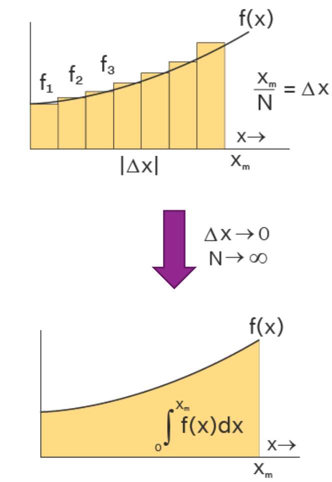
CHAPTER 5
CENTROIDS &
CENTERS OF GRAVITY

What is an Integral?

- Integral is the representation of the area of a region under a curve.
- We approximate the actual value of an integral by drawing rectangles under the curve. The area of each rectangle is summed to calculate the total area under the curve. This is called a Riemann Sum.



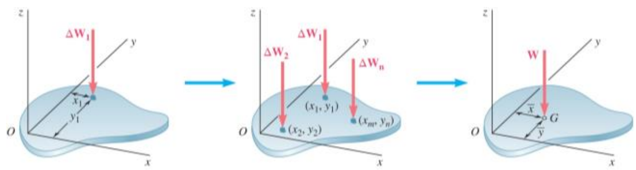
$$\text{Area} = \int_0^{x_m} f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f_i(x)\Delta x$$

https://www.cuemath.com/calculus/integral/

Center of Gravity (CG)

- The center of gravity (CG) is a point, often shown a G, which locates the resultant weight of a system of particles or a solid body.
- Consider the flat plate below made of many particles. The net or resultant weight of all particles is given as $W = \sum W_n = \int dW$.
- The center of gravity (G) is determined using the following equations:

$$\bar{x} = \frac{\sum \bar{x}_i W_i}{\sum W_i} \qquad \bar{y} = \frac{\sum \bar{y}_i W_i}{\sum W_i} \qquad \bar{z} = \frac{\sum \bar{z}_i W_i}{\sum W_i}$$



(a) Single element of the plate (b) Multiple elements of the plate (c) Center of gravity

https://static.boredpanda.com/blog/wp-content/uploads/2014/10/gravity-stone-balancing-michael-grab-4.jpg

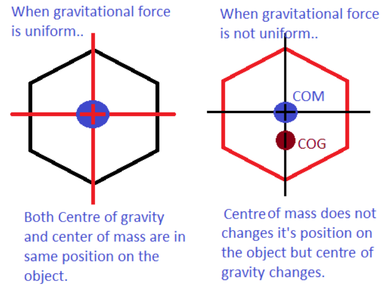
Center of Mass (CM)

- Since W=mg, by replacing the weights W with the masses m in the equations, the coordinates of the center of mass can be found.

$$\bar{x} = \frac{\sum \bar{x}_i W_i}{\sum W_i} \qquad \bar{x} = \frac{\sum \bar{x}_i m_i}{\sum m_i}$$

$$\bar{y} = \frac{\sum \bar{y}_i W_i}{\sum W_i} \quad \xrightarrow{W = mg} \quad \bar{y} = \frac{\sum \bar{y}_i m_i}{\sum m_i}$$

$$\bar{z} = \frac{\sum \bar{z}_i W_i}{\sum W_i} \qquad \bar{z} = \frac{\sum \bar{z}_i m_i}{\sum m_i}$$

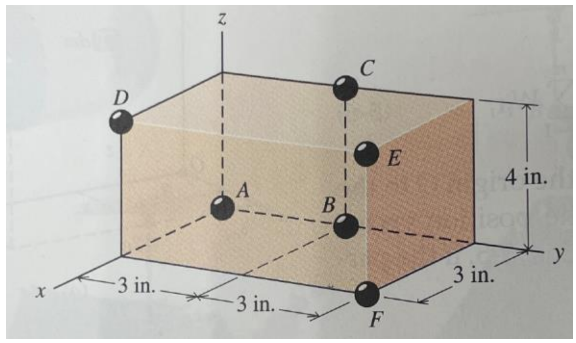


$\bar{x}W$ is the first moment about an axis or plane.

https://www.mechanicaleducation.com/wp-content/uploads/2019/05/Centre-of-gravity-mass.png

Example 1

- Locate the center of gravity for the six particles shown if W_A=50 lb, W_B=25 lb, W_C=30 lb, W_D=35 lb, W_E=20 lb, and W_F=40 lb.



