Quantization dimension for fractals of overlapping construction

Károly Simon

Joint with Mrinal Kanti Roychowdhury

The term "quantization" originates from signal processing. In signal processing the quantization is a process of discretizing signals.

In mathematics Quantization: The best approximation of a Borel probability measure P on $\mathbb{R}^d$ by a discrete probability measure supported by n points $\{a_1 \ldots, a_n\} \subset \mathbb{R}^d$, for a given n.

(a) Information theory (signal compression): Shannon (1959), Gersho and Gray (1992)
(b) Cluster analysis (quantization of empirical measures), pattern recognition, speech recognition: Anderberg (1973), Bock (1974), Diday and Simon (1976), Tou and Gonzales (1974)
(c) Numerical integration: Pagès (1997)
(d) Stochastic processes (sampling design): Bucklew and Cambanis (1988), Benhenni and Cambanis (1996)
(e) Mathematical models in economics (optimal location of service centers): Bollobás $(1972, 1973)$

Wasserstein-Kantorovitch L_r-metric for probability measures μ, ν

$$\boxed{W_r\left(\mu,\nu\right)} = \inf\left\{\left(\int \|x-y\|^r d\mathfrak{m}(x,y)\right)^{1/r} : \mathfrak{m} \text{ probability on } \mathbb{R}^d \times \mathbb{R}^d \text{ with marginals } \mu \text{ and } \nu\right\}$$

In particular if $r = 1$ then we get the well known Kantorovich-Rubenstein metric:

$$\boxed{W_1(\mu,\nu)} := \left\{\int f(x)d\mu(x) - \int f(x)d\nu(x) \ : \ \mathrm{Lip}(f) \leqslant 1\right\}, \quad \text{where } 0 <$$

Given a probability measure μ on $\mathbb{R}^d$, a number $r \in (0, \infty)$ and a natural number n .

We define the `nth quantization error of order r for μ`

$$\boxed{e_{n,r}(\mu)} := \inf\left\{\left(\int d(x, \alpha)^r d\mu(x)\right)^{1/r} : \alpha \subset \mathbb{R}^d, 1 \leqslant \#(\alpha) \leqslant n\right\}.$$

Let $\mathcal{P}_n$ be the set of all discrete probabilities on $\mathbb{R}^d$ with cardinality at most n. Then we have

$$e_{n,r}(\mu) = \inf_{\nu \in \mathcal{P}_n} W_r(\mu, \nu). \tag{1}$$

Recall that W_r is the Wasserstein-Kantorovich L_r metric.

The *lower* and the upper quantization dimensions of μ of order r are defined by

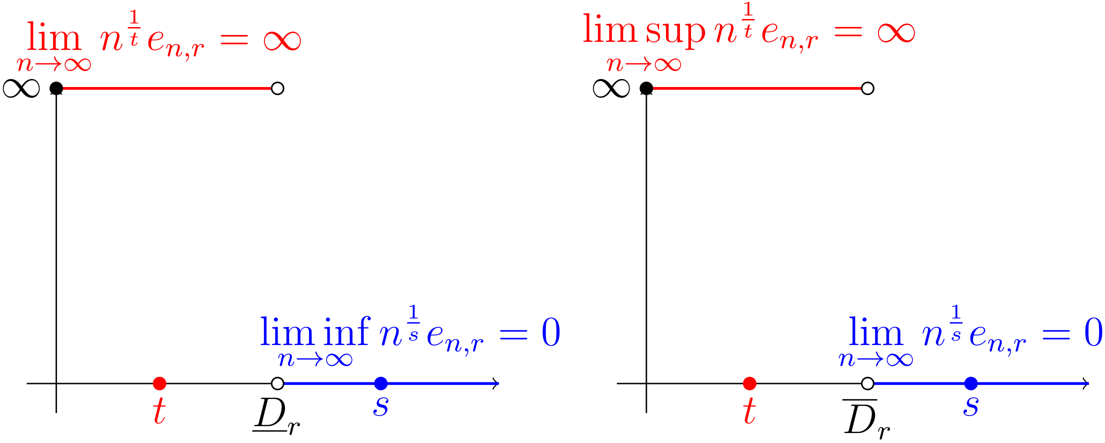
$$\underline{D}_r(\mu) := \liminf_{n\to\infty} \frac{\log n}{-\log e_{n,r}(\mu)}, \qquad \overline{D}_r(\mu) := \limsup_{n\to\infty} \frac{\log n}{-\log e_{n,r}(\mu)},$$

