MECE 3320 – Measurements & Instrumentation

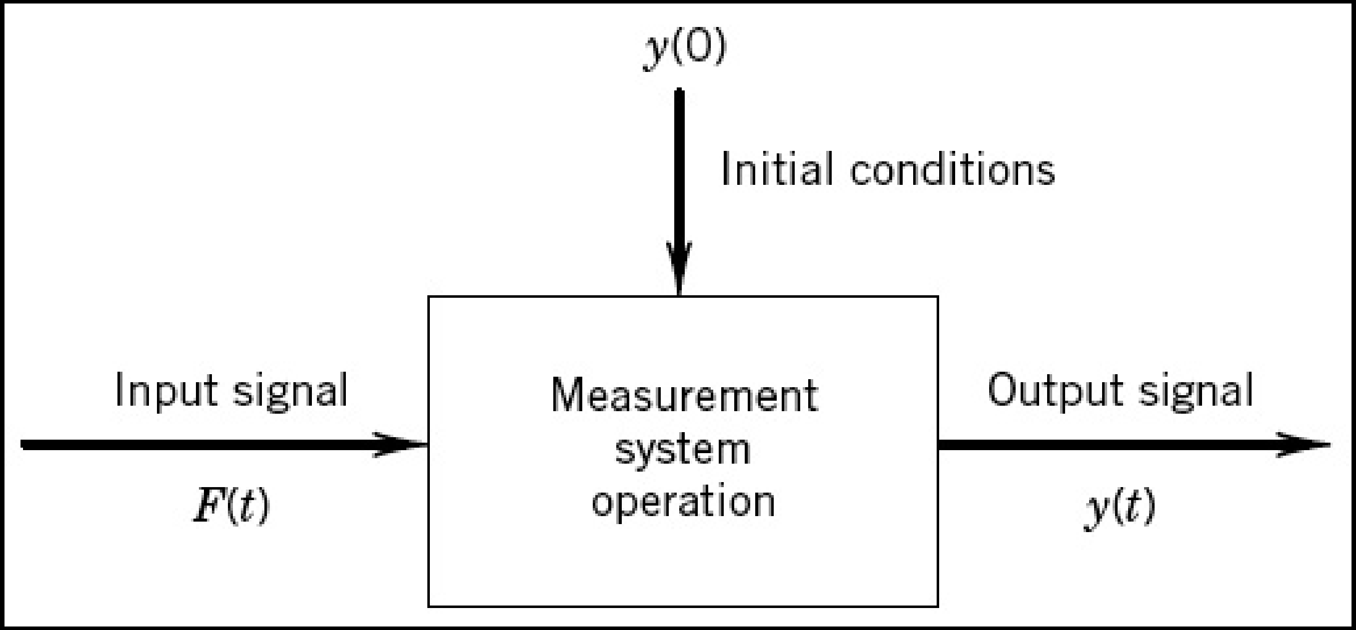
Measurement System Behavior

Dr. Isaac Choutapalli

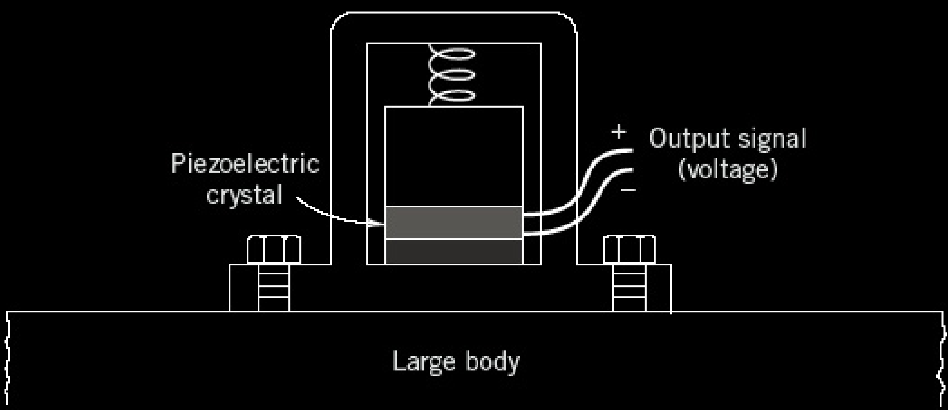
Department of Mechanical Engineering

University of Texas – Pan American

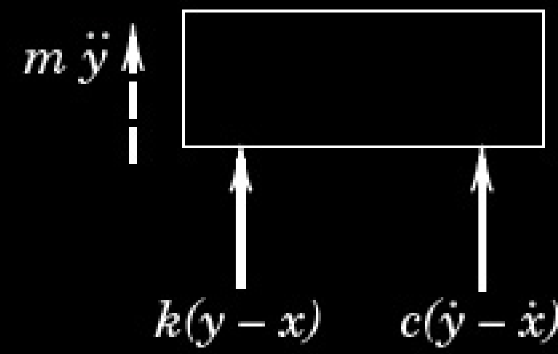
Measurement System Behavior+



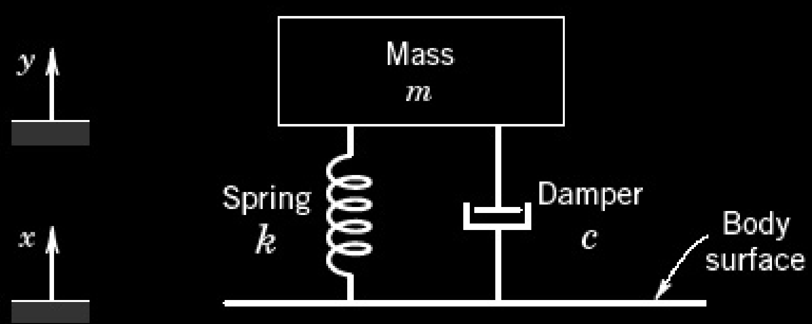
Response of a Seismic Accelerometer



(*a*) Piezoelectric accelerometer attached to large body



(*c*) Free-body diagram



(*b*) Representation using mass, spring, and damper

The response can be modeled by the following differential equation:

$$m\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky=c\frac{dx}{dt}+kx$$

Most measurement systems can be modeled as:

