Sol$^n$ #9

Page 170 (a) Let $(s_n)$ converges to $s$. Then for $\varepsilon > 0$, $\exists$ a +ve integer $N$ such that $\forall n \geq N$, with $n \in \mathbb{N}$, we have

$$|s_n - s| < \varepsilon.$$

Then, $\big||s_n| - |s|\big| \leq |s_n - s| < \varepsilon \quad \forall n \geq N$ with $n \in \mathbb{N}$. Hence $|s_n| \to |s|$ as $n \to \infty$.

(b) ~~Assume that $|s_n|$~~:

It is not correct. For example, take $s_n = (-1)^n$. Then $|s_n| = 1 \ \forall n \in \mathbb{N}$. Here $(|s_n|)$ is convergent, but $(s_n)$ is not convergent.

(c) Notice that $\big||s_n|\big| = |s_n|$, and so

$|s_n - 0| < \varepsilon$ ~~iff~~ if and only if

$$\big||s_n| - 0\big| < \varepsilon$$

Hence, the statement is true.

Sol$^n$ 12

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(a) Let $(t_n)$ is a bounded sequence, then $\exists M > 0$ such that

$$|t_n| \leq M \quad \forall n \in \mathbb{N}.$$

Suppose that $\lim_{n\to\infty} s_n = 0$. Then, for $\frac{\varepsilon}{M} > 0$, $\exists$ a +ve integer $N$ such that $\forall n \geq N$, $n \in \mathbb{N}$, we have

$$|s_n - 0| < \frac{\varepsilon}{M}, \text{ i.e. } |s_n| < \frac{\varepsilon}{M}.$$

Then,

$$|s_n t_n - 0| = |s_n||t_n| < \frac{\varepsilon}{M} \cdot M = \varepsilon$$

$\forall n \geq N$ with $n \in \mathbb{N}$.

Hence, $\lim_{n\to\infty} s_n t_n = 0$.

(b) Suppose that $(t_n)$ is not a bonded sequene.

Let $t_n = n \quad \forall n \in \mathbb{N}$.

Take $s_n = \frac{1}{n} \quad \forall n \in \mathbb{N}$.

Then, we ~~see~~ that

$\lim_{n\to\infty} s_n = 0$, but $\lim_{n\to\infty} s_n t_n = 1 \neq 0$

i.e. boundedness is necessary.