If $\underline{D}_r(\mu) = \bar{D}_r(\mu)$, the common value is denoted by $D_r(\mu)$. That is if the quantization domension $D_r(\mu)$ exists then

$$\log e_{n,r}(\mu) \sim \log \left(\frac{1}{n}\right)^{1/D_r(\mu)}.$$

Recall : $e_{n,r}(\mu) = \inf\left\{\left(\int d(x,\alpha)^r d\mu(x)\right)^{1/r} : \alpha \subset \mathbb{R}^d, 1 \leqslant \#(\alpha) \leqslant n\right\}.$

S. Graf and H. Luschgy [2, Proposition 11.3]



Recall : $\boxed{e_{n,r}(\mu)} = \inf\left\{\left(\int d(x,\alpha)^r d\mu(x)\right)^{1/r} : \alpha \subset \mathbb{R}^d, 1 \leqslant \#(\alpha) \leqslant n\right\}.$

Theorem 1.1 (Pozelberger & Graf-Luschgy)

Let μ be a probability measure of compact support $K \subset \mathbb{R}^d$ and let $1 \leqslant r \leqslant s < \infty$. Then

(2) $\overline{D}_r(\mu) \leqslant \overline{D}_s(\mu) \leqslant \overline{\dim}_{\mathrm{B}} K$ *and* $\underline{D}_r(\mu) \leqslant \underline{D}_s(\mu) \leqslant \underline{\dim}_{\mathrm{B}} K$

(3) $\dim_H^* \mu \leqslant \underline{D}_r(\mu)$ *and* $\dim_P^* \mu \leqslant \overline{D}_r(\mu)$.

If μ is an absolute continuous Borel measure on $\mathbb{R}^d$ then $D_r(\mu) = d$, for $0 < r < \infty$.

Self-similar Iterated Function Systems (SS-IFS)

A Self-Simlar Iterated Function System (SS-IFS) on $\mathbb{R}^d$ is a finite list $\mathcal{S} = \{S_1, \dots, S_m\}$ of strict contractions of $\mathbb{R}^d$ with contraction ratios $\lambda_1, \dots, \lambda_m$, $\lambda_i \in (0,1)$. The attractor of the IFS Λ is a unique non-empty compact set satisfying $\Lambda = \bigcup_{i=1}^{m} S_i(\Lambda)$.

The similarity dimension of $\mathcal{S}$ is the solution $s = s(\boldsymbol{\lambda})$ of the equation $\lambda_1^s + \cdots + \lambda_m^s = 1.$ Then we have $\dim_{\mathrm{H}} \Lambda \leqslant \min\{d, s\}$.

Self-similar measures I

As above, let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a SS-IFS with contraction ratios $\lambda_1, \dots, \lambda_m \in (0,1)$. Let $\mathbf{p} = (p_1, \dots, p_m) \in (0,1)^m$ be a probability vector. Then there exists a unique Borel probability measure $\nu = \nu_{\mathcal{S},\mathbf{p}}$ satisfying

(4) $$\nu_{\mathcal{S},\mathbf{p}}(H) = \nu(H) = \sum_{i=1}^{m} p_i \nu(S_i^{-1} H), \quad \forall H \subset \mathbb{R}^d, \text{ Borel.}$$

This is the self-similar measure correspoinding to the probability vector $\mathbf{p}$ (and also to the SS-IFS $\mathcal{S}$.)

