distributed loading is a force that is spread over a length, area, or volume.

$$R = \int_L w dx$$

$$M_O = \int x dR \qquad x_C = \frac{M_O}{R}$$

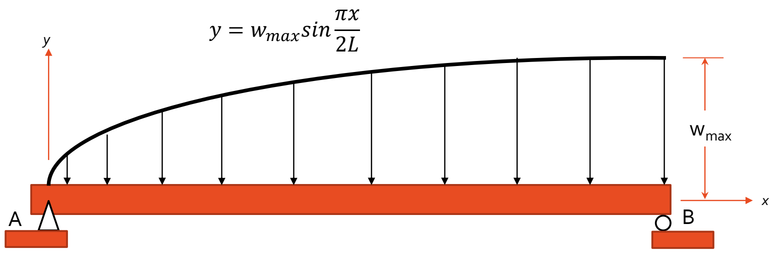




http://hyperphysics.phy-astr.gsu.edu/hbase/Fluids/airfoil.html
https://res.cloudinary.com/internachi/image/fetch/f_auto,q_auto:best/https://s3.amazonaws.com/uploads-east-1.nachi.org/gallery-images/exterior/general/hydrostatic-pressure.jpg
https://encrypted-tbn0.gstatic.com/images?q=tbn:ANd9GcTnM1-MJo9oZD2Bs6WWlLnWHsvN4ejs-FyoTA&usqp=CAU

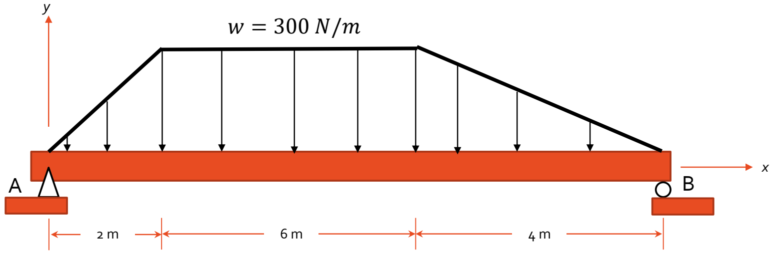
Example 6

- A beam is subjected to the load shown. Determine the resultant of this distributed load and locate its line of action with respect to the left support of the beam.



Example 7

- A beam is subjected to the loading shown. Determine the resultant of this system of distributed loads and locate its line of action with respect to the left support of the beam.



Centroid of a Submerged Surface

- Hydrostatic pressure is the pressure exerted by a fluid on an immersed body

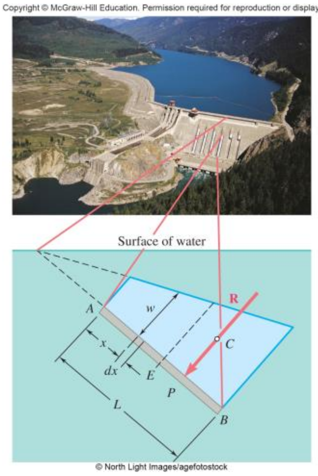
$$P_h = \rho g h$$

  - Linear relationship between pressure and height/depth of a fluid

- Resultant force acting on a submerged surface

$$R = \int_A \rho dA = \int_V dV_{ps} = V_{ps}$$

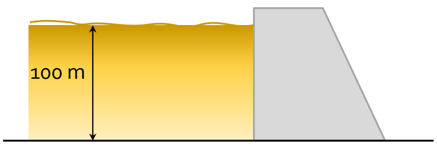
- The principle of moments is used to determine the centroid location of the resultant force.



© North Light Images/agefotostock

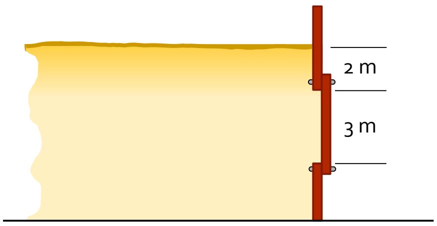
Example 8

- The water behind a dam is 100 m deep. Determine a) the magnitude of the resultant force **R** exerted on a 30-m length of dam by the water pressure and b) the distance from the water surface to the center of pressure.



Example 9

- Determine the magnitude of the resultant force acting on the submerged rectangular plate AB which has a width of 1.5 m and its location. ρ_w = 1000 kg/m³.



