Lemma 3.5.4 page 145) If S is a nonempty closed and bounded subset of $\mathbb{R}$, then S has a maximum and a minimum element.

Proof Let S be a bounded subset of $\mathbb{R}$. Then, S is both bounded above and bounded below. ~~Let~~ First assume that S is bounded above. Since S is nonempty and bounded above by completeness axiom (see page 126) $\sup(S)$ exists in $\mathbb{R}$. Let $m = \sup(S)$. Then If m is an accumulation point of S, then since S is closed, we have $m \in S$ i.e. $m = \max(S)$. Suppose that m is not an accumulation point of S. We will still show that ~~$m = \max(S)$~~ $m \in S$. ~~Assume~~ We will show it by contradiction. ~~Since~~ Assume that $m \notin S$. Again, m is not an accumulation point of S, and so $\exists$ an $\varepsilon > 0$ such that

$$N(m; \varepsilon) \cap S = \varphi \quad [\text{as } N^*(m;\varepsilon) \cap S = \varphi \text{ and } m \notin S]$$

$$\Rightarrow (m-\varepsilon, m+\varepsilon) \cap S = \varphi$$

~~i.e. $\Rightarrow m - \varepsilon, m$~~ i.e. there does not exist any $y \in S$ such that $m - \varepsilon < y \leq m + \varepsilon$

i.e. all $y \in S$, we have $y \leq m - \varepsilon$

i.e. $m - \varepsilon$ is an upper bound of S

i.e. $m - \varepsilon$ is an upper bound of S which contradicts the fact that $m = \sup(S)$.

Hence, we can assume that $m \in S$. ~~i.e.~~ ~~m~~ Since $m = \sup(S)$ and $m \in S$, we have $m = \max(S)$, i.e. S has a maximum element.

Similarly, we can show that ~~S is~~ as S is also bounded below, if $m = \inf(S)$, then $m = \cancel{\min} \in S$ ~~and m~~ i.e. m is the minimum element in S.

Thus, the proof is complete.