Self-similar measures II

Let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a SS-IFS with contraction ratios $\lambda_1, \dots, \lambda_m \in (0, 1)$. Let $\mathbf{p} = (p_1, \dots, p_m) \in (0, 1)^m$ be a probability vector. The similarity dimension of the corresponding self-similar measure $\nu = \nu_{\mathcal{S},\mathbf{p}}$ is

$$\dim_S \nu = \frac{\sum_{i=1}^{m} p_i \log p_i}{\sum_{i=1}^{m} p_i \log \lambda_i}.$$

Recall : $\nu(H) = \sum_{i=1}^{m} p_i \nu(S_i^{-1} H)$

Self-similar measures III

Let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a SS-IFS with contraction ratios $\lambda_1, \dots, \lambda_m \in (0,1)$ and let s be the similarity dimension. That is $\sum_{i=1}^{m} \lambda_i^s = 1$. Consider the

$$\mathbf{p} = (\lambda_1^s, \dots, \lambda_m^s).$$

The self-similar measure that corresponds to this probability vector is called natural measure.

Open Set Condition (OSC)

The so-called Open Set Condition (OSC) if the cylinders $\{\Lambda_i\}_{i=1}^m$ are well separated. More precisely, the OSC holds if there there exists an open set $V \subset \mathbb{R}^d$ such that

(a) $S_i(V) \subset V$ for all i and

(b) $S_i(V) \cap S_j(V) = \emptyset$ for all $i \neq j$.

If the OSC holds then the dimension of the attractor and any self-similar measures are equal to their similarity dimension.

Theorem (Graf, Luschgy, 2001) Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a SS-IFS on $\mathbb{R}^d$ satisfying the OSC with contraction ratios $\lambda_1, \ldots, \lambda_m \in (0,1)$. Let $\mathbf{p} = (p_1, \ldots, p_m) \in (0,1)^m$ be a probability vector. Let $\nu = \nu_{\mathcal{S},\mathbf{p}}$ be the corresponding self-similar measure. For every $r \in (0, \infty)$ the quantization dimension $D_r = D_r(\nu)$ exists and satisfies

(5) $$\boxed{\sum_{i=1}^{m} \left(p_i \lambda_i^r\right)^{\frac{D_r}{r+D_r}} = 1}.$$

If ν_{nat} is the natural measure (the invariant measure corresponding to $\mathbf{p}_{\text{nat}} = (\lambda_1^s, \ldots, \lambda_m^s)$) then $D_r(\nu_{\text{nat}}) = s$ for all $r > 0$, where s is the similarity dimension. $Recall:$ $\nu(H) = \sum_{i=1}^{m} p_i \nu(S_i^{-1} H), \quad \forall H \subset \mathbb{R}^d,$ Borel.

Until now whatever I have said it was on $\mathbb{R}^d$ from now on we confine ourself to $\mathbb{R}$. A self similar IFS on $\mathbb{R}$ is of the form

$$\mathcal{S} = \{S_i(x) = \lambda_i x + t_i\}_{i=1}^m, \quad \lambda_i \in (-1,1) \setminus \{0\} \tag{6}$$

We have seen that OSC implies that the Hausdorff dimension of a self-similar measure $\nu_{\mathcal{S},\mathbf{p}}$ corresponding to the probability vector $\mathbf{p} = (p_1, \ldots, p_m)$ is equal to the similarity dimension of $\nu_{\mathcal{S},\mathbf{p}}$:

$$\dim_S \nu_{\mathcal{S},\mathbf{p}} = \frac{\sum_{i=1}^m p_i \log p_i}{\sum_{i=1}^m p_i \log |\lambda_i|}.$$

$Recall:$ $\nu(H) = \sum_{i=1}^m p_i \nu(S_i^{-1} H).$

We know that $\dim_{\mathrm{H}} \nu_{\mathcal{S},\mathbf{p}} \leqslant \min\{1, \dim_{\mathrm{S}} \nu_{\mathcal{S},\mathbf{p}}\}$ always holds. It follows from a Theorem of Hochman that "typically" we have equality. For

$$\mathcal{S}_{\boldsymbol{\lambda}}^{\mathbf{t}} = \{S_i(x) = \lambda_i \cdot x + t_i\}_{i=1}^{m}$$

