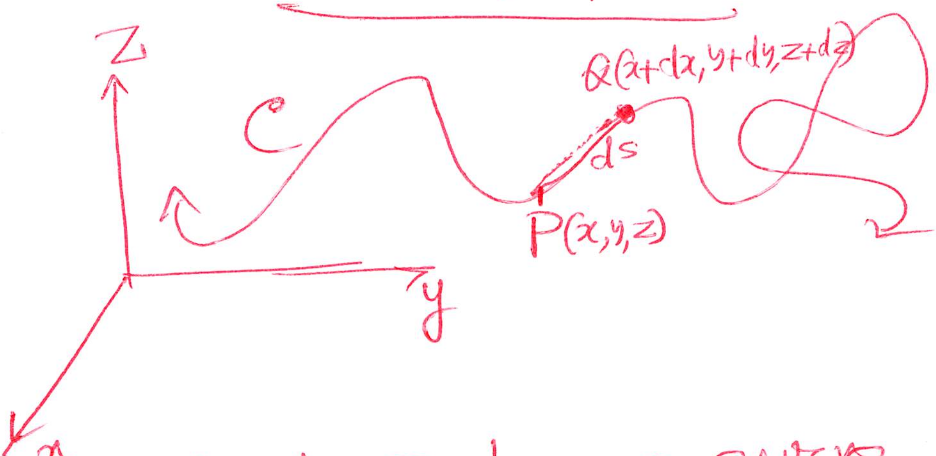
Section 13.2

1) 

Let C be a curve in the space. Sometimes, it is not possible to represent the equation of the curve in the form $Z = f(x, y)$, because Z may not be a function of x, y. We always represent the equation of a curve C as follows

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

where $t \in \mathbb{R}$. The above equations are called the parametric equations of the curve, and t is called the parameter.

2) Suppose C is a bounded curve, i.e. the parametric equations of the curve C are given by

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

for $a \leq t \leq b$.

Let ds be the length of the curve between the two points (x, y, z) and $(x+dx, y+dy, z+dz)$. Then

$$ds = \sqrt{(x+dx-x)^2 + (y+dy-y)^2 + (z+dz-z)^2}$$

$$\Rightarrow (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

$$\Rightarrow \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

$$\Rightarrow \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$\Rightarrow ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Hence, the length L of the curve C ~~is given~~ ~~by~~ for $a \le t \le b$ is given by

$$L = \int_C ds = \int_{t=a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

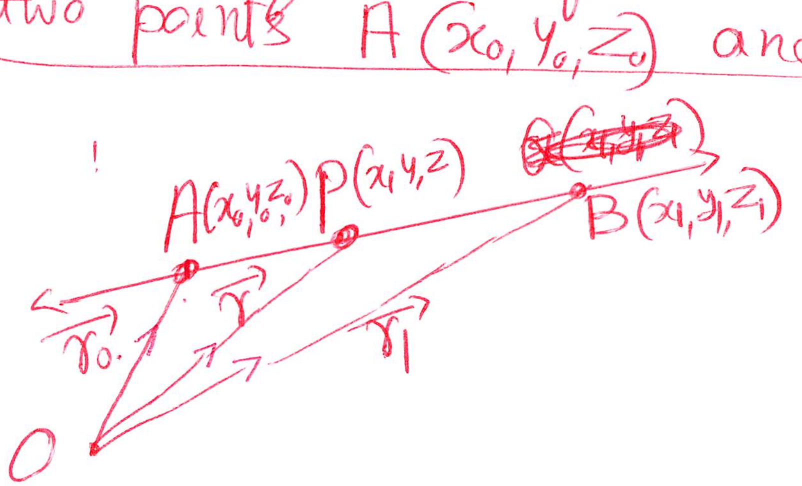
In fact, for the curve C given by the parametric equations

$$x = x(t),\ y = y(t),\ z = z(t),\quad a \le t \le b$$

we have

$$\int_C f(x,y,z)\, ds = \int_{t=a}^{b} f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

3) Vector equation of the line joining the two points $A(x_0, y_0, z_0)$ and $B(x_1, y_1, z_1)$.



