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A characterization of exponential distribution and the Sukhatme–Rényi decomposition of exponential maxima

George P. Yanev *, Santanu Chakraborty

School of Mathematical and Statistical Sciences, The University of Texas Rio Grande Valley, United States

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ABSTRACT

A new characterization of the exponential distribution is established. It is proven that the well-known Sukhatme–Rényi necessary condition is also sufficient for exponentiality. A method of proof due to Arnold and Villasenor based on the Maclaurin series expansion of the density is utilized.

1. Introduction and main results

A number of characterizations of the exponential distribution are based on the distributional equation $X + T \stackrel{d}{=} Y$ involving a pair of random variables (X, Y) and a random translator (shift) variable T, independent of X. Characterizations making use of this equation when X, Y, and T are either order statistics or record values were obtained in Wesolowski and Ahsanullah (2004), Castano-Martinez et al. (2012), and Shah et al. (2014) among others. In all studies so far the translator T was assumed to follow a certain distribution. This restriction is removed in our theorem below.

Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size $n \geq 2$ from a parent X with absolutely continuous cdf F, such that $F(0) = 0$. Denote the maximum order statistic by $X_{n:n}$.

Arnold and Villaseñor (2013) obtained a series of characterizations of the exponential distribution based on a random sample of size two. In particular, they proved that, under some additional conditions on the cdf F,

$$X_1 + \frac{1}{2}X_2 \stackrel{d}{=} X_{2:2}$$

characterizes the exponential distribution with some positive parameter. They also made conjectures for extensions to larger sample sizes. In Chakraborty and Yanev (2013) and Yanev and Chakraborty (2013) some of the results from Arnold and Villaseñor (2013) were generalized to random samples of size $n \geq 3$. For instance, it was proven in Chakraborty and Yanev

* Corresponding author.
E-mail address: george.yanev@utrgv.edu (G.P. Yanev).

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(2013), under the same assumptions on the cdf F as in the case $n = 2$, that for a fixed $n \geq 2$

$$X_{n-1:n-1} + \frac{1}{n}X_n \stackrel{d}{=} X_{n:n} \tag{1}$$

characterizes the exponential distribution.

The contribution of the present paper is twofold. (i) The characterization equation (1) is extended to the case of maxima of n and $n - s$ random variables for $1 \leq s \leq n$. (ii) The technique of proof from Arnold and Villaseñor (2013) for a random sample of size two is expanded to the case of sample size $n \geq 2$ for any fixed n. The proof of the main result makes use of combinatorial identities, which might be of independent interest. We believe that this technique will be useful in obtaining other characterization results in the future.

Theorem. *Let X be a non-negative random variable with pdf $f(x)$. Assume that $f(x)$ is complex analytic for every x and $f(0) > 0$. Let n and s be fixed integers such that $1 \leq s \leq n - 1$. If*

$$X_{n-s:n-s} + \frac{1}{n-s+1}X_{n-s+1} + \cdots + \frac{1}{n}X_n \stackrel{d}{=} X_{n:n}, \tag{2}$$

then X is exponential with some positive parameter.

It is well-known (cf. Conway, 1978, p. 35) that every complex analytic function is infinitely differentiable and, furthermore, has a power series expansion about each point of its domain.

Note that the theorem has been applied in constructing goodness-of-fit tests for exponential distribution in Jovanovic et al. (2015) and Volkova (2015).

The following direct corollary of the theorem is of its own interest.

Corollary. *Let X be a non-negative random variable with pdf $f(x)$. Assume that $f(x)$ is complex analytic for every x and $f(0) > 0$. If for fixed n*

$$X_1 + \frac{1}{2}X_2 + \frac{1}{3}X_3 + \cdots + \frac{1}{n}X_n \stackrel{d}{=} X_{n:n}, \tag{3}$$

then X is exponential with some positive parameter.