- ❖ Zero-Order
- ❖ First-Order
- ❖ Second Order

Zero-Order Systems

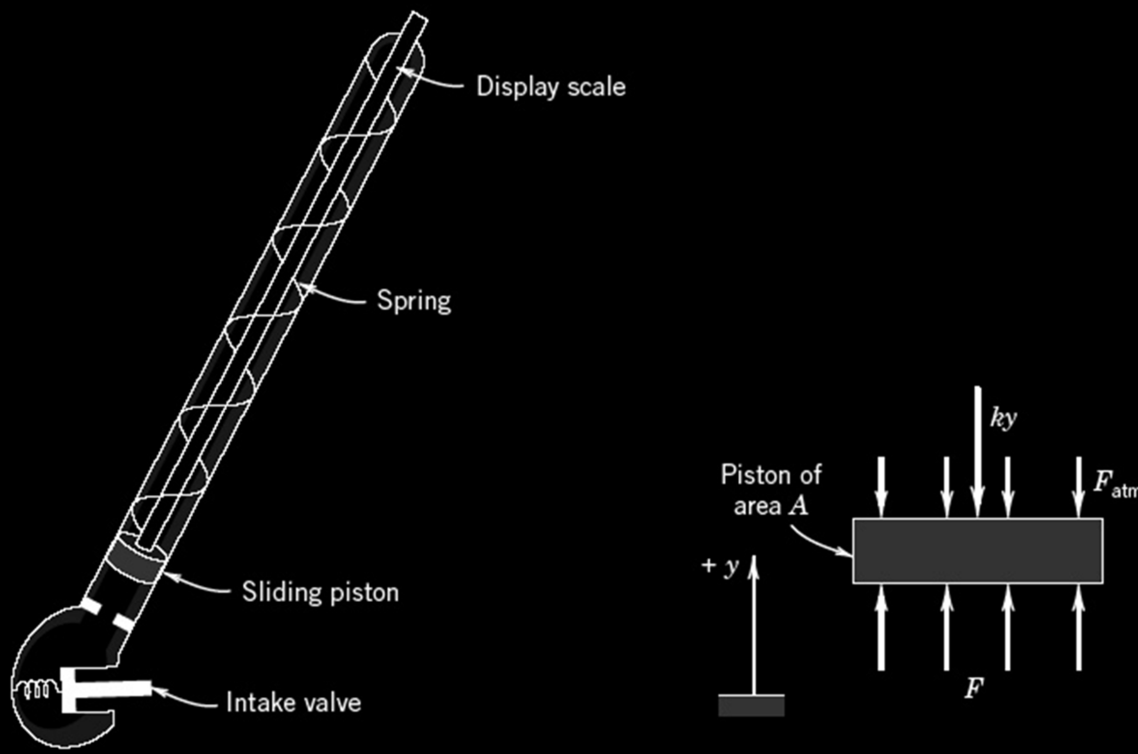
Zero-order system model is represented by:

$$a_0 y(t) = b_0 x(t) \rightarrow y(t) = Kx(t)$$

where K = static sensitivity = b_0/a_0

- The behavior is characterized by its static sensitivity, K and remains constant regardless of input frequency (ideal dynamic characteristic).
- In zero-order systems, the measurement system responds to an input instantly.
- It is useful for static inputs or static calibration.
- Dynamic signals can also be measured but only at equilibrium conditions.

Pencil-Type Pressure Gauge



(a) Pencil-style pressure gauge

(b) Free-body diagram

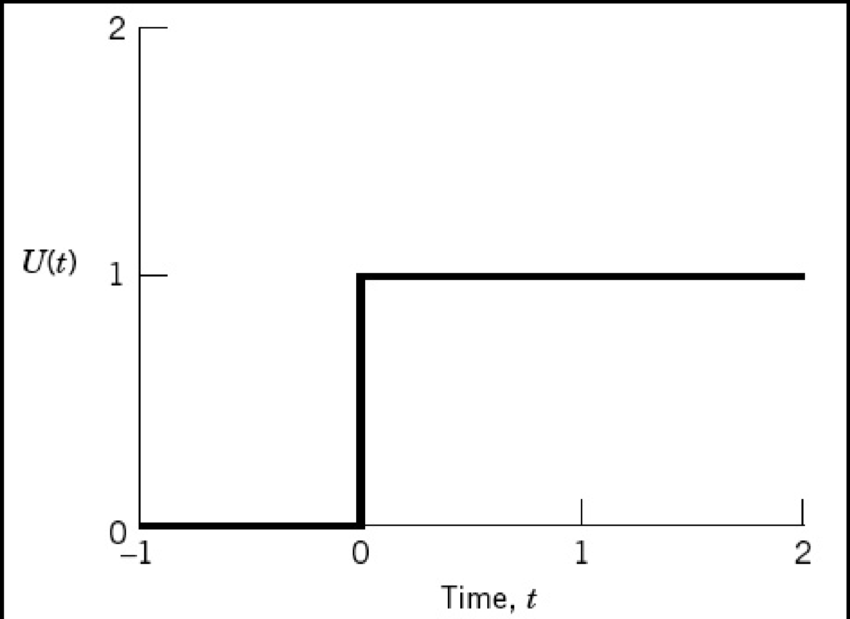
$$Ky = (p - p_{atm})A$$

Zero-order response equation is given by: $y = (A/K)(p - p_{atm})$

First-Order Systems: Step Input

- A first-order system is a measurement system that cannot respond to a change in input instantly.

