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Section 11.7

Find three positive numbers whose sum is 100 and whose product is a maximum.

Sol$^n$ Let the three positive numbers be $x, y, z$.

Given $x+y+z=100$.

We need to find the maximum value of

$xyz = xy(100-x-y)$.

Let $f(x,y) = xy(100-x-y) = 100xy - x^2y - xy^2$

Here $f_x = 100y - 2xy - y^2$

and $f_y = 100x - x^2 - 2xy$.

Now $f_x = 0$ & $f_y = 0$

$\Rightarrow 100y - 2xy - y^2 = 0$ & $100x - x^2 - 2xy = 0$

$\cancel{\Rightarrow 100 - 2x - y = 0 \text{ \& } 100 - x - 2y = 0 \left[\text{as } x>0 \text{ and } y>0\right]}$

$\Rightarrow y(100-2x-y) = 0$ & $x(100-x-2y)=0$

$\Rightarrow 100 - 2x - y = 0$ & $100 - x - 2y = 0$ $\left[\begin{array}{l}\text{as } x>0 \\ \text{and } y>0\end{array}\right]$

$\cancel{\Rightarrow y = 2x - 100 \cdots (1)}$ & $x = 100 - 2y \cdots (2)$

$\Rightarrow y = 100 - 2x \cdots (1)$

By (1) & (2) we have

$y = 100 - 2(100-2y)$

$\Rightarrow y = 100 - 200 + 4y$

$\Rightarrow 3y = 100$

$\Rightarrow y = \frac{100}{3}$. Putting $y = \frac{100}{3}$ in (2) we have

$$x = 100 - 2 \cdot \frac{100}{3} = \frac{300 - 200}{3} = \frac{100}{3}.$$

Hence, $\left(\frac{100}{3}, \frac{100}{3}\right)$ is a critical point.

~~We need to~~ First, we need to check whether $\left(\frac{100}{3}, \frac{100}{3}\right)$ ~~is~~ gives a local minimum or a local maximum value of the function.

We have

$$f_{xx} = -2y, \quad f_{xy} = 100 - 2x - 2y$$

$$f_{yy} = -2x$$

We see that

$$D\left(\frac{100}{3}, \frac{100}{3}\right) = f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) f_{yy}\left(\frac{100}{3}, \frac{100}{3}\right) - \left[f_{xy}\left(\frac{100}{3}, \frac{100}{3}\right)\right]^2$$

$$= \left(-2 \cdot \frac{100}{3}\right)\left(-2 \cdot \frac{100}{3}\right) - \left[100 - \frac{200}{3} - \frac{200}{3}\right]^2$$

$$= \frac{40000}{9} - \left(\frac{300 - 400}{3}\right)^2$$

$$= \frac{40000}{9} - \frac{10000}{9}$$

$$= \frac{30000}{9} = \frac{10000}{3} > 0$$

& $f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) = -2 \cdot \frac{100}{3} = -\frac{200}{3} < 0$

Hence, we can say that the point $\left(\frac{100}{3}, \frac{100}{3}\right)$ gives a local maximum value of the function $f(x, y)$. Since there is no other local maximum value, we can say that the point $\left(\frac{100}{3}, \frac{100}{3}\right)$ gives the maximum value (the absolute maximum) of the function $f(x, y)$.

Hence, the thee positive numbers whose product is maximum ~~is gi~~ are given by

$$x = \frac{100}{3}, \quad y = \frac{100}{3}, \quad \& \quad z = 100 - x - y$$

$$= 100 - \frac{100}{3} - \frac{100}{3}$$

$$= \frac{100}{3}$$

(Ans)