MECE 3320 - Measurements & Instrumentation

Static and Dynamic Characteristics of Signals

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A signal is the physical information about a measured variable being transmitted between a process and a measurement system.

**Characteristics of Signals**

- **Magnitude** - generally refers to the maximum value of a signal.
- **Range** - difference between maximum and minimum values of a signal.
- **Amplitude** - indicative of signal fluctuations relative to the mean.
- **Frequency** - describes the time variation of a signal.
Characteristics of Signals Contd.

- **Dynamic** - signal is time varying
- **Static** - signal does not change over the time period of interest
- **Deterministic** - signal can be described by an equation (other than a Fourier series or integral approximation)
- **Non-deterministic** - describes a signal which has no discernible pattern of repetition and cannot be described by a simple equation.
Signal characteristics to be considered in a measurement system

Range

Temporal variation
Signal Classification

Analog Signal
(continuous in time)

Discrete Signal
(information available at
discrete points in time)

(a) Analog signal representation
(b) Analog display

(a) Discrete time signal
(b) Discrete time waveform
**Quantization:** Assigning a single value to a range of magnitudes of a continuous signal. E.g. a digital watch displays a single numerical value for the entire duration of 1 min until it is updated at the next discrete time step.
I. Static
   \[ y(t) = A_0 \]

II. Dynamic
   Periodic waveforms
     Simple periodic waveform
     \[ y(t) = A_0 + C \sin(\omega t + \phi) \]
     Complex periodic waveform
     \[ y(t) = A_0 + \sum_{n=1}^{\infty} C_n \sin(n\omega t + \phi_n) \]
   Aperiodic waveforms
     Step
     \[ y(t) = A_0 U(t) \]
     \[ = A_0 \quad \text{for} \ t > 0 \]
     Ramp
     \[ y(t) = Kt \quad \text{for} \ 0 < t < t_f \]
     Pulse
     \[ y(t) = A_0 U(t) - A_0 U(t - t_1) \]
   III. Nondeterministic waveform
     \[ y(t) \approx A_0 + \sum_{n=1}^{\infty} C_n \sin(\omega_n t + \phi_n) \]
Mean value \( \bar{y} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} y(t) \, dt \)

RMS value \( y_{rms} = \sqrt{\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} y^2(t) \, dt} \)

Root Mean Square (RMS) value is an indication of the amount of variation in the dynamic portion of the signal.
Consider the function \( y(t) = 30 + 2 \cos 6\pi t \)

The mean is given by:

\[
\bar{y} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (30 + 2 \cos 6\pi t) \, dt
\]

\[
= \frac{1}{t_2 - t_1} \left[ 30t + \frac{2}{6\pi} \sin 6\pi t \right]_{t_1}^{t_2}
\]

\[
= \frac{1}{t_2 - t_1} \left[ 30(t_2 - t_1) + \frac{2}{6\pi} \left( \sin 6\pi t_2 - \sin 6\pi t_1 \right) \right]
\]

The RMS is given by:

\[
y_{rms} = \left\{ \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (30 + 2 \cos 6\pi t)^2 \, dt \right\}^{\frac{1}{2}}
\]

\[
= \left\{ \frac{1}{t_2 - t_1} \left[ 900t + \frac{120}{6\pi} \sin 6\pi t + 4 \left( \frac{1}{12\pi} \sin 6\pi t \cos 6\pi t + \frac{1}{2} t \right) \right] \right\}^{\frac{1}{2}}
\]
As the averaging time period becomes long relative to signal period, the resulting values will accurately represent the signal.

Non-deterministic signals have a range of frequencies and no single averaging period will produce an exact representation of the signal. In such a case, the signal should be averaged for a period longer than the longest time period contained within the signal.
DC offset may be removed from a dynamic signal to accentuate the characteristics of the fluctuating component of the signal.
A complex waveform, represented by white light, can be transformed into simpler components, represented by the colors in the spectrum.