we fix the contraction ratios $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \in (0,1)^m$ and consider the translations $\mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{R}^m$ as parameters. Then for every fixed $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \in ((-1,1)\backslash\{0\})^m$, the corresponding translation family is $\{\mathcal{S}_{\boldsymbol{\lambda}}^{\mathbf{t}}\}_{\mathbf{t}\in\mathbb{R}^m}$. Hochman Theorem says that for all but a set of at most $m-1$ packing dimension of $\mathbf{t} \in \mathbb{R}^m$, $\forall \mathbf{p}$ there in an equality in the yellow formula. In particular, for all $\boldsymbol{\lambda} \in ((-1,1)\backslash\{0\})^m$, for Lebesgue a.e. $\mathbf{t} \in \mathbb{R}^m$ we have equality in the yellow formula.

Now we recall that Theorem 1.1 impilies that

$$\min\{1, \dim_{\mathrm{H}} \nu_{\mathcal{S},\mathbf{p}}\} \leqslant \underline{D}_r(\nu_{\mathcal{S},\mathbf{p}}) \leqslant \overline{D}_r(\nu_{\mathcal{S},\mathbf{p}}) \leqslant \min\{1, s\}, \quad \sum_{i=1}^{m} |\lambda_i|^s = 1.$$

Putting together this with the Hochman Theorem mentioned on the previous slide: we get that $\forall\, \boldsymbol{\lambda} \in ((-1,1)\backslash\{0\})^m$, for all but a set of packing dimension at most $m-1$ set of $\mathbf{t}$ and finally for all probability vector $\mathbf{p} = (p_1, \ldots, p_m)$,

$$\text{(7)} \quad \min\left\{1, \frac{\sum_{i=1}^{m} p_i \log p_i}{\sum_{i=1}^{m} p_i \log|\lambda_i|}\right\} \leqslant \underline{D}_r(\nu_{\mathcal{S}_{\lambda}^{\mathbf{t}},\mathbf{p}}) \leqslant \overline{D}_r(\nu_{\mathcal{S}_{\lambda}^{\mathbf{t}},\mathbf{p}}) \leqslant \min\{1, s\}.$$

The case of the natural measure

In the previous formula, which was:

$$\min\left\{1, \frac{\sum_{i=1}^{m} p_i \log p_i}{\sum_{i=1}^{m} p_i \log|\lambda_i|}\right\} \leqslant \underline{D}_r(\nu_{\mathcal{S}_{\lambda}^{\mathbf{t}},\mathbf{p}}) \leqslant \overline{D}_r(\nu_{\mathcal{S}_{\lambda}^{\mathbf{t}},\mathbf{p}}) \leqslant \min\{1, s\}.$$

all inequalities are equalities if the measure is the natural self-similar measure that is $\mathbf{p}_{\text{nat}} = (|\lambda_1|^s, \dots, |\lambda_m|^s)$. So, $\forall\ \boldsymbol{\lambda} \in ((-1,1)\backslash\{0\})^m$, for all but a set of packing dimension at most $m-1$ set of $\mathbf{t} \in \mathbb{R}^m$, for $\mathbf{p}_{\text{nat}} = (|\lambda_1|^s, \dots, |\lambda_m|^s)$, the quantization dimension $D_r(\nu_{\mathcal{S}_{\lambda}^{\mathbf{t}},\mathbf{p}_{\text{nat}}})$ exists, and $\boxed{D_r(\nu_{\mathcal{S}_{\lambda}^{\mathbf{t}}\mathbf{p}_{\text{nat}}}) = \min\{1, s\}}$.

Recall : $\sum_{i=1}^{m} |\lambda_i|^s = 1$

S. Graf and H. Luschgy's result on slide 14 computes the quantization dimension $D_r(\nu)$ of a self-similar measure ν for a SS-IFS with slight overlapping between the cylinders. That is in the case when the OSC holds. In the overlapping case little is known. With Mrinal Kanti Roychowdhury (Univ of Texas Rio Grande Valley) we considered a special family self-similar fractals on the line with heavy overlaps and computed the quantization dimension for this special case in the hope that our method canbe generalized at least for SS-IFS on the line satisfying the so-called Weak Separation Property (WSP). For simplicity we define the WSP (Weak Separation Property) for homogeneous SS-IFS.