Eq. (3) is a particular case (for maxima) of the well-known Sukhatme–Rényi decomposition (cf. Arnold et al., 2008, p. 73) of the kth order statistic in a random sample $X_1, X_2, \ldots, X_n$ from an exponential distribution. It is known (cf. Arnold and Villaseñor, 2013) that if (3) holds for every n, then necessarily $X_1, X_2, \ldots, X_n$ have a common exponential distribution. Under the assumptions of the corollary, for $X_1, X_2, \ldots, X_n$ to be exponential it is sufficient that (3) holds for one fixed n only.

In the next section we state three lemmas, to be used in the proof of the theorem. The main steps in the proof of the theorem are outlined in Section 3. Details of the proof of the theorem are given in Section 4. Section 5 contains the proofs of Lemmas 1 and 2. Concluding remarks are given in Section 6.

2. Preliminaries

Introduce, for all non-negative integers n and i, and any real number x,

$$H_{n,i}(x) := \sum_{j=0}^{n}(-1)^j \binom{n}{j} (x-j)^i. \tag{4}$$

We start with identities involving $H_{n,i}(x)$, which may be of independent interest.

Lemma 1. *Let s and r be positive integers. Then*

$$\text{(i)} \sum_{j=0}^{r-1}\binom{r}{j} H_{s-1,j}(s) = H_{s,r}(s+1). \tag{5}$$

$$\text{(ii)} \sum_{j=0}^{r-1}\binom{r}{j+1} H_{s,j}(s+1) = \frac{1}{s+1}H_{s+1,r}(s+2). \tag{6}$$

$$\text{(iii)} \sum_{j=0}^{r-1}(s+2)^{r-1-j}H_{s,j}(s+1) = \frac{1}{s+1}H_{s+1,r}(s+2). \tag{7}$$

Define $G_m(x) := F^m(x)f(x)$ for $m \geq 1$; $G_0(x) := F(x)$. Assuming (8), we calculate the derivatives of $G_m(x)$ at 0 for $m \geq 1$.

Lemma 2. *Let $m \geq 1$ and d be integers, such that $d \geq -m$. Assume $F(0) = 0$. In case d is positive, also assume,*

$$f^{(k)}(0) = \left[\frac{f'(0)}{f(0)}\right]^{k-1} f'(0) \quad k = 1, \ldots, d, \tag{8}$$

then

$$G_m^{(m+d)}(0) = \begin{cases} \left[\frac{f'(0)}{f(0)}\right]^{d} f^{m+1}(0) H_{m,m+d}(m+1) & \text{if } d \geq 0; \\ 0 & \text{if } -m \leq d < 0. \end{cases} \tag{9}$$

The third lemma, extracted from the proof of Theorem 1 in Arnold and Villaseñor (2013), plays a central role in the proof of the theorem.

Lemma 3. *Let X be a non-negative random variable with pdf $f(x)$. Assume that $f(x)$ is complex analytic for every x and $f(0) > 0$. If*

$$f^{(k)}(0) = \left[\frac{f'(0)}{f(0)}\right]^{k-1} f'(0), \quad k = 1, 2, \ldots, \tag{10}$$

then X is exponential with some positive parameter.

Note that the assumptions for analyticity of $f(x)$ and $f(0) > 0$ are implicitly used in the proof of Lemma 3 given in Arnold and Villaseñor (2013).

3. Outline of the main steps in the proof of the theorem

The proof of the theorem can be divided into four steps as follows.

