Section 4.1

#7(c) Page 170

If $|x|<1$, then $\lim_{n\to\infty} x^n = 0$.

Sol^n ~~If~~ we know $0 \le |x| < 1$. If $|x|=0$, there is nothing to prove. Assume that $0<|x|<1$. Then $\exists$ a +ve real number h such that $|x| = \frac{1}{1+h}$. We first show that $\lim_{n\to\infty} |x|^n = 0$. Choose any $\varepsilon > 0$.

Now, $|x|^n = \left(\frac{1}{1+h}\right)^n = \frac{1}{(1+h)^n} < \frac{1}{1+nh} < \varepsilon$

Whenever

$1+nh > \frac{1}{\varepsilon}$

$\Rightarrow$ i.e. if $n > \left(\frac{1}{\varepsilon}-1\right)\frac{1}{h}$

Since $(1+h)^n = 1 + {}^nC_1 h + {}^nC_2 h^2 + \cdots$
$> 1 + {}^nC_1 h$
$= 1+nh$,
we have
$\frac{1}{(1+h)^n} < \frac{1}{1+nh}$

Let N be a +ve integer such that $N > \left(\frac{1}{\varepsilon}-1\right)\frac{1}{h}$.

Then, for all $n \ge N$, we have $n > \left(\frac{1}{\varepsilon}-1\right)$, and so

$|x|^n < \varepsilon$, i.e. $|x^n| < \varepsilon$.

Since $|x^n| = |x.x.x\cdots x|$
$= |x|.|x|\cdots|x|$
$= |x|^n$

Hence, $\lim_{n\to\infty} |x^n| = 0$.

~~As~~ Again,

$-|x^n| \le x^n \le |x^n|$

$\Rightarrow -\lim_{n\to\infty} |x^n| \le \lim_{n\to\infty} x^n \le \lim_{n\to\infty} |x^n|$

$\Rightarrow 0 \le \lim_{n\to\infty} x^n \le 0$

$\Rightarrow \lim_{n\to\infty} x^n = 0$ (Proved)

#10 Find an example of each of the following.

(a) A convergent sequence of rational numbers having an irrational limit.

(b) A convergent sequence of irrational numbers having a rational limit.

Solⁿ (a) We know $\sqrt{3}$ is an irrational number, and $\sqrt{3} = 1.732050808\cdots$

Let x_n be the first n digits in the decimal expansion of $\sqrt{3}$. Then,

$x_1 = 1$, $x_2 = 1.7$, $x_3 = 1.73$, $x_4 = 1.732$, and so on. Then, (x_n) is a sequence of ~~lim~~ $\cancel{\lim_{n\to\infty} x_n = 1.732050808\cdots = \sqrt{3}}$, rational numbers such that

$$\lim_{n\to\infty} x_n = 1.732050808\cdots = \sqrt{3},$$

which is an irrational number.

(Ans)

(b) Take $x_n = \frac{\sqrt{2}}{n}$.

Then, $\lim_{n\to\infty} x_n = 0$,

i.e. (x_n) is a sequence of irrational numbers having a rational ~~number~~ limit.

—x—

#14
Page 170

Suppose that $\lim_{n\to\infty} s_n = s$, with $s > 0$. Prove that there exists $N \in \mathbb{N}$ such that $s_n > 0$ for all $n \geq N$.

Proof: Since $\lim_{n\to\infty} s_n = s$ with $s > 0$, for $\varepsilon = \frac{s}{2} > 0$, $\exists$ a +ve integer N such that $\forall\, n \geq N$, we have

$$|s_n - s| < \varepsilon$$

$$\Rightarrow -\varepsilon < s_n - s < \varepsilon$$

$$\Rightarrow s - \varepsilon < s_n < s + \varepsilon$$

$$\Rightarrow \frac{s}{2} < s_n < \frac{3s}{2}$$

$$\Rightarrow s_n > \frac{s}{2} > 0 \quad \forall\, n \geq N$$

(Proved)

Section 4.2

#8 page 178 (a) Give an example of a convergent sequence (s_n) of positive numbers such that

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = 1.$$

(b) Give an example of a divergent sequence (t_n) of positive numbers such that

$$\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.$$

Sol^n (a) Take $s_n = \frac{1}{n}$

Then (s_n) is a convergent sequence converging to zero. Here

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

(b) Take $t_n = n$.

Then, (t_n) is a divergent sequence, with $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = \lim_{n\to\infty} \frac{n+1}{n} = 1.$

#11 page 178 Prove that if $\lim_{n\to\infty} s_n = 0$ and $s_n > 0$ for all n, then for any $k > 0$, $\lim_{n\to\infty} s_n^k = 0$.

Proof Choose $\varepsilon > 0$. Since $\lim_{n\to\infty} s_n = 0$, for $\varepsilon^{\frac{1}{k}} > 0$, $\exists$ a +ve integer N such that $\forall n \geq N$,

$$|s_n - 0| < \varepsilon^{\frac{1}{k}} \Rightarrow |s_n| < \varepsilon^{\frac{1}{k}} \Rightarrow |s_n|^k < \varepsilon$$

Thus, for $\varepsilon > 0$, $\exists$ a +ve integer N such that for all $n \geq N$,

$$0 \leq s_n^k \leq |s_n|^k < \varepsilon \quad [\text{as } 0 \leq s_n \leq |s_n|]$$

$$\Rightarrow \lim_{n\to\infty} s_n^k = 0. \quad \text{(Proved)}$$

#12 Page 178

Suppose that (s_n) converges to s. Prove that (s_n^2) converges to s^2 ~~wit~~ directly using the definition of limit.