That is when all the contraction ratios are the same $\lambda \in (0,1)$.

$$\mathcal{S} = \{S_i(x) = \lambda \cdot x + t_i\}_{i=1}^{m}, \qquad x \in \mathbb{R}. \tag{8}$$

Let I be the interval spanned by the smallest and largest fixed points of the mappings S_i. We assume that I is not a point. Let $I_{\mathbf{i}} := S_{\mathbf{i}}(I)$, where $\mathbf{i} = (i_1, \dots, i_n) \in \{1, \dots, m\}^n$, $S_{\mathbf{i}} := S_{i_1} \circ \cdots \circ S_{i_n}$. We say that the WSP holds if there exists an $\varepsilon > 0$ such that

$$\forall n, \forall \mathbf{i}, \mathbf{j} \in \{1, \dots, m\}^n \ , \mathbf{i} \neq \mathbf{j} \ \text{ either } I_{\mathbf{i}} = I_{\mathbf{j}} \ \text{ or } \frac{|I_{\mathbf{i}} \cap I_{\mathbf{j}}|}{|I_{\mathbf{i}}|} < 1 - \varepsilon.$$

We remark that for a SS-IFS on the line given in the form

$$\boxed{\mathcal{S} = \{S_i(x) = \lambda \cdot x + t_i\}_{i=1}^{m}}, \qquad x \in \mathbb{R}.$$

the OSC holds if and only if

(9) $$\forall n, \forall \mathbf{i}, \mathbf{j} \in \{1, \dots, m\}^n, \mathbf{i} \neq \mathbf{j}, \qquad \frac{|I_{\mathbf{i}} \cap I_{\mathbf{j}}|}{|I_{\mathbf{i}}|} < 1 - \varepsilon.$$

So, the difference is that in the case of the WSP the total overlap between the cylinders is allowed. We consider the following self-similar IFS on $\mathbb{R}$

(10) $$\mathcal{S} = \left\{ S_i(x) = \frac{1}{3}x + i \right\}_{i \in \{0,1,3\}}.$$

$\mathcal{S} = \left\{ S_i(x) = \frac{1}{3}x + i \right\}_{i \in \{0,1,3\}}$. We write $\mathcal{A} := \{0, 1, 3\}$, and Σ (Σ^*) for the set of infinite (finite) words above the alphabet $\mathcal{A}$, respectively.

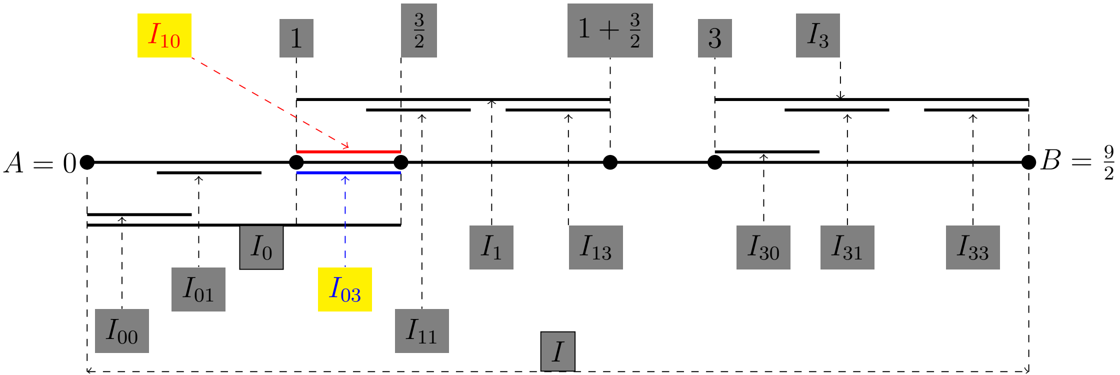


Figure: $I_{i_1 \dots i_n} := S_{i_1 \dots i_n}(I)$, $(i_1, \dots, i_n) \in \mathcal{A}^n$. $I_{10} = I_{03}$

$\mathcal{S} = \left\{ S_i(x) = \frac{1}{3}x + i \right\}_{i \in \{0,1,3\}}$. The orange ones are the level 1, the red ones are the level 2 and the blue ones are the level 3 cylinders.