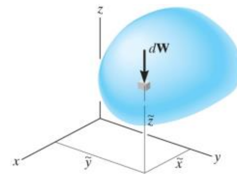
Center of Gravity (CG)

- A rigid body can be considered as made up of an infinite number of particles. Hence, using the same principles as with the center of mass, we get the coordinates of G by simply replacing the discrete summation sign (Σ) by the continuous summation sign (∫) and W by dW.

$$\bar{x} = \frac{\sum \bar{x}_i W_i}{\sum W_i} \qquad \bar{y} = \frac{\sum \bar{y}_i W_i}{\sum W_i} \qquad \bar{z} = \frac{\sum \bar{z}_i W_i}{\sum W_i}$$

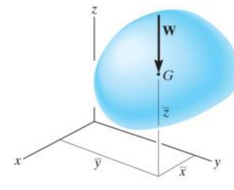
$$\Sigma = \int \qquad W = dW$$

$$\bar{x} = \frac{\int \tilde{x} dW}{\int dW} \qquad \bar{y} = \frac{\int \tilde{y} dW}{\int dW} \qquad \bar{z} = \frac{\int \tilde{z} dW}{\int dW}$$



(a)

Figure: 09_001a



(b)

Figure: 09_001b

Center of Mass (CM) & Centroid

- By replacing W with m in these equations, the coordinates of the center of mass can be found.

$$\bar{x} = \frac{\int \tilde{x} dW}{\int dW} \qquad \bar{x} = \frac{\int \tilde{x} dm}{\int dm}$$

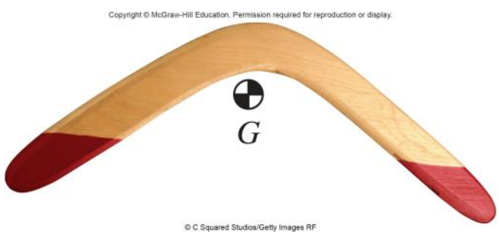
$$\bar{y} = \frac{\int \tilde{y} dW}{\int dW} \quad \xrightarrow{W = mg} \quad \bar{y} = \frac{\int \tilde{y} dm}{\int dm}$$

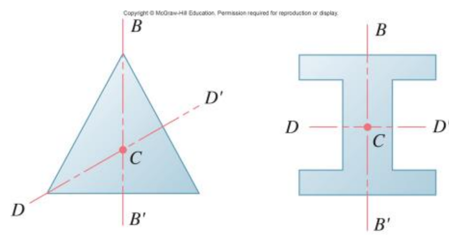
$$\bar{z} = \frac{\int \tilde{z} dW}{\int dW} \qquad \bar{z} = \frac{\int \tilde{z} dm}{\int dm}$$

- Similarly, the coordinates of the centroid of a volume, area, or length can be obtained by replacing W with V, A, or L, respectively.

Centroid of an Area

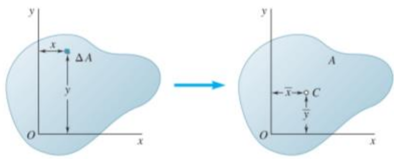
- The centroid is the geometric center of a body
  - This coincides with the center of mass and the center of gravity only if the material is uniform or homogenous (density and specific weight is constant throughout the body).
- The centroid may be located at a point that is not on the object.
- If there is a plane of symmetry, the centroid will lie upon that plane.





Centroids and First Moments of

AREAS



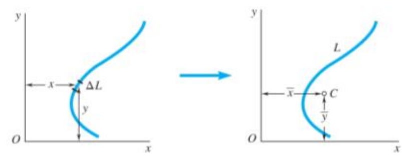
- Centroid

$$\bar{x} = \frac{\int xdA}{A} \qquad \bar{y} = \frac{\int ydA}{A}$$

- First Moment of Area

$$Q_x = \int ydA \qquad Q_y = \int xdA$$

LINES



- Centroid

$$\bar{x} = \frac{\int xdL}{L} \qquad \bar{y} = \frac{\int ydL}{L}$$

- First Moment of Line

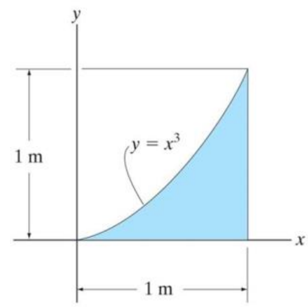
$$Q_x = \int ydL \qquad Q_y = \int xdL$$