A very complex signal, even one that is non-deterministic in nature, can be approximated as an infinite series of sine and cosine function – *Fourier Series*.
Spring-Mass System

Governing Equation:

\[ m \frac{d^2 y}{dt^2} + ky = 0 \]

Solution:

\[ y = A \cos(\omega t) + B \sin(\omega t); \quad \omega = \sqrt{\frac{k}{m}} \]

The solution can also be expressed as:

\[ y = C \cos(\omega t - \phi) = C \sin(\omega t - \phi^*) \]

\[ C = \sqrt{A^2 + B^2} \]

\[ \phi = \tan^{-1}\left(\frac{B}{A}\right); \quad \phi^* = \tan^{-1}\left(\frac{A}{B}\right) \]

\[ \phi^* = \frac{\pi}{2} - \phi \]

The time period \( T \) is given by:

\[ T = \frac{2\pi}{\omega} = \frac{1}{f} \]

- Amplitude of oscillation is \( C \)
- Frequency of oscillation is \( f \)
Any complex signal can be thought of as made up of sines and cosines of different amplitudes and time periods, which are added together in an infinite trigonometric series – *Fourier Series.*

*Modes of vibration for a string plucked at its center*
A function $y(t)$ with a period $T = 2\pi / \omega$ can be represented by a trigonometric series, such that for any $t$,

$$
y(t) = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos n\omega t + B_n \sin n\omega t \right)
$$

$$
A_0 = \frac{1}{T} \int_{-T/2}^{T/2} y(t) \, dt
$$

$$
A_n = \frac{1}{T} \int_{-T/2}^{T/2} y(t) \cos n\omega t \, dt
$$

$$
B_n = \frac{1}{T} \int_{-T/2}^{T/2} y(t) \sin n\omega t \, dt
$$

If $y(t)$ is even function,  $y(t) = \sum_{n=1}^{\infty} (A_n \cos n\omega t)$

If $y(t)$ is odd function,  $y(t) = \sum_{n=1}^{\infty} (B_n \sin n\omega t)$
Determine the Euler coefficients of the Fourier series of the function \( f(x) = x \) for \( -\pi \leq x \leq \pi \):

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \, dx = 0
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx) \, dx
\]

\[
= \left[ \frac{1}{n^2 \pi} \cos(nx) + \frac{x}{n\pi} \sin(nx) \right]_{-\pi}^{\pi} = 0
\]

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) \, dx
\]

\[
= \left[ \frac{1}{n^2 \pi} \sin(nx) - \frac{x}{n\pi} \cos(nx) \right]_{-\pi}^{\pi} = -\frac{2}{n} \cos(n\pi) = \frac{2}{n} (-1)^{n+1}
\]
Fourier Series Example

\[ f(x) = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin(nx) = \]

\[ 2 \sin(x) - \sin(2x) + \frac{2}{3} \sin(3x) - \frac{1}{2} \sin(4x) + \cdots \]
The partial sum of a Fourier series shows oscillations near a discontinuity point as shown from the previous example. These oscillations do not flatten out even when the total number of terms used is very large.

As an example, the real function \( f(x) = x \) has a value of 3.14 when \( x = 3.14 \). On the other hand, the partial sum solution using 100 terms has a value of 0.318 when \( x = 3.14 \). It is drastically different from the real value.

In general, these oscillations worsen when the number of terms used decrease. In the previous example, the value of the five terms solution is only 0.016 when \( x = 3.14 \).

Therefore, extreme attention has to be paid when using the Fourier analysis on discontinuous functions.
Fourier Transform and Frequency Spectrum

The Fourier coefficients are given by:

\[
A_n = \frac{1}{T} \int_{-T/2}^{T/2} y(t) \cos n \omega t \, dt
\]
\[
B_n = \frac{1}{T} \int_{-T/2}^{T/2} y(t) \sin n \omega t \, dt
\]

If the period \( T \) of the function approaches infinity, the spacing between the frequency components become infinitely small. In such a case, the coefficients \( A_n \) and \( B_n \) become continuous functions of frequency and can be expressed as:

\[
A(\omega) = \int_{-\infty}^{\infty} y(t) \cos \omega t \, dt
\]
\[
B(\omega) = \int_{-\infty}^{\infty} y(t) \sin \omega t \, dt
\]

Source: Cambridge University
Fourier Transform and Frequency Spectrum

Now, consider a complex number defined by \( Y(\omega) \).

\[
Y(\omega) = A(\omega) - iB(\omega) = \int_{-\infty}^{\infty} y(t)(\cos \omega t - i \sin \omega t) dt
\]

Introducing \( e^{-i\theta} = \cos \theta - i \sin \theta \)

\[
Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-i\omega t} dt
\]

The above equation can be rewritten as

\[
Y(f) = \int_{-\infty}^{\infty} y(t)e^{-i2\pi ft} dt
\]

The above equation provides the \textit{two-sided Fourier transform of} \( y(t) \).

