Example

Take $\varepsilon = 1$. Then,

$$N(3;\varepsilon) = N(3;1) = (3-1,\ 3+1) = (2,4)$$

$$N(0;5) = (0-5,\ 0+5) = (-5,5).$$

Example of deleted nbd

$$N^*(3;1) = (2,4) \setminus \{3\} = (2,3) \cup (3,4)$$

$$N^*(0;5) = (-5,0) \cup (0,5).$$

Consider the open interval $(3,5)$

Here 3, and 5 are boundary points. Interior of a set S is denoted by int(S), and the ~~boundary of is~~

set of all boundary points of a set S is denoted by ~~bd(S)~~. bd(S).

Here $(3,5)$ is an open set, since all the points of the interval $(3,5)$ are interior points..

The interval $[3,5]$ is a closed set, since $\mathbb{R} \setminus [3,5] = (-\infty,\infty) \setminus [3,5]$

$$= (-\infty, 3) \cup (5, \infty)$$

which is open.

The empty ϕ is ~~open~~ closed since $\mathbb{R} \setminus \phi = (-\infty,\infty) \setminus \phi = (-\infty, \infty)$ which is open.

The empty set ϕ is also open since there are no point in ϕ which are not interior of point of ϕ, i.e. ~~ϕ~~ so we can say ϕ is an open set.

Theorem 3.4.10

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(a) The union of any collection of open sets is an open set.

Proof Let $\mathcal{A}$ be an arbitrary collection of open sets. Let

$$S = \bigcup_{A \in \mathcal{A}} A.$$

We need to show S is open. ~~Let~~ Take any $x \in S$. Then

$$x \in \bigcup_{A \in \mathcal{A}} A$$

$\Rightarrow x \in A$ for some $A \in \mathcal{A}$

$\Rightarrow$ Since A is open, $\exists$ an $\varepsilon > 0$, such that $N(x; \varepsilon) \subset A \subset \bigcup_{A \in \mathcal{A}} A = S$

$\Rightarrow$ i.e. x is an interior point of S. i.e. Hence S is an open set.

Thus, the proof is complete.

Example.
Take $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$

Then
$$\bigcup_{A \in \mathcal{A}} A = A_1 \cup A_2 \cup A_3 \cup A_4$$

Similarly
$$\bigcap_{A \in \mathcal{A}} A = A_1 \cap A_2 \cap A_3 \cap A_4$$

(b) The intersection of any finite collection of open sets is an open set.

Proof Let $\{A_1, A_2, \ldots, A_n\}$ be a finite collection of open set.

Let $S = A_1 \cap A_2 \cap \ldots \cap A_n$.

We need to show S is open. Take any $x \in S$.

$\Longrightarrow x \in A_1 \cap A_2 \cap \ldots \cap A_n$

$\Longrightarrow x \in A_1, x \in A_2, \ldots, x \in A_n$

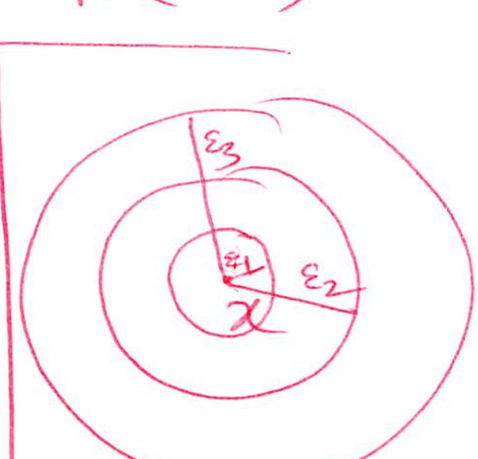
$\Longrightarrow \exists\, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n > 0$ ~~sucth~~ such that

$N(x;\varepsilon_1) \subset A_1,\ N(x;\varepsilon_2) \subset A_2, \ldots, \cancel{N(x,\varepsilon_n)}$

$N(x;\varepsilon_n) \subset A_n$

Take

$\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$



~~$N(x;\varepsilon) \in N(x$~~

Then

$N(x;\varepsilon) \subset N(x;\varepsilon_1) \subset A_1$

$N(x;\varepsilon) \subset N(x;\varepsilon_2) \subset A_2$

$\vdots$

$N(x;\varepsilon) \subset N(x;\varepsilon_n) \subset A_n$

$\Longrightarrow N(x;\varepsilon) \subset A_1 \cap A_2 \cap \cdots \cap A_n$

$\Longrightarrow N(x;\varepsilon) \subset S$

$\Rightarrow x$ is an interior point of S.

Hence, S is open.

Thus, the proof is complete

Corollary 3.4.11

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(a) The intersection of any collection of closed sets is closed.

(b) The union of any finite collection of closed sets is closed.

Proof of (a)

Let $\mathcal{A}$ be an arbitrary collection of closed sets.

Let $S = \cancel{\bigcup_{A\in\mathcal{A}}} \bigcap_{A\in\mathcal{A}} A$

We need to show S is closed.

Notice that

$$\mathbb{R}\setminus S = \mathbb{R}\setminus \bigcap_{A\in\mathcal{A}} A$$

$$= \bigcup_{A\in\mathcal{A}} (\mathbb{R}\setminus A).$$

Again each $\mathbb{R}\setminus A$ is open, and so

~~$\bigcup_{A\in\mathcal{A}} (\mathbb{R}\setminus A)$ is open~~

We can say $\bigcup_{A\in\mathcal{A}} (\mathbb{R}\setminus A)$ is ~~th~~ a union of an arbitrary collection of open sets. Hence $\bigcup_{A\in\mathcal{A}} (\mathbb{R}\setminus A)$ is open, i.e. $\mathbb{R}\setminus S$ is open, i.e. S is closed.

(Proved)

Proof of (b)

Let $\{A_1, A_2, \dots, A_n\}$ be a finite collection of closed sets. Let

~~$S = A_1 \cap A_2 \cap$~~

$$S = A_1 \cup A_2 \cup \dots \cup A_n$$

We need to show S is closed.

Notice that

$$\mathbb{R}\setminus S = \mathbb{R}\setminus (A_1 \cup A_2 \cup \dots \cup A_n)$$

$$= (\mathbb{R}\setminus A_1) \cap (\mathbb{R}\setminus A_2) \cap \dots \cap (\mathbb{R}-A_n)$$

which is the intersection of a finite collection of open sets, and so $\mathbb{R}\setminus S$ is open, i.e. S is closed.

Thus, the proof is complete.

Example

Let $S = \{x_1, x_2, x_3\}$