$$\tau \frac{dy(t)}{dt} + y(t) = Kx(t)$$ ← *forcing function*



$$x(t) = AU(t) = \begin{cases} 0 & t \le 0 \\ A & t > 0 \end{cases}$$

$$\tau \frac{dy(t)}{dt} + y(t) = KAU(t)$$

$$y(t) = \underbrace{Ce^{-t/\tau}}_{y_{cf}} + \underbrace{KA}_{y_{pi}}$$

y_{cf}: Transient response

y_{pi}: Steady state response

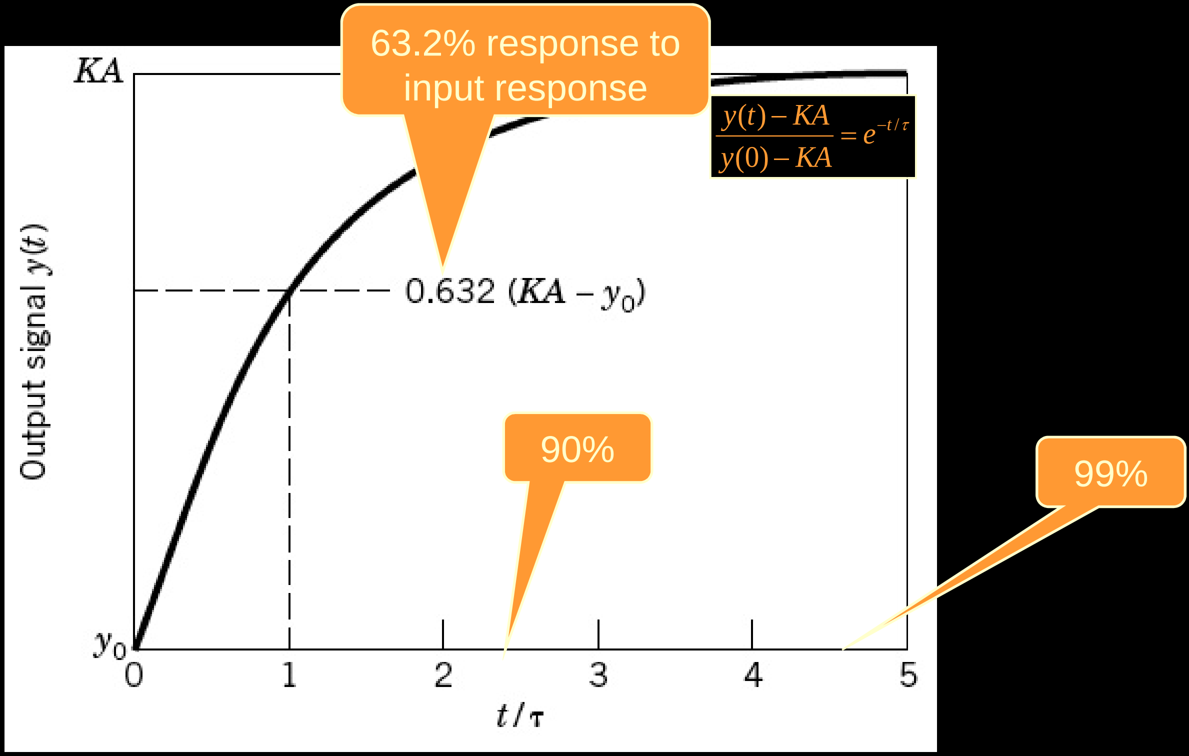
First-Order Systems: Step Response

Suppose we rewrite the equation as

Steady-state response

Error Fraction

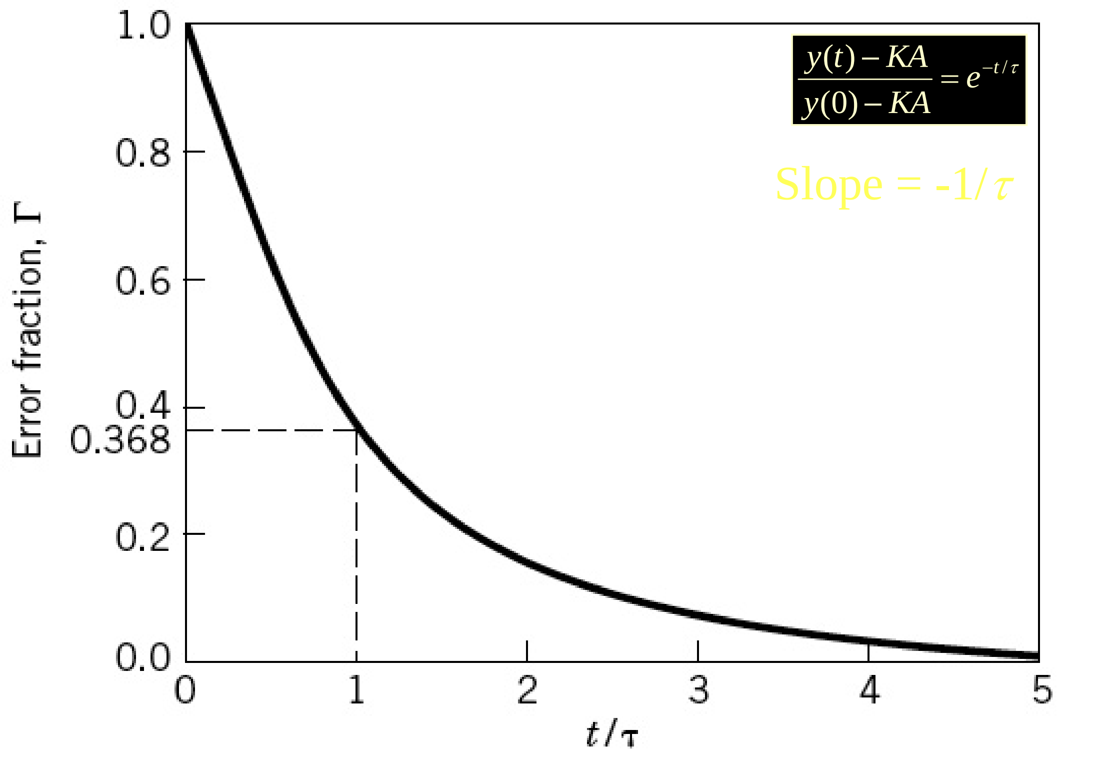
$$e_m(t) = \frac{y(t) - KA}{y_0 - KA} = \frac{y(t) - y(\infty)}{y(0) - y(\infty)} = e^{-t/\tau}$$