Sol^n Since (s_n) converges, the sequence (s_n) is bounded, i.e. $\exists$ a +ve real number M_1 such that $|s_n| \leq M_1$ $\forall n \in \mathbb{N}$.

Let $M = M_1 + |s|$.

Now, since (s_n) converges to s, for $\varepsilon > 0$, $\exists$ a +ve integer N such that $\forall n \geq N$, we have

$$|s_n - s| < \frac{\varepsilon}{M_1}$$

Then

$$
\begin{aligned}
|s_n^2 - s^2| &= |(s_n+s)(s_n-s)| \\
&= |s_n+s| \cdot |s_n - s| \\
&\leq (|s_n| + |s|)|s_n - s| \\
&\leq (M_1 + |s|)|s_n - s| \\
&< M_1 \frac{\varepsilon}{M_1} = \varepsilon
\end{aligned}
$$

[As $|s_n+s| \leq |s_n| + |s|$]

i.e. ~~$|s_n$~~ for $\varepsilon > 0$, $\exists$ a +ve integer N such that $\forall n \geq N$,

$|s_n^2 - s^2| < \varepsilon$. Hence, $\lim_{n\to\infty} s_n^2 = s^2$

(Proved).

Section 4.3

#7) Let $s_1 = \sqrt{6}$, $s_2 = \sqrt{6+\sqrt{6}}$, $s_3 = \sqrt{6+\sqrt{6+\sqrt{6}}}$, and in general define $s_{n+1} = \sqrt{6+s_n}$. Prove that the sequence (s_n) converges, and find its limit.

Sol^n We have

$$0 < s_1 = \sqrt{6} = \sqrt{2\cdot 3} \leq \sqrt{3\cdot 3} = 3$$

$$0 < s_2 = \sqrt{6+\sqrt{6}} \leq \sqrt{6+3} = \sqrt{9} = 3$$

$$0 < s_3 = \sqrt{6+\sqrt{6+\sqrt{6}}} \leq \sqrt{6+3} = \sqrt{9} = 3$$

and so on.

Thus, (s_n) is a bounded sequence.

We now show that (s_n) is an increasing sequence. Given

$$s_{n+1} = \sqrt{6+s_n}$$

and so

$$s_{n+2}^2 - s_{n+1}^2 = \left(\sqrt{6+s_{n+1}}\right)^2 - \left(\sqrt{6+s_n}\right)^2$$

$$= 6 + s_{n+1} - 6 - s_n$$

$$= s_{n+1} - s_n$$

$$\Rightarrow (s_{n+2} + s_{n+1})(s_{n+2} - s_{n+1}) = s_{n+1} - s_n$$

$$\Rightarrow s_{n+2} - s_{n+1} = \frac{s_{n+1} - s_n}{s_{n+2} + s_n}$$

Hence $s_{n+2} > s_{n+1}$ if and only if $s_{n+1} > s_n$.

Notice that

$s_2 > s_1$, and so $s_3 > s_2$, and so $s_4 > s_3$,

and so on.

i.e. $s_1 < s_2 < s_3 < s_4 < \cdots$

i.e. (s_n) is an increasing sequence.

~~i.e. (s_n)~~

Since (s_n) is bounded and a monotone sequence, ~~b~~ we can say that (see Theorem 4.3.3, page 180), (s_n) converges.

Let $\lim_{n\to\infty} s_n = l$.

Since $s_n \geq 0$, we can say that $l \geq 0$.

Again, given

$$s_{n+1} = \sqrt{6+s_n}$$

$$\Rightarrow \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \sqrt{6+s_n}$$

$$\Rightarrow l = \sqrt{6+l}$$

$$\Rightarrow l^2 = 6+l$$

$$\Rightarrow l^2 - l - 6 = 0$$

$$\Rightarrow (l-3)(l+2) = 0$$

$$\Rightarrow l = 3, -2$$

since $l \geq 0$, we have $l = 3$

i.e. $\lim_{n\to\infty} s_n = 3$ Ans

Section 4.4

#7) Prove or give a counterexample.

(a) Every bounded sequence has a Cauchy subsequence.

(b) Every monotone sequence has a bounded subsequence.

(c) Every convergent sequence can be represented as the sum of two ~~oscillating~~ oscillating sequences.

Solⁿ (a) True. Theorem 4.4.7 says that every bounded sequence has a convergent subsequence. Theorem 4.3.10 says that every convergent sequence is a Cauchy sequence.

(b) False. Take $S_n = n \; \forall n$. Then (S_n) is a monotone sequence, it does not have a bounded subsequence.

(c) True. Let (S_n) be a convergent sequence such that $\lim_{n\to\infty} S_n = S$. Let $a_n = (-1)^n + \frac{S_n}{2}$ and $b_n = (-1)^{n+1} + \frac{S_n}{2}$. Notice that here both (a_n) & (b_n) are oscillating sequences, since they each have two subsequences converging to $-1+\frac{S}{2}$ and $1+\frac{S}{2}$. But

$$S_n = a_n + b_n.$$