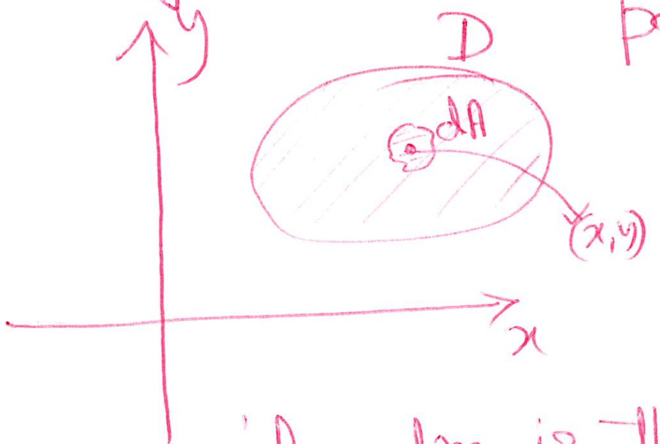
Section 12.4

~~In chapter 7, we~~

1) In this section, we consider a thin plate or lamina with variable density. Suppose the lamina occupies a region D of the xy-plane, and its density (in units of mass per unit area) at a point (x, y) in D be given by $\rho(x,y)$, where ρ is a continuous function of (x,y) in D. That means, if dm is the mass of the lamina in a very small area dA around the point (x,y), then we have



$$\rho(x,y) = \frac{dm}{dA}$$

i.e. $dm = \rho(x,y)\, dA$.

Thus, the mass m of the lamina is given by

$$m = \iint_D \rho(x,y)\, dA.$$

2)

2) Moments and center of mass

We know from chapter 7 that the moment of a particle about an axis is the product of the mass of the particle and its directed distance from the axis. Notice that ~~a~~ the area dA is very very small. So, we can consider the mass dm of dA works at the point (x,y).

Hence, its moment about the x-axis is $y\,dm = y\rho(x,y)\,dA$, and about the y-axis is

$$x\,dm = x\rho(x,y)\,dA.$$

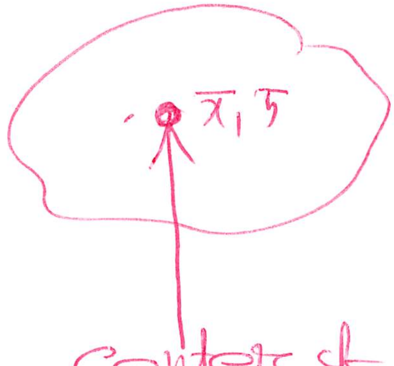
Thus, the moment of the entire lamina about the x-axis is given by

$$M_x = \iint_D y\,\rho(x,y)\,dA,$$

and the moment of the entire lamina about the ~~y-axis~~ x-axis given by

$$M_y = \iint_D x\rho(x,y)\,dA.$$

3)



center of mass.

Let $(\bar{x}, \bar{y})$ be the coordinates of the center of mass of the lamina, i.e. the mass m of the lamina works at the point $(\bar{x}, \bar{y})$. Then,

we know

$$m\bar{x} = \text{Moment of the lamina about } y\text{-axis}$$

$$= M_y$$

$$= \iint_D x \rho(x,y)\, dA$$

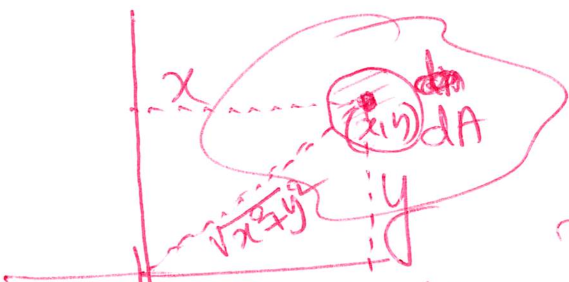
$$\Rightarrow \bar{x} = \frac{1}{m} \cdot \iint_D x\rho(x,y)\, dA = \frac{M_y}{m}.$$

Similarly, we have

$$\bar{y} = \frac{1}{m} \cdot \iint y\rho(x,y)\, dA = \frac{M_x}{M}.$$

4) Moment of ~~in~~ inertia

We know that the moment of inertia (also called the second moment) of a particle of mass m about an axis is defined by mr^2, where r is the distance from the particle to the axis.



Recall that dm is the mass of the very small area dA of the lamina about the point (x, y). So, we can assume that the mass dm works at the point (x, y). Its moment of inertia about the x-axis is given by $y^2 dm = y^2 \rho(x,y) dA$

Hence, the moment of the lamina about the x-axis is given by

$$I_x = \iint y^2 dm = \iint_D y^2 \rho(x,y) dA.$$

Similarly, the moment of inertia of the lamina about the y-axis is given by

$$I_y = \iint_D x^2 \rho(x,y)\, dA$$

Again, the moment of inertia about the origin of the mass dm is given by $dm.(x^2+y^2)$. Hence, the moment of inertia about the origin, also called the polar moment of inertia, of the lamina is given by

$$I_0 = \iint_D (x^2+y^2)\, dm$$

$$= \iint_D (x^2+y^2)\, \rho(x,y)\, dA$$

Notice that

$$I_0 = \iint_D x^2 \rho(x,y)\, dA + \iint_D y^2 \rho(x,y)\, dA$$

$$= I_y + I_x$$

i.e. $I_0 = I_x + I_y$.