First-Order Systems: Step Response

$$e_m = \frac{y(t) - KA}{y(0) - KA} = e^{-t/\tau}$$

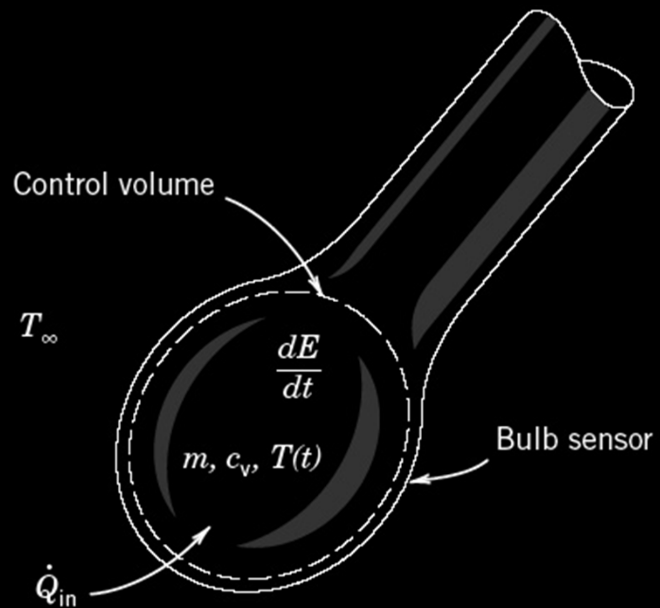
$$\ln e_m = 2.3 \log e_m = -\frac{t}{\tau}$$



τ Is called the time-constant – time it takes for the measurement system to respond to 63.2% of the input signal.

First-Order Systems: Step Response

Thermometer & Energy Balance:



Assumptions:

- ❖ Uniform temperature within the bulb (lumped analysis)
- ❖ Constant mass

Energy Balance:

$$\frac{dE}{dt} = \dot{Q}$$

$$mc_v \frac{dT(t)}{dt} = hA_s[T_\infty - T(t)]$$

Rewriting, we obtain

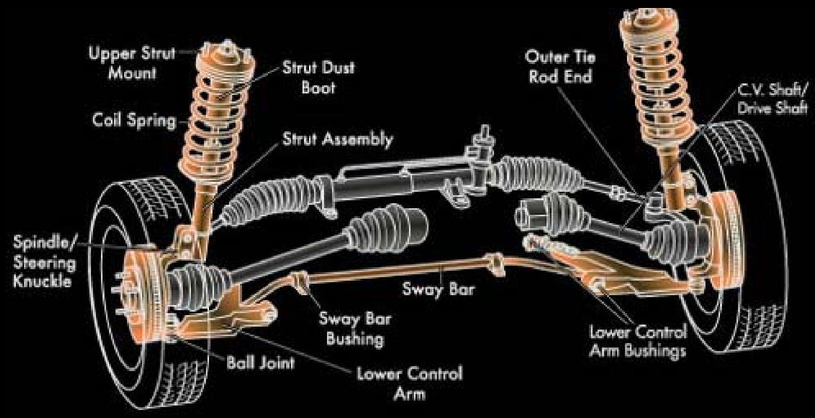
$$\frac{mc_v}{hA_s}\frac{dT(t)}{dt} + T(t) = T_\infty$$

Solving the above differential equation gives,

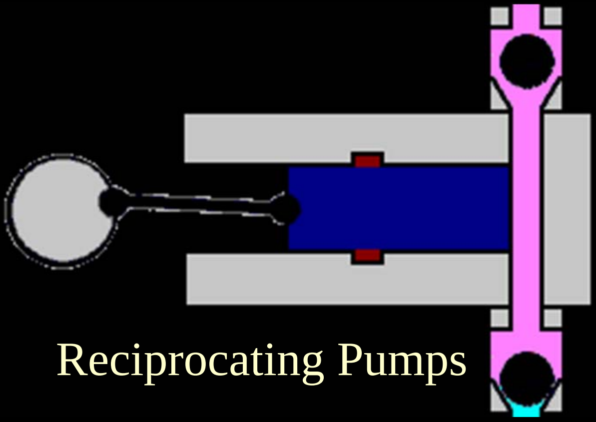
$$T(t) = T_\infty + [T(0) - T_\infty]e^{-t/\tau}$$

First-Order Systems: Frequency Response

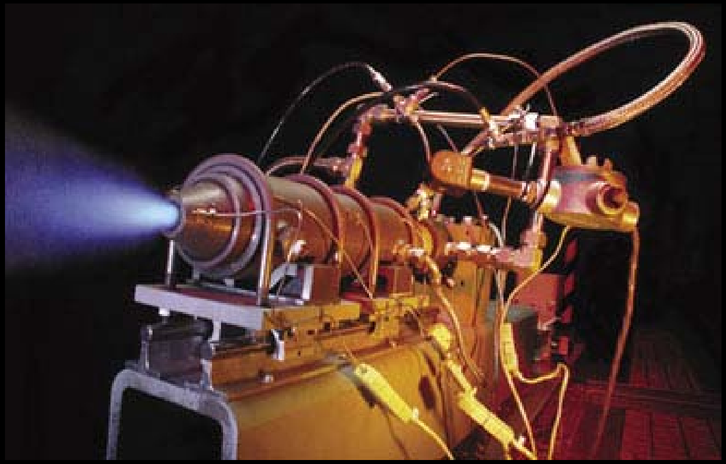
Periodic signals are encountered in many applications. Some examples are:



