Section 5.1

Example. Let S be a nonempty subset of the set of real numbers $\mathbb{R}$. Let $c \in S'$. Then, there exists a sequence (s_n) of elements in S with $s_n \neq c$ for all $n \in \mathbb{N}$, such that $s_n \to c$ as $n \to \infty$.

Solution. Since $c \in S'$, for $\epsilon = \frac{1}{n} > 0$, where $n \in \mathbb{N}$, we have $N^*(c; \frac{1}{n}) \cap S \neq \emptyset$. Let $s_n \in N^*(c; \frac{1}{n}) \cap S$. Then, $s_n \in S$, $s_n \neq c$, and $0 < |s_n - c| < \frac{1}{n}$. This implies that

$$0 \leq \lim_{n\to\infty} |s_n - c| \leq \lim_{n\to\infty} \frac{1}{n} = 0,$$

and so, $\lim_{n\to\infty} |s_n - c| = 0$ implying $\lim_{n\to\infty}(s_n - c) = 0$, and hence, $\lim_{n\to\infty} s_n = c$. Thus, the statement is true. $\square$

(Solution 8, page 204)

(a) Given $\lim_{x\to c} f(x) = L$. Then, for $\epsilon > 0$, there exists a $\delta > 0$, such that $|f(x) - L| < \epsilon$, whenever $x \in D$, and $0 < |x - c| < \delta$. We know that $\Big||f(x) - |L|\Big| \leq |f(x) - L|$. Hence, we can say that for $\epsilon > 0$, there exists a $\delta > 0$, such that $\Big||f(x)| - |L|\Big| < \epsilon$, whenever $x \in D$, and $0 < |x - c| < \delta$. Thus, we have $\lim_{x\to c} |f(x)| = |L|$.

(b) Since $c \in D'$, there exists a sequence (s_n) of elements in D, with $s_n \neq c$ for all $n \in \mathbb{N}$, such that $s_n \to c$ as $n \to \infty$. Given, $\lim_{x\to c} f(x) = L$. Hence, we have $\lim_{n\to\infty} f(s_n) = L$, which by Theorem 5.1.8, implies that

$$\lim_{n\to\infty} \sqrt{f(s_n)} = \sqrt{\lim_{n\to\infty} f(s_n)} = \sqrt{L},$$

which again, by Theorem 5.1.8, implies that $\lim_{x\to c} \sqrt{f(x)} = \sqrt{L}$. $\square$

(Solution 13, page 204)

Since $c \in D'$, there exists a sequence (s_n) of elements in D, with $s_n \neq c$ for all $n \in \mathbb{N}$, such that $s_n \to c$ as $n \to \infty$. Since, $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, by Theorem 5.1.8, we have

$$\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} h(s_n) = L.$$

Again, by the given condition, we have $f(s_n) \leq g(s_n) \leq h(s_n)$, as $s_n \in D$ for all $n \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} f(s_n) \leq \lim_{n\to\infty} g(s_n) \leq \lim_{n\to\infty} h(s_n),$$

which implies $L \leq \lim_{n\to\infty} g(s_n) \leq L$, i.e., $\lim_{n\to\infty} g(s_n) = L$. Hence, by applying Theorem 5.1.8 again, we have $\lim_{x\to c} g(x) = L$. $\square$

Section 5.2

(Solution 10, page 214)

(a) Let (x_n) be a sequence in D converging to c. Since f is continuous at c, by Theorem 5.2.2, we have $\lim_{n\to\infty} f(x_n) = f(c)$. Then,

$$\lim_{n\to\infty} |f|(x_n) = \lim_{n\to\infty} |f(x_n)| = |\lim_{n\to\infty} f(x_n)| = |f(c)|.$$

Hence, by applying Theorem 5.2.2 again, we see that $|f|$ is continuous at c. $\square$

(b) False. Define

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ -1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then, $|f|(x) = 1$ for all $x \in \mathbb{R}$. Here, $|f|$ is continuous on $\mathbb{R}$, but f is not continuous anywhere. $\square$

(Solution 18, page 214)

Let $x \in (a, b)$. Then, x is an interior point of (a, b), and so there exists an $\epsilon > 0$, such that $N(x; \epsilon) \subset (a, b)$. Now, by Theorem 3.3.10, there exists a positive integer N such that $0 < \frac{1}{N} < \epsilon$. Then, notice that for all $n \in \mathbb{N}$ with $n \geq N$, we have $0 < \frac{1}{n} \leq N$, and so for all $n \geq N$, we have

$$N(x; \frac{1}{n}) \subseteq N(x; \frac{1}{N}) \subseteq N(x; \epsilon) \subset (a, b).$$

Take $r_n \in \mathbb{Q}$, such that $r_n \in N(x; \frac{1}{n})$. Then, for $n \geq N$, (r_n) is a sequence of rational numbers in (a, b) such that as $r_n \in N(x; \frac{1}{n})$, we have

$$0 \leq |r_n - x| < \frac{1}{n}.$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, by squeeze theorem, we have $\lim_{n\to\infty} |r_n - x| = 0$, i.e., $\lim_{n\to\infty}(r_n - x) = 0$, i.e., $x = \lim_{n\to\infty} r_n$. Again f is continuous on (a, b). Hence, by applying Theorem 5.2.2, we have

$$f(x) = \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} 0 = 0.$$

i.e., $f(x) = 0$ for all $x \in (a, b)$. Thus, the proof is complete. $\square$