Step 1: Define $d_j := n - j + 1$ and $y_j := z - x_s - \sum_{k=1}^{j-1} x_k$ for $1 \leq j \leq s$. Show that the equality in distribution (2) is equivalent to

$$\int_0^z G_{n-s-1}(x_s) \int_0^{y_1} \cdots \int_0^{y_{s-1}} \left(\prod_{j=1}^{s-1} f\left(d_j x_j\right)\right) f\left(d_s y_s\right)\, dx_{s-1} \ldots dx_1 dx_s$$
$$= f(z) \int_0^z \int_0^{x_1} \cdots \int_0^{x_{s-1}} \left(\prod_{k=1}^{s-1} f(x_k)\right) G_{n-s-1}(x_s)\, dx_s \ldots dx_1. \tag{11}$$

Step 2: Denote

$$r_j(t) := n - s + t + 1 - \sum_{k=1}^{j-1} i_k \quad 1 \leq j \leq s,\ t \geq 1,$$

where i_k are integers. We shall write r_j instead of $r_j(t)$. Also, introduce

$$a_{i_1,\ldots,i_s} := d_s^{\,r_s - i_s - 1} \prod_{j=1}^{s-1} d_j^{\,i_j}, \qquad b_{i_1,\ldots,i_s} := \binom{r_s}{i_s + 1} \prod_{j=1}^{s-1} \binom{r_j + s - j}{i_j}.$$

Prove (by differentiating (11) $(n+t)$ times with respect to z and setting $z = 0$) that (11) implies

$$\sum_{i_1=0}^{r_1} \cdots \sum_{i_{s-1}=0}^{r_{s-1}} \sum_{i_s=0}^{r_s - 1} a_{i_1,\ldots,i_s} \left(\prod_{j=1}^{s-1} f^{(i_j)}(0)\right) f^{(r_s - i_s - 1)}(0) G_{n-s-1}^{(i_s)}(0)$$
$$= \sum_{i_1=0}^{r_1} \cdots \sum_{i_{s-1}=0}^{r_{s-1}} \sum_{i_s=0}^{r_s - 1} b_{i_1,\ldots,i_s} \left(\prod_{j=1}^{s-1} f^{(i_j)}(0)\right) f^{(r_s - i_s - 1)}(0) G_{n-s-1}^{(i_s)}(0). \tag{12}$$

Step 3: Using Lemma 1, prove that

$$\sum_{i_1=0}^{r_1} \cdots \sum_{i_{s-1}=0}^{r_{s-1}} \sum_{i_s=0}^{r_s - 1} a_{i_1,\ldots,i_s} H_{n-s-1,i_s}(n-s) = \sum_{i_1=0}^{r_1} \cdots \sum_{i_{s-1}=0}^{r_{s-1}} \sum_{i_s=0}^{r_s - 1} b_{i_1,\ldots,i_s} H_{n-s-1,i_s}(n-s). \tag{13}$$

Step 4: Prove (10) by induction using the results from Step 2 and Step 3.

The statement of the theorem follows by Step 4 and Lemma 3.

4. Proofs of the steps in Section 3

Let $F_n(x)$ and $f_n(x)$ denote the cdf and pdf, respectively, of the maximum $X_{n:n}$. Obviously, $F_n(x) = F^n(x)$.

4.1. Proof of Step 1

Let $f_{n-1,n}(x)$ denote the density of $X_{n-1}/(n-1) + X_n/n$. Setting $s = 2$ in (2), for the density $f_{LHS}(z \mid s = 2)$, say, of the left-hand side of (2) we find

$$\begin{aligned} f_{LHS}(z \mid s=2) &= \int_0^z f_{n-2}(x_2) f_{n-1,n}(z-x_2)\, dx_2 \\ &= \int_0^z (n-2)G_{n-3}(x_2)n(n-1)\int_0^{z-x_2} f(nx_1)f((n-1)(z-x_2-x_1))\, dx_1 dx_2 \\ &= (n)_3 \int_0^z G_{n-3}(x_2)\int_0^{z-x_2} f(nx_1)f((n-1)(z-x_2-x_1))\, dx_1dx_2, \end{aligned}$$

where $(n)_3 := n(n-1)(n-2)$. Setting $s = 3$ in (2), we have

$$\begin{aligned} & f_{LHS}(z \mid s=3) \\ &= (n)_4 \int_0^z G_{n-4}(x_3)\int_0^{z-x_3}\int_0^{z-x_3-x_1}\int_0^{z-x_3-x_1-x_2} f(nx_1)f((n-1)x_2)f((n-2)(z-x_3-x_1-x_2))\, dx_2dx_1dx_3. \end{aligned}$$

