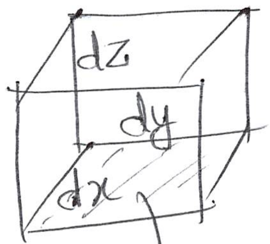
Section 12.6

We know that the volume of a solid with height dz and base area dA is given by

$$dV = dz \cdot dA.$$
$$= \text{height} \times \text{area}$$



Area $= dA$
$= \text{length} \times \text{width}$
$= dx\,dy$

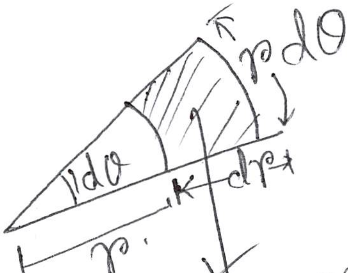
If the area dA is rectangular as shown in the figure, we have

$$dA = dx\,dy,$$

and so $dV = dz \cdot dA = dz \cdot dx \cdot dy$
$$= dx \cdot dy \cdot dz.$$

If the area dA can be represented by polar coordinates as shown in the figure, then $dA = r\,dr\,d\theta$, and ~~Hence~~, then we have

$$dV = dz \cdot dA$$
$$= r\,dz \cdot dr \cdot d\theta.$$



Area $= dA = \text{length} \times \text{width}$
$= dr \cdot r\,d\theta$
$= r\,dr \cdot d\theta$

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Evaluate

$$\iiint z\,dV,$$

Where E is the solid enclosed by the paraboloid $z=x^2+y^2$ and the plane $z=4$.

Sol^n since $z=x^2+y^2$ and $z=4$;

we have $x^2+y^2=4$

$\Rightarrow r^2=4$ [As $x=r\cos\theta$, $y=r\sin\theta$

$\Rightarrow r=2$

Hence,

$$\iiint_E z\,dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=r^2}^{4} z\,r\,dz\,dr\,d\theta$$

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\left[\frac{rz^2}{2}\right]_{z=r^2}^{4} dr\,d\theta$$

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\left(8r-\frac{r^5}{2}\right)dr\,d\theta$$

$$=\left[\int_0^{2\pi} 1\,d\theta\right]\left[\int_0^2\left(8r-\frac{r^5}{2}\right)dr\right]$$

$$=2\pi\left[4r^2-\frac{r^6}{12}\right]_0^2$$

$$=2\pi\left[16-\frac{64}{12}\right]$$

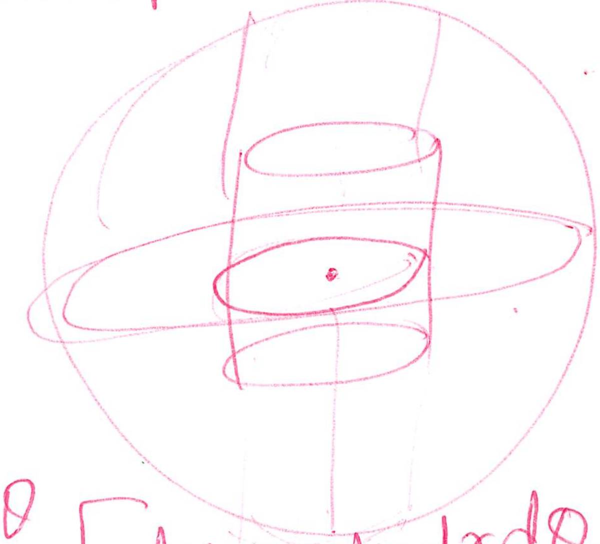
$$=2\pi\left[16-\frac{16}{3}\right]=\frac{64\pi}{3}$$ (Ans)

#22) Find the volume of the solid that lies within both the cylinder $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$.

Sol^n

Volume

$$= \iiint_E dV$$



$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

[$dV = r\,dz\,dr\,d\theta$. We have $x^2+y^2+z^2=4$

$\Rightarrow r^2+z^2=4$

$\Rightarrow z^2=4-r^2$

$\Rightarrow z=\pm\sqrt{4-r^2}$]

$$= \int_0^{2\pi}\int_{r=0}^{1}\Big[rz\Big]_{z=-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\Big[r\cdot\sqrt{4-r^2}-\left(-r\sqrt{4-r^2}\right)\Big]dr\,d\theta$$

$$= \left[\int_{\theta=0}^{2\pi} d\theta\right]\int_{r=0}^{1} 2r\sqrt{4-r^2}\,dr$$

$$= -2\pi\int_{r=0}^{1}(\underbrace{4-r^2}_{x})^{\frac{1}{2}}\,d(\underbrace{4-r^2}_{x})$$

[as $d(4-r^2) = -2r\,dr$]

$$= -2\pi \left[\frac{(4-r^2)^{\frac{3}{2}}}{\frac{3}{2}} \right]_{r=0}^{1}$$

$$= -\frac{4\pi}{3} \left[(4-r^2)^{\frac{3}{2}} \right]_0^1$$

$$= -\frac{4\pi}{3} \left[3^{\frac{3}{2}} - 4^{3/2} \right]$$

$4 = 2^2$

$$= -\frac{4\pi}{3} \left[3\sqrt{3} - 2^3 \right]$$

$$= \frac{4\pi}{3} \left[8 - 3\sqrt{3} \right] \quad \text{Ans.}$$

23) Find the volume of the solid that is enclosed by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=2$.

Sol^n volume

$$= \iiint_E dV$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=r}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left[rz \right]_{r}^{\sqrt{2-r^2}} dr\,d\theta$$

We have

$x^2+y^2+z^2=2$

$\Rightarrow x^2+y^2+x^2+y^2=2$

$\Rightarrow x^2+y^2=1$

$0 \le r \le 1,$

$\to 0 \le \theta \le 2\pi$

$z^2 = 2 - x^2 - y^2$

$= 2 - r^2$

$$= \int_0^{2\pi} \int_{r=0}^{1} \left[r\sqrt{2-r^2} - r^2 \right] dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_{r=0}^{1} \left[r\sqrt{2-r^2} - r^2 \right] dr$$

$$= 2\pi \left[\int_{r=0}^{1} r\sqrt{2-r^2}\, dr - \int_{r=0}^{1} r^2 dr \right]$$

$$= 2\pi \left[\left(-\frac{1}{2}\right) \int_{r=0}^{1} (2-r^2)^{\frac{1}{2}}\, d(2-r^2) - \int_{r=0}^{1} r^2 dr \right]$$

Since $d(2-r^2) = -2r\,dr$

$$= 2\pi \left[\left(-\frac{1}{2}\right) \cdot \left[\frac{(2-r^2)^{\frac{3}{2}}}{\frac{3}{2}} \right]_{r=0}^{1} - \left[\frac{r^3}{3} \right]_0^1 \right]$$

$$= 2\pi \left[\left(-\frac{1}{2}\right) \cdot \frac{2}{3} \left[1 - 2^{\frac{3}{2}} \right] - \frac{1}{3} \right]$$

$$= 2\pi \left[-\frac{1}{3}\left(1 - 2\sqrt{2}\right) - \frac{1}{3} \right]$$

$$= \cancel{2\pi \left[-\frac{1}{3} + \frac{2}{3}\sqrt{\ } \right]}$$

$$= \frac{2\pi}{3} \left[-1 + 2\sqrt{2} - 1 \right]$$

$$= \frac{2\pi}{3} \left[2\sqrt{2} - 2 \right]$$

$$= \frac{4\pi}{3} \left(\sqrt{2} - 1\right)$$

Ans