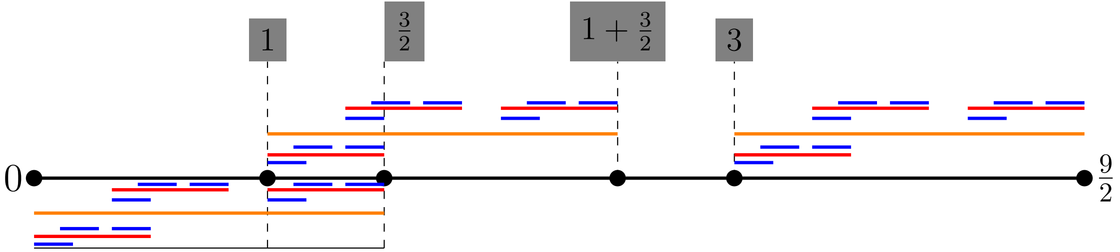


Figure: $I_{i_1 \dots i_n} := S_{i_1 \dots i_n}(I)$, $(i_1, \dots, i_n) \in \mathcal{A}^n$. $I_{10} = I_{03}$

$$A := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad B := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{11}$$

Given a probability vector $\mathbf{p} = (p_0, p_1, p_3)$. Let $\nu = \nu_{\mathcal{S},\mathbf{p}}$ be the corresponding self-similar measure: $\nu(H) = \sum_{i \in \mathcal{A}} p_i \nu(S_i^{-1} H)$, $\forall\ H \subset \mathbb{R}$.

$$\Sigma_A := \{\mathbf{i} \in \Sigma : (i_k, i_{k+1}) \neq (0,3)\}, \ \Sigma_B := \{\mathbf{i} \in \Sigma : (i_k, i_{k+1}) \neq (1,0)\}.$$

If $p_1 \geqslant p_3$ then we work on Σ_A, otherwise we work on Σ_B. From now on we always assume that $p_1 \geqslant p_3$.

Let

$$\boxed{\mathcal{T}_n := \{\mathbf{i} \in \mathcal{A}^n : (i_k, i_{k+1}) \neq (0,3), \forall k < n\}}, \qquad \boxed{\Sigma_A^* = \bigcup_{n=1}^{\infty} \mathcal{T}_n \cup \flat},$$

where $\flat$ is the empty word.

(a) For an $\mathbf{i} \in \mathcal{T}_n$ there can be exponentially many $\mathbf{j} \in \mathcal{A}^n$ with $I_{\mathbf{i}} \cap I_{\mathbf{j}} \neq \emptyset$.

(b) If $\mathbf{i} \in \mathcal{T}_n$ then there is at most one $\mathbf{j} \in \mathcal{T}_n \setminus \{\mathbf{i}\}$ such that $I_{\mathbf{i}} \cap I_{\mathbf{j}} \neq \emptyset$.

(12)

$$\boxed{\mathcal{I}_{\mathbf{i}} := \{\boldsymbol{\eta} \in \mathcal{A}^n : S_{\boldsymbol{\eta}} = S_{\mathbf{i}}\}} \quad \text{and} \quad \boxed{\psi(\mathbf{i}) := \sum_{\boldsymbol{\eta} \in \mathcal{I}_{\mathbf{i}}} p_{\boldsymbol{\eta}}}, \quad \text{for every } \mathbf{i} \in \Sigma_A^*,$$

and we define

(13) $$\psi(\flat) := 1, \text{ where } \flat \text{ is the empty word.}$$

We prove that the limit in the following definition exists:

(14) $$\boxed{p(t) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{i} \in \mathcal{T}_n} \psi^t(\mathbf{i})}, \quad t \geqslant 0.$$

Moreover, we verify that the function $t \mapsto p(t)$, is continuous and strictly decreasing. In this way the following function