Here $\vec{r_0} = \vec{OA} = \langle x_0, y_0, z_0 \rangle$

$\vec{r_1} = \vec{OB} = \langle x_1, y_1, z_1 \rangle$

$\vec{r} = \vec{OP} = \langle x, y, z \rangle$.

Since $\vec{AP} \parallel \vec{AB}$, there exists a scalar $t \in \mathbb{R}$ such that

$$\vec{AP} = t(\vec{AB})$$

$$\Rightarrow \vec{OP} - \vec{OA} = t(\vec{OB} - \vec{OA})$$ [by triangle law of vector addition]

$$\Rightarrow \vec{r} - \vec{r_0} = t(\vec{r_1} - \vec{r_0})$$

$$\Rightarrow \vec{r} = \vec{r_0} + t(\vec{r_1} - \vec{r_0})$$

$$\Rightarrow \vec{r} = (1-t)\vec{r_0} + t\vec{r_1}$$

The above equation is called the vector equation of the line joining the two points $A(x_0, y_0, z_0)$ and $B(x_1, y_1, z_1)$.

Notice that the vector representation or the vector equation of the line segment joining A and B is given by

$$\vec{r} = (1-t)\vec{r_0} + t\vec{r_1} \quad \text{for } 0 \le t \le 1.$$

4) We know $\vec{r} = \langle x, y, z \rangle$.
In the case of parametric representation,
$x = x(t)$, $y = y(t)$, $z = z(t)$ for $t \in \mathbb{R}$,
and so

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

and $\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

$$\Rightarrow |\vec{r}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$
$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Hence, $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$
$$= |r'(t)|\, dt$$

Again,
$$\vec{T}(t) = \text{the unit tangent vector}$$
$$= \frac{\vec{r}'(t)}{|\vec{r}'(t)|}.$$

Definition
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Let F be a continuous vector field defined on a smooth curve C given by a vector function $\vec{r}(t)$, $a \le t \le b$. Then, the line integral of $\vec{F}$ along C is given by

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$
$$= \int_C \vec{F}(\vec{r}(t)) \cdot \vec{T}(t)\, |r'(t)|\, dt$$
$$= \int_C \vec{F} \cdot \vec{T}\, ds.$$

Exercises Section 13.2

Evaluate the line integral, where c is the given curve.

#2) $\int_c xy\,ds$, $c: x=t^2,\ y=2t,\ 0\le t\le 1$

Sol^n $\int_c xy\,ds = \int_{t=0}^{1} t^2 \cdot 2t \cdot 2\sqrt{t^2+1}\,dt$

$\left[ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{(2t)^2+(2)^2}\,dt = \sqrt{4t^2+4}\,dt = 2\sqrt{t^2+1}\,dt\right.$

$= 4\int_{t=0}^{1} t^3\sqrt{t^2+1}\,dt$

$= 4\int_{t=0}^{1} t^2\sqrt{t^2+1}\cdot t\,dt$

$\left[\begin{array}{l}\text{Put} \\ u = t^2+1 \\ du = 2t\,dt \\ \Rightarrow t\,dt = \frac{1}{2}du \\ \text{when } t=0,\ u=1 \\ \text{when } t=1,\ u=2\end{array}\right.$

$= 4\int_{1}^{2} (u-1)\sqrt{u}\,\frac{du}{2}$

$= 2\int_{1}^{2} \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$

$= 2.\left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{3/2}}{3/2}\right]_1^2$

$= 2\left[\frac{2}{5}\left(2^{\frac{5}{2}}-1\right) - \frac{2}{3}\left(2^{3/2}-1\right)\right]$

$= 2\left[\frac{2}{5}\left(4\sqrt{2}-1\right) - \frac{2}{3}\left(2\sqrt{2}-1\right)\right]$

$= 2\left[\left(\frac{8}{5}\sqrt{2} - \frac{4}{3}\sqrt{2}\right) - \left(\frac{2}{5}-\frac{2}{3}\right)\right]$

$= 2\left[\frac{4\sqrt{2}}{15} + \frac{4}{15}\right] = \frac{8}{15}(\sqrt{2}+1)$ Ans