The magnitude of \( Y(f) \), also called modulus, is given by

\[
|Y(f)| = \sqrt{\text{Re} Y(f)^2 + \text{Im} Y(f)^2}
\]

The phase of \( Y(f) \) is given by

\[
\phi(f) = \tan^{-1} \frac{\text{Im} Y(f)}{\text{Re} Y(f)}
\]
Consider a signal represented by the Fourier series:

\[ y(t) = 5 \sin(2\pi t) + 3 \sin(6\pi t + 0.2) + \sin(10\pi t + 0.1) \]

The frequency spectrum is shown below:

\[ C_1 = 5 \text{ V} \quad C_2 = 0 \text{ V} \quad C_3 = 3 \text{ V} \quad C_4 = 0 \text{ V} \quad C_5 = 1 \text{ V} \]

\[ f_1 = 1 \text{ Hz} \quad f_2 = 2 \text{ Hz} \quad f_3 = 3 \text{ Hz} \quad f_4 = 4 \text{ Hz} \quad f_5 = 5 \text{ Hz} \]

\[ \phi_1 = 0 \text{ rad} \quad \phi_2 = 0 \text{ rad} \quad \phi_3 = 0.2 \text{ rad} \quad \phi_4 = 0 \text{ rad} \quad \phi_5 = 0.1 \text{ rad} \]
Consider a discrete signal with \( N \) data points with \( y(t) \) as the source. So, we have \( y(0), y(1), \ldots, y(N-1) \).

We know the Fourier transform of \( y(t) \) is

\[
Y(f) \equiv \int_{-\infty}^{\infty} y(t)e^{-i2\pi ft} dt
\]

Now, each discrete point can be represented by a delayed impulse function, i.e

\[
y(r\delta t) = y(t)\delta(t - r\delta t)
\]

This implies that integral above is only evaluated at definite points. So, we can rewrite the integral as:

\[
Y(f) \equiv \int_{-\infty}^{\infty} y(t)e^{-i2\pi ft} dt = \int_{0}^{(N-1)\delta t} y(t)e^{-i2\pi ft} dt = y(0)e^{-i0} + y(1)e^{-i2\pi f\delta t} + y(2)e^{-i4\pi f\delta t} + \ldots + y(N-1)e^{-i2\pi f(N-1)\delta t}
\]
Therefore
\[
Y(f) = \sum_{k=0}^{N-1} y(k) e^{-i2\pi fk\delta t}
\]

This can be evaluated for any \( f \). But, we want to evaluate it for the fundamental frequency and its harmonics, i.e. \( f, 2f, 3f, \ldots \), or \( 1/N\delta t, 2/N\delta t, 3/N\delta t, \ldots, n/N\delta t \) Therefore
\[
Y(n) = \sum_{k=0}^{N-1} y(k) e^{-i2\pi \frac{n}{N} k}, n = 0, 1, \ldots
\]

In matrix form, it can be represented as
\[
\begin{bmatrix}
Y(0) \\
Y(1) \\
\vdots \\
Y(N-1)
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & W & W^2 & W^{N-1} & \ldots & 1 \\
1 & W^2 & W^4 & W^{2(N-1)} & \ldots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & W^{N-1} & W^{2(N-1)} & \ldots & W^{(N-1)(N-1)} & 1
\end{bmatrix}
\begin{bmatrix}
y(0) \\
y(1) \\
\vdots \\
y(N-1)
\end{bmatrix}
\]

where \( W = e^{-i\frac{2\pi}{N}} \)
Consider a continuous signal:

\[ y(t) = 5 + 2\cos(2\pi t - \frac{\pi}{2}) + 3\cos(4\pi t) \]