(15) $$t \mapsto \widetilde{p}(t) := p(t) - rt \log 3$$

has a unique zero which we call t_0 . That is

(16) $$\widetilde{p}(t_0) = 0.$$

Then we define χ_r such that

(17) $$\boxed{t_0 = \frac{\chi_r}{r+\chi_r}} \quad \text{that is} \quad \boxed{\chi_r = \frac{t_0 r}{1-t_0}}.$$

The new result

Recall: $\mathcal{A} = \{0,1,3\}$, $\mathcal{T}_n := \{\mathbf{i} \in \mathcal{A}^n : (i_k, i_{k+1}) \neq (0,3), \forall k < n\}$.

For every $\mathbf{i} \in \Sigma_A^*$, $\mathcal{I}_{\mathbf{i}} := \{\boldsymbol{\eta} \in \mathcal{A}^n : S_{\boldsymbol{\eta}} = S_{\mathbf{i}}\}$ and $\psi(\mathbf{i}) := \sum_{\boldsymbol{\eta} \in \mathcal{I}_{\mathbf{i}}} p_{\boldsymbol{\eta}}$.

Let $p(t) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{i} \in \mathcal{T}_n} \psi^t(\mathbf{i})$, t_0 is defined such that $p(t_0) = rt_0 \log 3$,

$\chi_r := \frac{t_0 r}{1 - t_0}$.

Theorem 2.1

The quantization dimension of the measure $\nu_{\mathcal{S},\mathbf{p}}$ exists and

$$D_r(\nu_{\mathcal{S},\mathbf{p}}) = \chi_r.$$

Definition 2.2

We say that a function $\phi : \Sigma_A^* \to [0, \infty)$ is a `weak quasi-multiplicative potential` on Σ_A if the following three conditions hold:

(a) There is an $\boldsymbol{\ell} \in \Sigma_A^*$ which is not the empty word such that $\phi(\boldsymbol{\ell}) > 0$.

(b) $\exists\ C_1 > 0$ such that $\phi(\mathbf{ij}) \leqslant C_1 \phi(\mathbf{i})\phi(\mathbf{j})$, $\forall \mathbf{ij} \in \Sigma_A^*$.

(c) $\exists\ z \in \mathbb{N}$, $C_2 > 0$ such that $\forall \mathbf{i}, \mathbf{j} \in \Sigma_A^*$, $\exists \mathbf{k} \in \bigcup_{\ell=1}^{z} \mathcal{T}_\ell \cup \flat$ such that $\mathbf{ikj} \in \Sigma_A^*$ and $\phi(\mathbf{i})\phi(\mathbf{j}) \leqslant C_2 \phi(\mathbf{ikj})$.

Theorem 2.3 (Feng)

Let ϕ be a weak quasi-multiplicative potential on Σ_A^*. Then there exists a unique invariant ergodic measure $\mathfrak{m}$ on Σ_A with the following property

$$\mathfrak{m}(\mathbf{i}) \approx \frac{\phi(\mathbf{i})}{\sum_{\mathbf{j}\in\mathcal{T}_n}\phi(\mathbf{j})} \approx \phi(\mathbf{i})\exp\left(-nP(\phi)\right), \quad \mathbf{i}\in\Sigma_A^* \tag{18}$$

where $a(\mathbf{i}) \approx b(\mathbf{i})$ if there exists a $c>0$ such that $\frac{1}{c}b(\mathbf{i}) \leqslant a(\mathbf{i}) \leqslant cb(\mathbf{i})$. Moreover, the pressure $P(\phi)$ of ϕ is

$$P(\phi) := \lim_{n\to\infty} \log \sum_{\mathbf{i}\in\mathcal{T}_n} \phi(\mathbf{i}). \tag{19}$$

Recall: $\mathcal{A} = \{0,1,3\}$, $\boxed{\mathcal{T}_n := \{\mathbf{i} \in \mathcal{A}^n : (i_k, i_{k+1}) \neq (0,3), \forall k < n\}}$.