Repeating this argument, we obtain for any s such that $2 \le s \le n-1$,

$$\frac{f_{LHS}(z)}{(n)_{s+1}} = \int_0^z G_{n-s-1}(x_s)\int_0^{y_1}\cdots\int_0^{y_{s-1}}\left(\prod_{j=1}^{s-1} f\left(d_jx_j\right)\right) f\left(d_sy_s\right)\, dx_{s-1}\dots dx_1dx_s. \tag{14}$$

For the density $f_{RHS}(z)$, say, of the right-hand side of (2), we have

$$\begin{aligned} f_{RHS}(z) = nf(z)F_{n-1}(z) &= nf(z)\int_0^z f_{n-1}(x_1)\, dx_1 \\ &= n(n-1)f(z)\int_0^z f(x_1)F_{n-2}(x_1)\, dx_1 \\ &= n(n-1)(n-2)f(z)\int_0^z\int_0^{x_1} f(x_1)f(x_2)F_{n-3}(x_2)\, dx_2dx_1. \end{aligned}$$

Repeating this argument $(s-2)$ more times we obtain

$$\frac{f_{RHS}(z)}{(n)_{s+1}} = f(z)\int_0^z\int_0^{x_1}\cdots\int_0^{x_{s-1}}\left(\prod_{k=1}^{s-1} f(x_k)\right) G_{n-s-1}(x_s)\, dx_s\dots dx_1. \tag{15}$$

Combining (14) and (15) we obtain (11).

4.2. Proof of Step 2

Define

$$K_{n,s-1}(y_1) := \int_0^{y_1}\cdots\int_0^{y_{s-1}}\left(\prod_{j=1}^{s-1} f\left(d_jx_j\right)\right) f\left(d_sy_s\right)\, dx_{s-1}\dots dx_1.$$

Observing that $K^{(i)}_{n,s-1}(0) = 0$ when $i < s-1$ and $G^{(i)}_{d_{s+2}}(0) = 0$ for $i < d_{s+2}$, for the $(n+t)$th derivative of the left-hand side of (11) at 0, we obtain

$$\begin{aligned} & \frac{d}{dz^{n+t}}\left\{\int_0^z G_{d_{s+2}}(x_s)K_{n,s-1}(z-x_s)dx_s\right\}\Bigg|_{z=0} \\ &= \frac{d}{dz^{n+t-1}}\left\{G_{d_{s+2}}(z)K_{n,\,s-1}(0) + \int_0^z G_{d_{s+2}}(x_s)K'_{n,s-1}(z-x_s)dx_s\right\}\Bigg|_{z=0} \\ &= \frac{d}{dz^{n+t-s}}\left\{G_{d_{s+2}}(z)K^{(s-1)}_{n,s-1}(0) + \int_0^z G_{d_{s+2}}(x_s)K^{(s)}_{n,s-1}(z-x_s)\, dx_s\right\}\Bigg|_{z=0} \\ &= \sum_{i=n-s-1}^{n-s+t} G^{(i)}_{d_{s+2}}(0)K^{(n+t-1-i)}_{n,s-1}(0). \end{aligned} \tag{16}$$