Vehicle Suspension System



Reciprocating Pumps



Pulsed Detonation Engines

First-Order Systems: Frequency Response

Consider a first-order measuring system to which an input represented by the following equation is applied.

$$x(t) = A \sin \omega t$$

$$\tau \frac{dy}{dt} + y = KA \sin \omega t$$

The complete solution:

$$y(t) = \underbrace{Ce^{-t/\tau}}_{\text{Transient response}} + \underbrace{\frac{KA}{\sqrt{1+(\omega\tau)^2}} \sin\left(\omega t - \tan^{-1} \omega\tau\right)}_{\text{Steady state response}}$$

Transient response

Steady state response = Frequency response

$$y(t) = Ce^{-t/\tau} + B(\omega) \sin[\omega t + \phi(\omega)]$$

Amplitude of steady state response

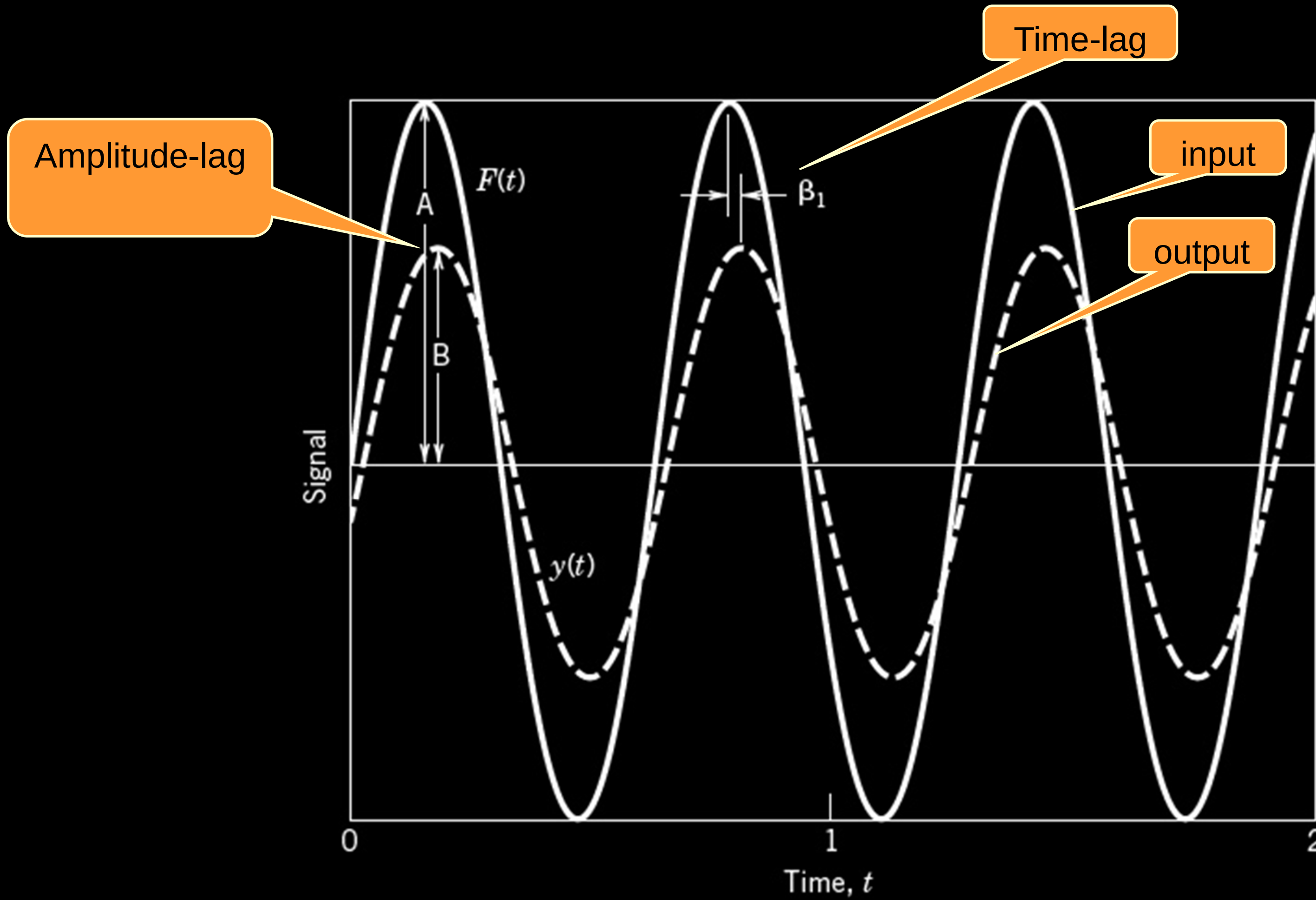
$$B(\omega) = \frac{KA}{\left[1+(\omega\tau)^2\right]^{1/2}}$$

Phase-Shift

$$\phi(\omega) = -\tan^{-1} \omega\tau$$

First-Order Systems: Frequency Response

The steady-state response of any system to which a periodic input of frequency, ω, is applied is known as the frequency response of that system.