Centroids of Areas via Integration

1. Sketch the body approximately to scale.
2. Establish a coordinate system.
    - Rectangular coordinates for flat planes for boundaries
    - Polar coordinates for circular boundaries
  - When a body has an axis of symmetry, the centroid is located on that axis
    - What happens if a body has two axes of symmetry?
3. Choose an appropriate differential element dA at a general point.
  - Generally, if y is easily express in terms of x (e.g. $y=x^2+1$), use a vertical rectangular element. If the opposite it true (e.g. $x=\sqrt{(y-1)}$), then use a horizontal element.
4. Express dA in terms of the differentiating elements dx (or dy).
5. Determine the coordinates $(\bar{x}, \bar{y})$ of the centroid of the rectangular element in terms of the general point (x,y).
6. Express all variables and integral limits in the formula using either x or y depending on whether the differential element is in terms of dx or dy, respectively, and integrate.

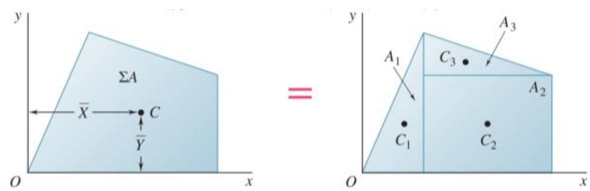
Example 2

- Locate the centroid of the area shown.

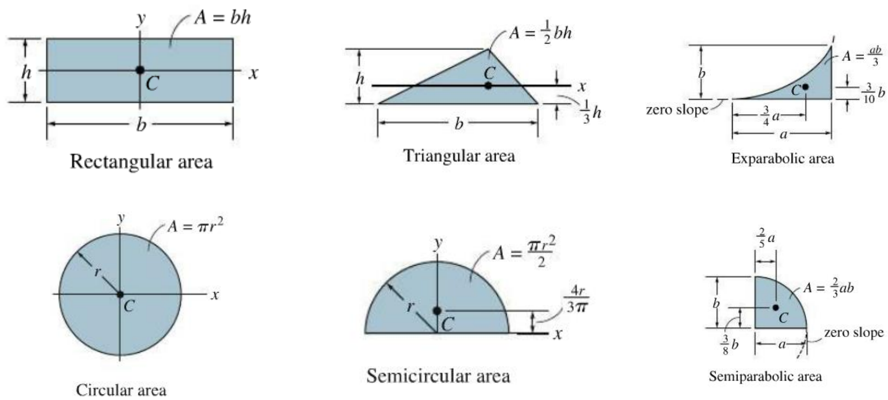


Composite Centroids and First Moments

- Some shapes can be broken up into simple shapes with known centroids and first moments.
- Take the quadrilateral shown below. It can be broken into two basic shapes (one rectangle and two triangles) with known centroids and first moments.
- The sum of the components of the individual centroids and first moments will yield the overall centroid and first moment of the quadrilateral

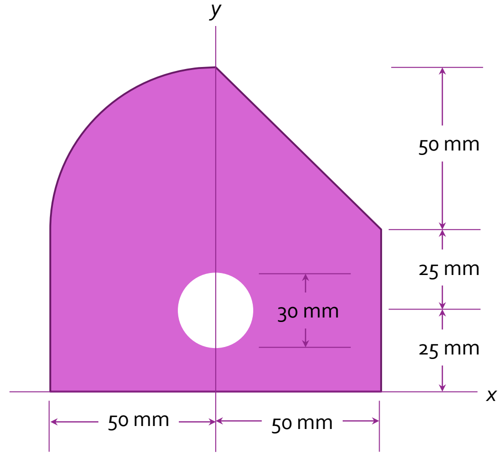


Centroids of Known Shapes



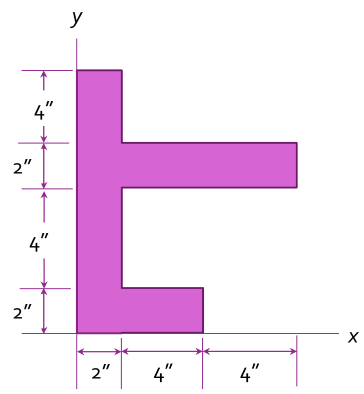
Example 3

- Locate the centroid of the composite area shown.