3-D Centers of Gravity and Centroids

Center of Gravity of a 3-D Body

$$\bar{x} = \frac{\int \tilde{x} dW}{W}$$

$$\bar{y} = \frac{\int \tilde{y} dW}{W}$$

$$\bar{z} = \frac{\int \tilde{z} dW}{W}$$

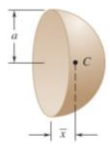
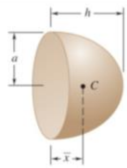
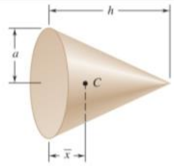
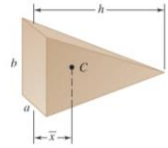
Centroid of a Volume

$$\bar{x} = \frac{\int \tilde{x} dV}{V}$$

$$\bar{y} = \frac{\int \tilde{y} dV}{V}$$

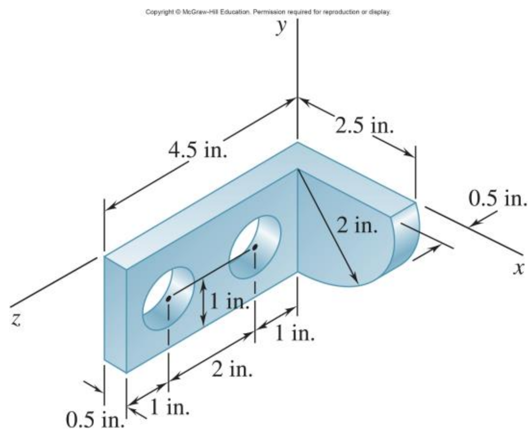
$$\bar{z} = \frac{\int \tilde{z} dV}{V}$$

Centroids & Volume of Common 3-D Shapes

Shape		$\bar{x}$	Volume
Hemisphere		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$
Cone		$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3}abh$

Example 10

- Locate the center of gravity of the steel machine part shown.

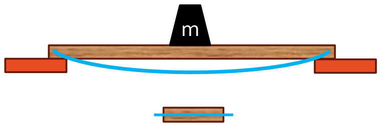


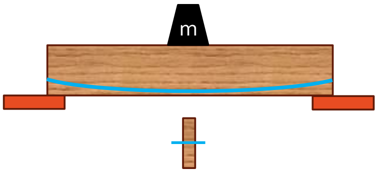
Moment of Inertia

aka Second Moment of Area

- A moment of inertia is a measure of how resistant an object is to changes in its rotational motion.
- The second moment of area or moment of inertia of an area with respect to the x and y axes are

$$I_x = \int_A y^2 dA \qquad I_y = \int_A x^2 dA$$



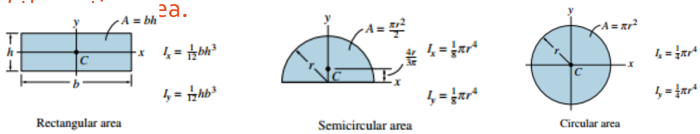


Moments of Inertia
Parallel-Axis Theorem

- A geometric property that is calculated about an axis.

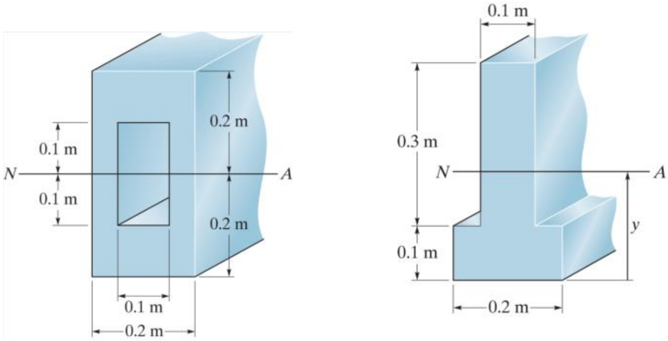
$$I_x = \bar{I}_{x'} + Ad_y^2 \qquad I_y = \bar{I}_{y'} + Ad_x^2$$

- The term Ad^2 is zero if the axis passes through the area's centroid
- Composite areas can be used to calculate the moment of inertia of complex shapes
- You can subtract the moment of inertia of an empty area from the moment of inertia [illegible] ea.



Example 11 & 12

- Determine the centroid and moment of inertia for each of the following cross-sections about the neutral axis.



Example 13 & 14

- Determine the centroid and moment of inertia for each of the following cross-sections about the indicated neutral axis.