Now, let us sample the signal 4 times, i.e. \( f_s = 4\text{Hz} \), at \( t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \). (Put \( t = k\delta t, k = 0, 1, 2, 3 \))
So the discrete signal is given by:

$$y(k) = 5 + 2\cos(2\pi k \delta t - \frac{\pi}{2}) + 3\cos 4\pi k \delta t$$

We can write the Fourier transform as

$$Y(n) = \sum_{k=0}^{N-1} y(k)e^{-i2\pi \frac{n}{N}k} = \sum_{k=0}^{N-1} y(k)e^{-i\frac{\pi}{2}\frac{n}{k}}$$

In matrix form, we can rewrite the transform as:

$$\begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 \\ 1 & W^2 & W^4 & W^6 \\ 1 & W^3 & W^6 & W^9 \end{bmatrix} \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix}$$

where $W = e^{-i\frac{\pi}{2}}$
Discrete Fourier Transform - Example

The matrix therefore becomes:

\[
\begin{bmatrix}
Y(0) \\
Y(1) \\
Y(2) \\
Y(3)
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -i & -1 & i \\
1 & -1 & 1 & -1 \\
1 & i & -1 & i
\end{bmatrix} \begin{bmatrix}
8 \\
4 \\
8 \\
0
\end{bmatrix} = \begin{bmatrix}
-20 \\
-4i \\
12 \\
4i
\end{bmatrix}
\]

The magnitude of the DFT coefficients is shown below:
The inverse transform of

\[ Y(f) = \sum_{k=0}^{N-1} y(k) e^{-i2\pi fk\delta t} \]

is

\[ y(k) = \frac{1}{N} \sum_{n=0}^{N-1} Y(n) e^{+i\frac{2\pi n k}{N}} \]

i.e. the inverse matrix is \((1/N)\) times the complex conjugate of the original matrix.

Since the original matrix is symmetrical, the spectrum is symmetrical about \(N/2\).

Therefore \(F(n)\) and \(F(N-n)\) produce two frequency components \((n \times 2\pi/\delta t, \ n \leq N/2; \ \text{and} \ n > N/2)\) one of which is only valid.

The higher of these two frequencies is called the aliasing frequency.
Because of symmetry, the contribution to $y(k)$ is both from $Y(n)$ and $Y(N-n)$. Therefore

$$y_n(k) = \frac{1}{N} \left\{ Y(n)e^{+\frac{2\pi}{N}nk} + Y(N-n)e^{i\frac{2\pi}{N}(N-n)k} \right\}$$

$$\forall \ y(k), Y(N-n) = \sum_{k=0}^{N-1} y(k)e^{-i\frac{2\pi}{N}(N-n)k}$$

But $e^{-i\frac{2\pi}{N}(N-n)k} = e^{-i2\pi k} e^{i\frac{2\pi}{N}nk} = e^{i\frac{2\pi}{N}nk}$

Therefore $F^*(n) = F(N-n)$ i.e. the complex conjugate.
Substituting into equation for $y_n(k)$,

$$y_n(k) = \frac{1}{N} \left\{ Y(n)e^{+\frac{2\pi}{N}nk} + Y^*(n)e^{-\frac{2\pi}{N}nk} \right\}$$

i.e. $y_n(k) = \frac{2}{N} \left\{ \text{Re}\{Y(n)\} \cos \frac{2\pi}{N} nk - \text{Im}\{Y(n)\} \sin \frac{2\pi}{N} nk \right\}$

or $y_n(k) = \frac{2}{N} |Y(n)| \cos \left\{ \frac{2\pi}{N\delta t} k\delta t + \text{arg}[Y(n)] \right\}$

This represents a samples sine/cosine wave at a frequency of $1/N\delta t$ Hz and a magnitude of $2Y(n)/N$. 
Interpretation of Example

1. \( Y(0) = 20 \) represents a d.c value of \( \frac{Y(0)}{N} = \frac{20}{4} = 5 \).

2. \( Y(1) = -4i \) represents a fundamental component with peak amplitude \( 2Y(1)/N = \frac{2 \times 4}{4} = 2 \) with phase given by \( \text{arg}[Y(1)] = -90 \, \text{deg} \), i.e. \( 2 \cos(\frac{\pi k}{2} - 90) \).

3. \( Y(2) = 12 \) \( (n = N/2) \) implies a component \( -3 \cos(\pi k) \).

In typical applications, \( N \) is generally 1024 or greater.

- \( Y(n) \) has 1024 components: 513 to 1023 are complex conjugates of 1 to 511.