For every $\mathbf{i} \in \Sigma_A^*$, $\boxed{\mathcal{I}_{\mathbf{i}} := \{\boldsymbol{\eta} \in \mathcal{A}^n : S_{\boldsymbol{\eta}} = S_{\mathbf{i}}\}}$ and $\boxed{\psi(\mathbf{i}) := \sum_{\boldsymbol{\eta} \in \mathcal{I}_{\mathbf{i}}} p_{\boldsymbol{\eta}}}$.

$\psi : \Sigma_A^* \to [0,\infty)$ is NOT quasi-multiplicative. Let

$$(20) \qquad \boxed{\widehat{\psi}(\mathbf{i})} := \begin{cases} \max\{\psi(\mathbf{i}), \psi(\mathbf{i}^- 0)\}, & \text{if } \mathbf{i}_{|\mathbf{i}|} = 1; \\ \psi(\mathbf{i}), & \text{if } \mathbf{i}_{|\mathbf{i}|} \neq 1, \end{cases} \qquad \text{for } \mathbf{i} \in \Sigma_A^*.$$

(a) $\widehat{\psi}$ is weak quasi-multiplacative .

(b) The potentials ψ and $\widehat{\psi}$ have the same pressure function. Namely,

$$(21) \qquad 1 \leqslant \frac{\widehat{\psi}(\mathbf{i})}{\psi(\mathbf{i})} \leqslant n, \qquad \forall \mathbf{i} \in \mathcal{T}_n.$$

$$p(t)=\lim_{n\to\infty}\frac{1}{n}\log\sum_{\mathbf{i}\in\mathcal{T}_n}(\psi(\mathbf{i}))^t=\lim_{n\to\infty}\frac{1}{n}\log\sum_{\mathbf{i}\in\mathcal{T}_n}\left(\widehat{\psi}(\mathbf{i})\right)^t. \tag{22}$$

$$\widehat{\phi}_t(\mathbf{i}):=\left(\widehat{\psi}(\mathbf{i})\cdot 3^{-|\mathbf{i}|r}\right)^t. \tag{23}$$

$$P(t):=\lim_{n\to\infty}\frac{1}{n}\log\sum_{\mathbf{j}\in\mathcal{T}_n}\widehat{\phi}_t(\mathbf{j})=p(t)-rt\log 3. \tag{24}$$

$P(t)$ strictly decreasing $P(0)=p(0)=0.876036$,

$P(1)=0-r\log 3<0$, there is a unique zero t_0. That is $P(t_0)=0$.

Let $\phi := \widehat{\phi}_{t_0}$. Then ϕ is a weak quasi-multiplicative potential whose pressure is zero. So, by Feng Theorem we get

Proposition 2.4

There is a $C_4 > 1$ and a unique invariant ergodic measure $\mathfrak{m}$ on Σ_A such that

$$C_4^{-1} < \frac{\mathfrak{m}([\mathbf{i}])}{\phi(\mathbf{i})} < C_4, \qquad \text{for all} \quad \mathbf{i} \in \Sigma_A^*. \tag{25}$$

Using this measure $\mathfrak{m}$ and the standard techniques of the theory of quantization dimension we get that Theorem 2.1 holds.

Some further references are given below.

[3] [4] [1] [5]

References

[1] S. Graf and H. Luschgy.
Asymptotics of the quantization errors for self-similar probabilities.
Real Analysis Exchange, pages 795–810, 2000.

[2] S. Graf and H. Luschgy.
Foundations of quantization for probability distributions.
FOUNDATIONS OF QUANTIZATION FOR PROBABILITY DISTRIBUTIONS, 1730:1–+, 2000.

[3] M. Kesseböhmer and S. Zhu.
Some recent developments in quantization of fractal measures.
Fractal Geometry and Stochastics V, pages 105–120, 2015.

[4] L. Lindsay and R. Mauldin.
Quantization dimension for conformal iterated function systems.
Nonlinearity, 15(1):189–199, 2002.

[5] M. K. Roychowdhury.
Quantization dimension and temperature function for bi-lipschitz mappings.
Israel Journal of Mathematics, 192(1):473–488, 2012.

Thank you for your attention!