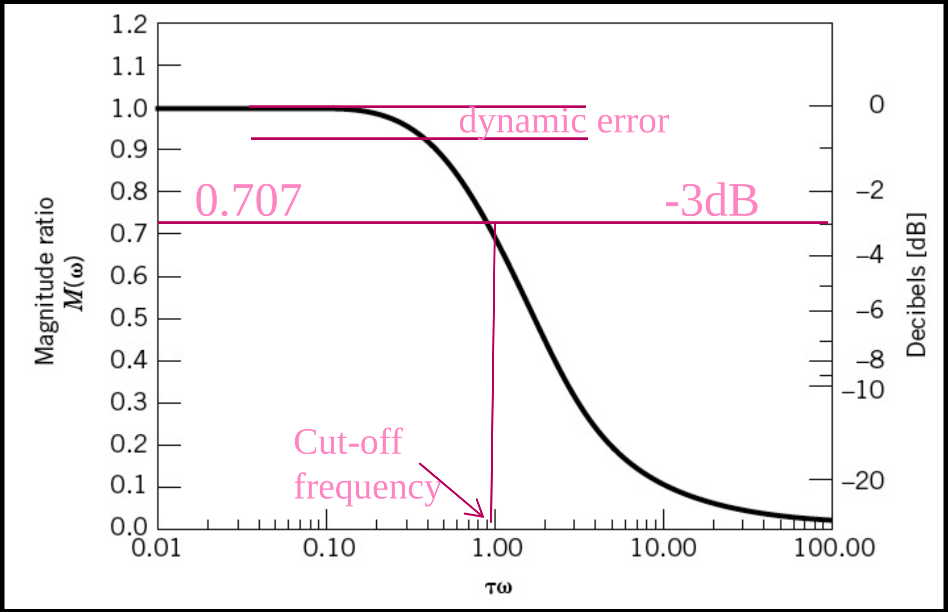


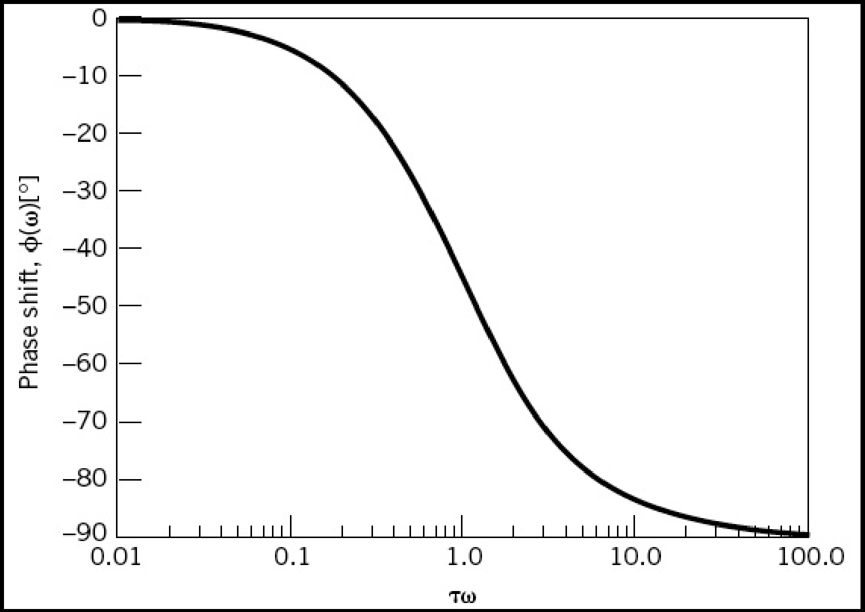
First-Order Systems: Frequency Response

The ratio of the output signal amplitude to the signal amplitude is called *magnitude ratio*.

$$M(\omega) = \frac{B}{KA} = \frac{1}{\left[1+(\omega\tau)^2\right]^{1/2}}$$

The phase angle is $\phi(\omega) = -\tan^{-1}(\omega\tau)$





Dynamic error, $\delta(\omega) = M(\omega) - 1$: a measure of an inability of a system to adequately reconstruct the amplitude of the input for a particular frequency.

First-Order Systems

A first order instrument is to measure signals with frequency content up to 100 Hz with an accuracy of 5%. What is the maximum allowable time constant? What will be the phase shift at 50 and 100 Hz?

$$\text{Dynamic error} = \left(\frac{1}{\sqrt{\omega^2\tau^2+1}} - 1 \right) \times 100\%$$

From the condition |Dynamic error| < 5%, it implies that $0.95 \le \frac{1}{\sqrt{\omega^2\tau^2+1}} \le 1.05$

But for the first order system, the term $1/\sqrt{\omega^2\tau^2+1}$ can not be greater than 1 so that the constrain becomes

$$0.95 \le \frac{1}{\sqrt{\omega^2\tau^2+1}} \le 1$$

Solve this inequality give the range $0 \le \omega\tau \le 0.33$

The largest allowable time constant for the input frequency 100 Hz is $\tau = \frac{0.33}{2\pi 100\ \text{Hz}} = 0.52\ \text{ms}$

The phase shift at 50 and 100 Hz can be found from $\phi = -\arctan \omega\tau$

This gives ϕ = -9.33^{o} and = -18.19^{o} at 50 and 100 Hz respectively

Second-Order Systems

Second order systems are modeled by second order differential equations.

$$a_2 \frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t)$$

$$\boxed{\frac{1}{\omega_n^2} \frac{d^2 y(t)}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dy(t)}{dt} + y(t) = Kx(t)}$$

where

$K = \dfrac{b_0}{a_0}$ = the static sensitivity

$\zeta = \dfrac{a_1}{2\sqrt{a_0 a_2}}$ = the damping ratio, dimensionless

$\omega_n = \sqrt{\dfrac{a_0}{a_2}}$ = the natural angular frequency

Second-Order Systems

The solution to the second order differential equation depends on the roots of the characteristic equation.