Example 4

- Locate the centroid of the composite area shown.



Theorems of Pappus-Guldinus

- Surface Area
  - The area A of a surface of revolution equals the product of the length of the generating curve and the distance traveled by the centroid of the curve in generating the surface area.

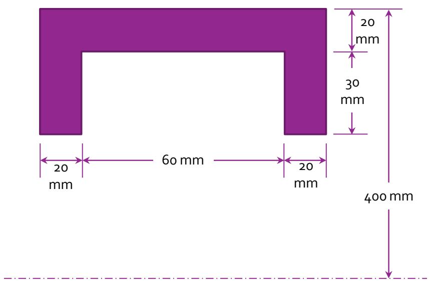
$$A = 2\pi\bar{y}L$$

- Volume
  - The volume V of a body of revolution equals the product of the generating area and the distance traveled by the centroid of the area in generating the volume.

$$V = 2\pi\bar{y}A$$

Example 5

- The outer diameter of a pulley is 0.8 m, and the cross-section of its rim is as shown. Knowing that the pully is made of steel which has a density of 7.85×10^3 kg/m^3, determine the mass and weight of the rim.

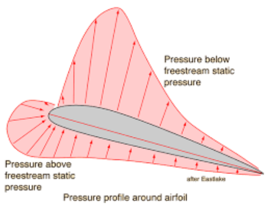


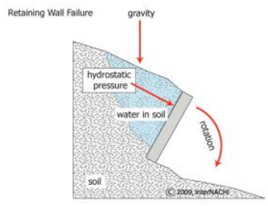
Centroid of a General Distributed Loading

- A distributed loading is a force that is spread over a length, area, or volume.

$$R = \int_L w dx$$

$$M_O = \int x dR \qquad x_C = \frac{M_O}{R}$$

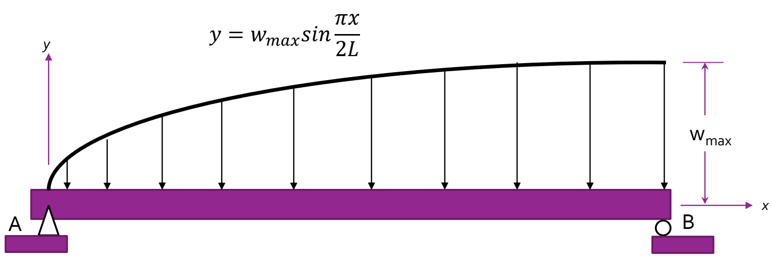




http://hyperphysics.phy-astr.gsu.edu/hbase/Fluids/airfoil.html https://res.cloudinary.com/internachi/image/fetch/f_auto,q_auto:best/https://s3.amazonaws.com/uploads-east-1.nachi.org/gallery-images/exterior/general/hydrostatic-pressure.jpg
https://encrypted-tbn0.gstatic.com/images?q=tbn:ANd9GcTnM1-MJo9oZD2Bs6WWILnWHsvN4ejs-FyoTA&usqp=CAU

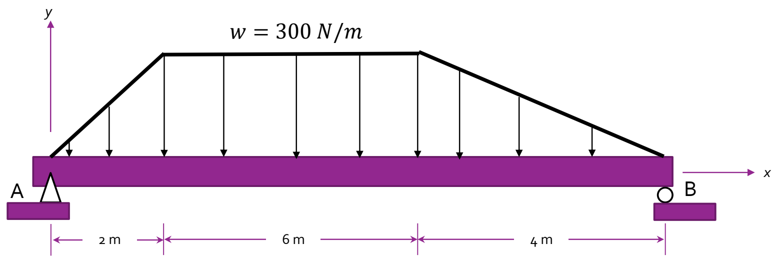
Example 6

- A beam is subjected to the load shown. Determine the resultant of this distributed load and locate its line of action with respect to the left support of the beam.