Using the recursive relation

$$K_{n,s-1}(u)=\int_0^u f(nx)K_{n-1,s-2}(u-x)\,dx \quad 3\le s\le n-1,$$

one can show by induction that the mth derivative of $K_{n,s-1}(u)$ at 0 for any $m\ge s-1$ and any fixed $n\ge 2$ is given by

$$K^{(m)}_{n,s-1}(0)=\sum_{i_1=0}^{l_1}\cdots\sum_{i_{s-1}=0}^{l_{s-1}}\left(\prod_{j=1}^{s-1}d_j^{i_j}f^{(i_j)}(0)\right)d_s^{l_s}f^{(l_s)}(0), \tag{17}$$

where $l_j=m-s+1-\sum_{l=1}^{j-1}i_l$ for $1\le j\le s$. We omit the derivation of (17) here. Substituting (17) into (16) and changing the indexes of summation, one can see that the last sum in (16) equals

$$\sum_{i_1=0}^{r_1}\cdots\sum_{i_{s-1}=0}^{r_{s-1}}\left(\prod_{j=1}^{s-1}d_j^{i_j}f^{(i_j)}(0)\right)\sum_{i_s=0}^{r_s-1}d_s^{r_s-i_s-1}f^{(r_s-i_s-1)}(0)G^{(i_s)}_{d_{s+2}}(0) \tag{18}$$

where, as before, $r_j=n-s+t+1-\sum_{k=0}^{j-1}i_k$ for $1\le j\le s$. Thus, we have obtained the left-hand side of (12).

We turn to the right-hand side of (12). Denote

$$L(x_1|x_2,\dots,x_s):=\int_0^{x_1}\cdots\int_0^{x_{s-1}}\left(\prod_{k=1}^{s-1}f(x_k)\right)G_{d_{s+2}}(x_s)\,dx_s\dots dx_2.$$

With this notation for the $(n+t)$th derivative of $f_{RHS}(z)/(n)_{s+1}$ at 0, we find

$$\begin{aligned}
&\frac{d}{dz^{n+t}}\left\{f(z)\int_0^z L(x_1|x_2,\dots,x_s)\,dx_1\right\}\Big|_{z=0}\\
&=\sum_{i_1=0}^{n+t}\binom{n+t}{i_1}f^{(i_1)}(0)\frac{d}{dz^{n+t-i_1}}\left\{\int_0^z f(x_1)L(x_1|x_2,\dots,x_s)\,dx_1\right\}\Big|_{z=0}\\
&=\sum_{i_1=0}^{n+t-1}\binom{n+t}{i_1}f^{(i_1)}(0)\frac{d}{dz^{n+t-1-i_1}}\left\{f(z)\int_0^z L(x_2|x_3,\dots,x_s)\,dx_2\right\}\Big|_{z=0}.
\end{aligned}$$

Recall that $r_j:=n+t-s+1-\sum_{k=1}^{j-1}i_k$ for $j=1,\dots,s$. Repeating the last argument, it is not difficult to obtain

$$\begin{aligned}
&\frac{d}{dz^{n+t}}\left\{f(z)\int_0^z L(x_1|x_2,\dots,x_s)\,dx_1\right\}\Big|_{z=0}\\
&=\sum_{i_1=0}^{r_1}\cdots\sum_{i_{s-1}=0}^{r_{s-1}}\left(\prod_{j=1}^{s-1}\binom{r_j+s-j}{i_j}f^{(i_j)}(0)\right)\frac{d}{dz^{r_s}}\left\{f(z)\int_0^z G_{d_{s+2}}(x_s)dx_s\right\}\Big|_{z=0}\\
&=\sum_{i_1=0}^{r_1}\cdots\sum_{i_{s-1}=0}^{r_{s-1}}\left(\prod_{j=1}^{s-1}\binom{r_j+s-j}{i_j}f^{(i_j)}(0)\right)\sum_{i_s=0}^{r_s-1}\binom{r_s}{i_s+1}f^{(r_s-i_s-1)}(0)G^{(i_s)}_{d_{s+2}}(0).
\end{aligned} \tag{19}$$

Combining (18) and (19) we prove Step 2.