$$\frac{1}{\omega_n^2}D^2+\frac{2\zeta}{\omega_n}D+1=0$$

This quadratic equation has two roots:

$$S_{1,2}=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$$

Depending on the value of ζ, three forms of complementary solutions are possible

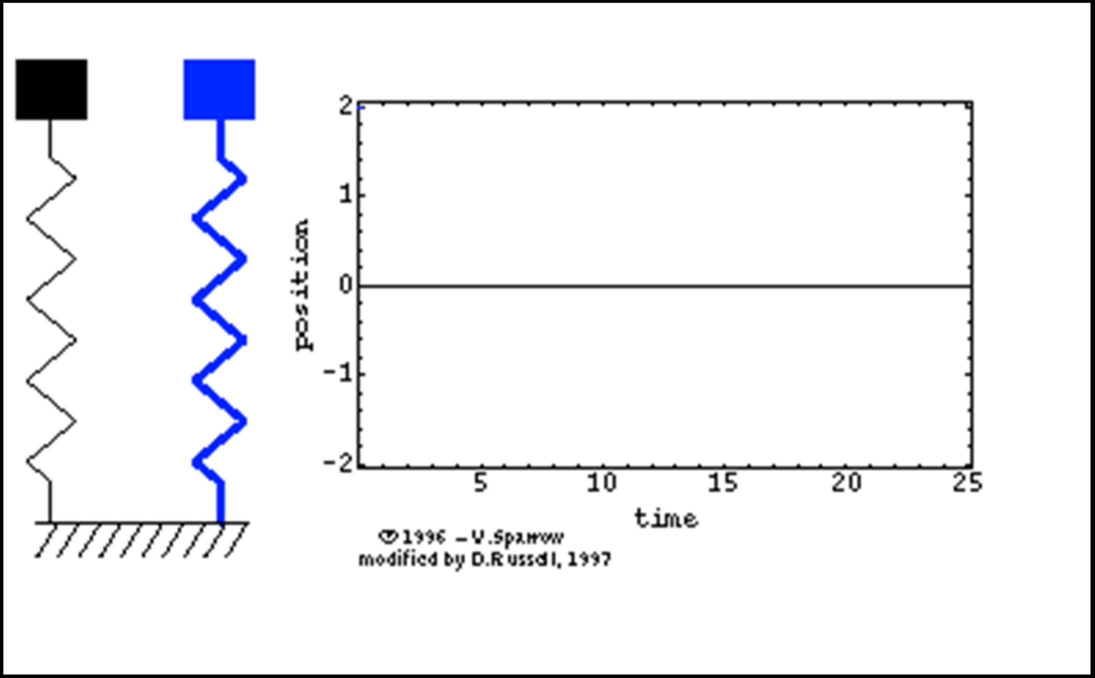
Overdamped ($\zeta > 1$): $y_{oc}(t)=C_1e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}+C_2e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$

Critically damped ($\zeta = 1$): $y_{oc}(t)=C_1e^{-\omega_n t}+C_2te^{-\omega_n t}$

Underdamped ($\zeta < 1$): $y_{oc}(t)=Ce^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\Phi\right)$

Second-Order Systems

Undamped & Underdamped Second Order-Systems



Animation courtesy of Dr. Dan Russell, Kettering University

Second-Order Systems: Step Input

For a step input $x(t)$ $$\frac{1}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y=KAU(t)$$

Initial conditions: $y = 0$ at $t = 0$, $dy/dt = 0$ at $t = 0$

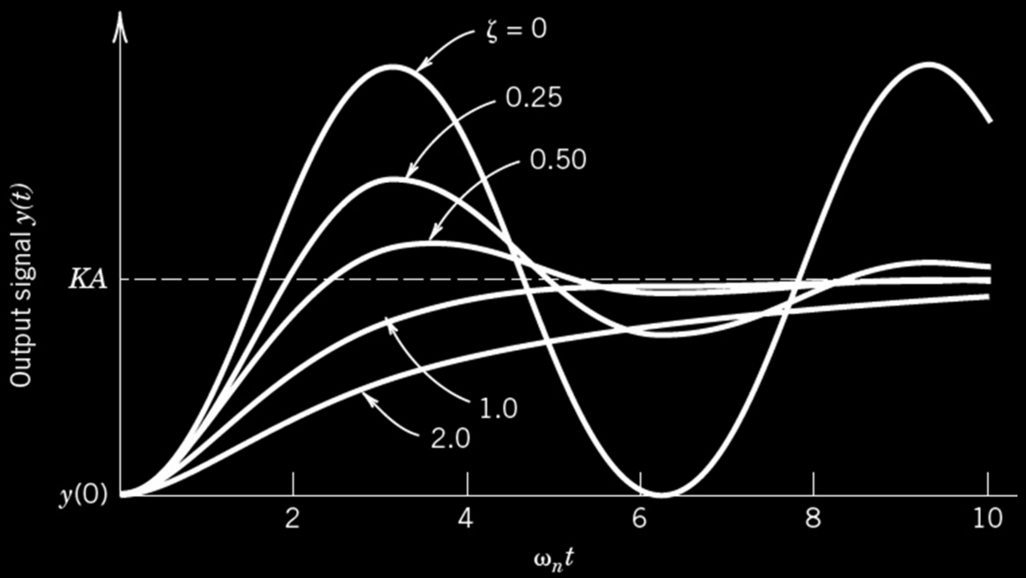
Solution:

Overdamped ($\zeta > 1$): $$\frac{y(t)}{KA}=-\frac{\zeta+\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}+\frac{\zeta-\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}+1$$

Critically damped ($\zeta = 1$): $$\frac{y(t)}{KA}=-(1+\omega_n t)e^{-\omega_n t}+1$$

Underdamped ($\zeta < 1$): $$\frac{y(t)}{KA}=-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t+\phi\right)+1$$