Example 7

- A beam is subjected to the loading shown. Determine the resultant of this system of distributed loads and locate its line of action with respect to the left support of the beam.

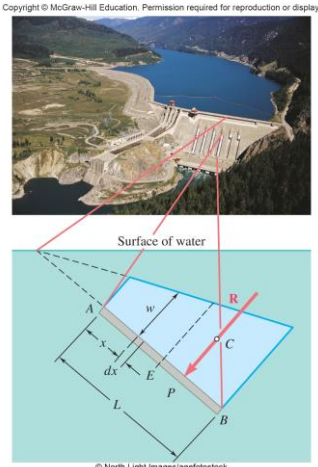


Centroid of a Submerged Surface

- Hydrostatic pressure is the pressure exerted by a fluid on an immersed body

$$P_h = \rho g h$$

  - Linear relationship between pressure and height/depth of a fluid



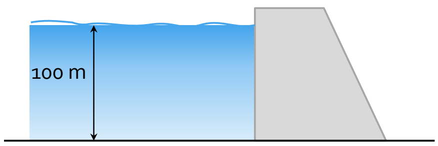
- Resultant force acting on a submerged surface

$$R = \int_A \rho dA = \int_V dV_{ps} = V_{ps}$$

- The principle of moments is used to determine the centroid location of the resultant force.

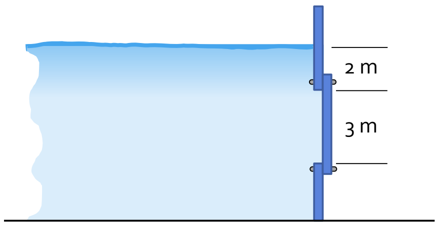
Example 8

- The water behind a dam is 100 m deep. Determine a) the magnitude of the resultant force **R** exerted on a 30-m length of dam by the water pressure and b) the distance from the water surface to the center of pressure.



Example 9

- Determine the magnitude of the resultant force acting on the submerged rectangular plate AB which has a width of 1.5 m and its location. ρ_w = 1000 kg/m^3.



3-D Centers of Gravity and Centroids

Center of Gravity of a 3-D Body

$$\bar{x} = \frac{\int \tilde{x} dW}{W}$$

$$\bar{y} = \frac{\int \tilde{y} dW}{W}$$

$$\bar{z} = \frac{\int \tilde{z} dW}{W}$$

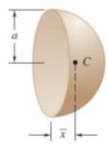
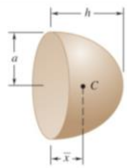
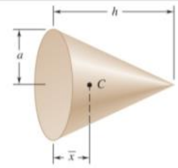
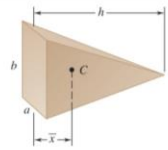
Centroid of a Volume

$$\bar{x} = \frac{\int \tilde{x} dV}{V}$$

$$\bar{y} = \frac{\int \tilde{y} dV}{V}$$

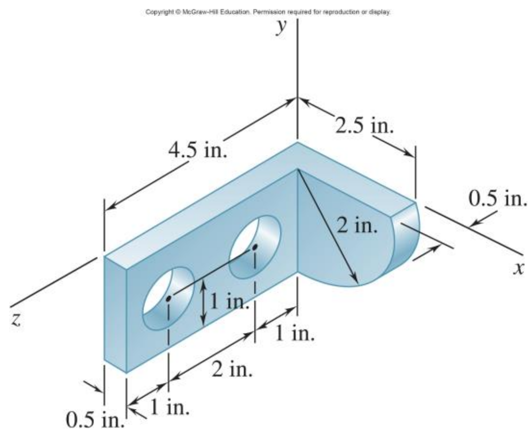
$$\bar{z} = \frac{\int \tilde{z} dV}{V}$$

Centroids & Volume of Common 3-D Shapes

Shape		$\bar{x}$	Volume
Hemisphere		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$
Cone		$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3} abh$

Example 10

- Locate the center of gravity of the steel machine part shown.



Example 11

- Locate the centroid of the part shown if all parts are made of steel (ρ=7860 kg/m3).
- Locate the center of gravity of the part shown if the triangular plate is made of steel (ρ=7860 kg/m3) and the rest of the components are made of aluminum (ρ=2700 kg/m3).