Second-Order Systems: Step Response

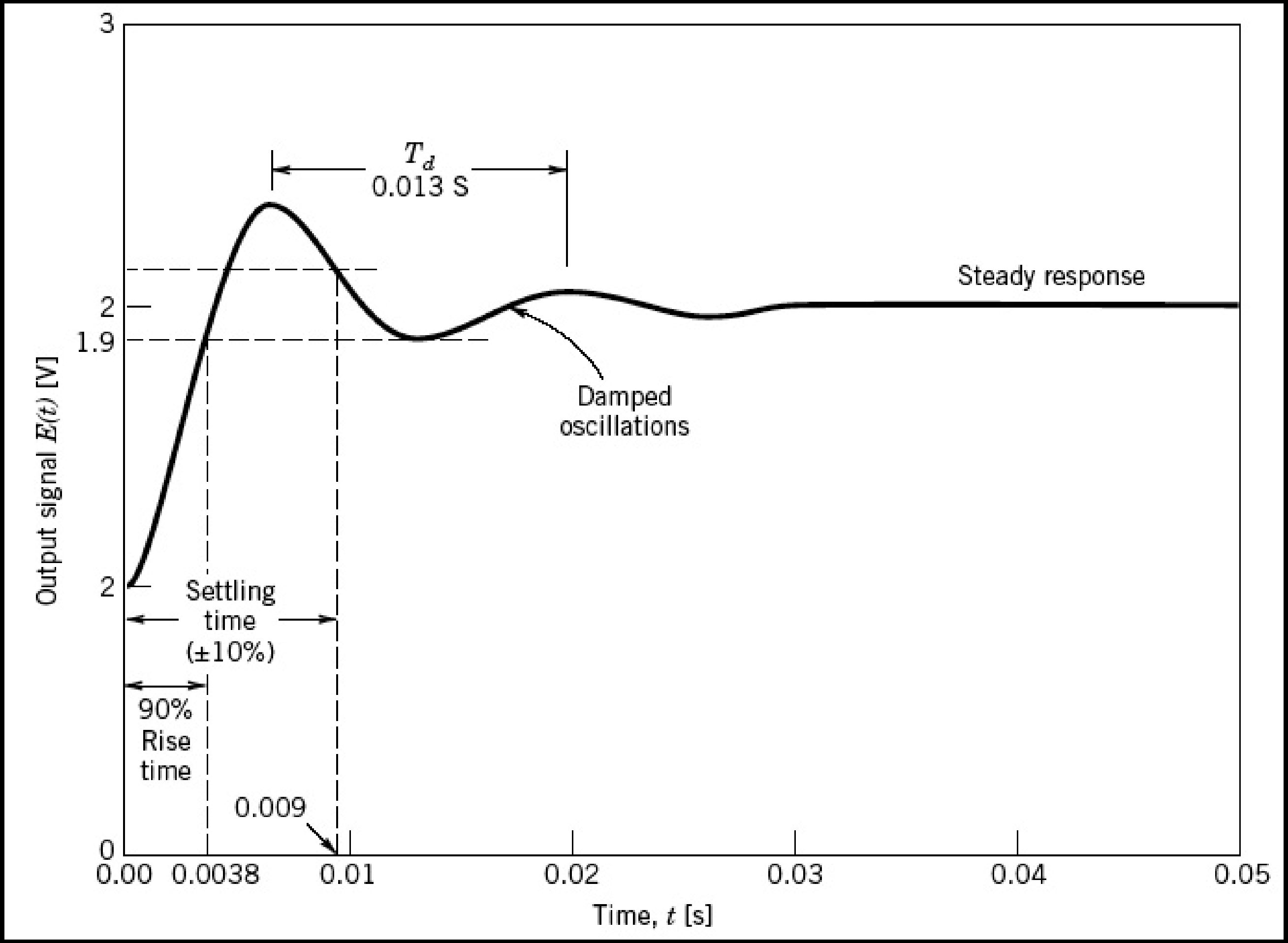


Ringing period: $T_d = \frac{2\pi}{\omega_d}$

Ringing frequency: $\omega_d = \omega_n\sqrt{1-\zeta^2}$

- Underdamped systems ($\zeta < 1$): Small rise time (90 % of input value), Large settling time
- (10% of steady-state value, *KA*)
- Overdamped systems ($\zeta > 1$): Large rise time, Small settling time
- Most measurement systems have damping ratios between 0.6 and 0.8.

Second-Order Systems: Step Response



Typical response of the 2nd order system

Second-Order Systems: Frequency Response

Let the input signal to the second-order system be of the form $x(t)$ = Asinωt

$$y(t) = y_c(t) + \frac{KA}{\left\{\left[1-(\omega/\omega_n)^2\right]^2 + (2\zeta\omega/\omega_n)^2\right\}^{1/2}} \sin[\omega t + \phi(\omega)]$$

where $\phi(\omega) = -\tan^{-1}\dfrac{2\zeta}{\omega/\omega_n - \omega_n/\omega}$

The steady state response is given by: $y_{\text{steady}}(t) = B(\omega)\sin[\omega t + \phi(\omega)]$

$$B(\omega) = \frac{KA}{\left\{\left[1-(\omega/\omega_n)^2\right]^2 + (2\zeta\omega/\omega_n)^2\right\}^{1/2}} \qquad \phi(\omega) = -\tan^{-1}\frac{2\zeta}{\omega/\omega_n - \omega_n/\omega}$$

The magnitude ratio is given by:

$$M(\omega) = \frac{B}{KA} = \frac{1}{\left\{\left[1-(\omega/\omega_n)^2\right]^2 + (2\zeta\omega/\omega_n)^2\right\}^{1/2}}$$

Second-Order Systems: Frequency Response

The magnitude ratio

$$M(\omega) = \frac{1}{\left\{\left[1-(\omega/\omega_n)^2\right]^2 + (2\zeta\omega/\omega_n)^2\right\}^{1/2}}$$

The phase shift

$$\phi(\omega) = -\tan^{-1}\frac{2\zeta}{\omega/\omega_n - \omega_n/\omega}$$