4.3. Proof of Step 3

We shall simplify the right-hand side of (13), working on the most inner sum first and moving to the outer ones later. Applying (6), we see that

$$\begin{aligned}
\sum_{i_{s-1}=0}^{r_{s-1}}\binom{r_s+1}{i_{s-1}}\sum_{i_s=0}^{r_s-1}\binom{r_s}{i_s+1}H_{n-s-1,i_s}(n-s)&=\frac{1}{n-s}\sum_{i_{s-1}=0}^{r_{s-1}}\binom{r_s+1}{i_{s-1}}H_{n-s,r_{s-1}-i_s}(n-s+1)\\
&=\frac{1}{(n-s)(n-s+1)}H_{n-s+1,r_{s-2}+1-i_{s-2}}(n-s+2).
\end{aligned}$$

Furthermore, since $H_{n-s+1,0}(n-s+1)=0$, applying (6) again, we have

$$\begin{aligned}
\sum_{i_{s-2}=0}^{r_{s-2}}\binom{r_{s-2}+2}{i_{s-2}}H_{n-s+1,r_{s-2}+1-i_{s-2}}(n-s+2)&=\sum_{i_{s-2}=0}^{r_{s-2}+1}\binom{r_{s-2}+2}{i_{s-2}}H_{n-s+1,r_{s-2}+1-i_{s-2}}(n-s+2)\\
&=\frac{1}{(n-s+2)}H_{n-s+2,r_{s-3}+2-i_{s-3}}(n-s+3).
\end{aligned}$$

Repeating this argument for the rest of the sums on the right-hand side of (13), we find

$$\sum_{i_1=0}^{r_1}\cdots\sum_{i_{s-1}=0}^{r_{s-1}}\left(\prod_{j=1}^{s-1}\binom{r_j+s-j}{i_j}\right)\sum_{i_s=0}^{r_s-1}\binom{r_s}{i_s+1}H_{n-s-1,i_s}(n-s)=\frac{H_{n-1,n+t}(n)}{(n-1)_s}, \tag{20}$$

where $d_{s+2}=n-s-1$. Let us turn to the left-hand side of (13). Recall that $H_{t,i_s}(\cdot)=0$ for $0\le i_s\le t-1$. Similarly to the arguments in the simplification of the right-hand side above, applying (7) instead of (6), we obtain

$$\sum_{i_1=0}^{r_1}\cdots\sum_{i_{s-1}=0}^{r_{s-1}}\left(\prod_{j=1}^{s-1}d_j^{i_j}\right)\sum_{i_s=0}^{r_s-1}d_s^{r_s-i_s-1}H_{n-s-1,i_s}(n-s)=\frac{H_{n-1,n+t}(n)}{(n-1)_s}. \tag{21}$$

Eqs. (20) and (21) imply (13), which completes the proof of Step 3.

4.4. Proof of Step 4

Denote $c_{i_1,\ldots,i_s}:=a_{i_1,\ldots,i_s}-b_{i_1,\ldots,i_s}$. With this notation and taking into account that $G_{n-s-1}^{(i_s)}(0)=0$ when $i_s<n-s-1$, we write (12) as

$$\sum_{i_1=0}^{r_1}\cdots\sum_{i_{s-1}=0}^{r_{s-1}}\sum_{i_s=n-s-1}^{r_s-1}c_{i_1,\ldots,i_s}\left(\prod_{j=1}^{s-1}f^{(i_j)}(0)\right)f^{(r_s-i_s-1)}(0)G_{n-s-1}^{(i_s)}(0)=0. \tag{22}$$

We shall prove (10) by (strong) induction on k. The base case $k=1$ is trivial. Assuming (10) for $k\le t$, we shall prove it for $k=t+1$, where t stands for any positive integer. First, observe that since the order of the derivative of $f(x)$ in (12) must be nonnegative, we have $r_s-i_s-1\ge 0$. Combining this with $i_s\ge n-s-1$, we see that

$$\sum_{k=1}^{s-1}i_k\le t+1. \tag{23}$$

To extract the terms with a factor $f^{(t+1)}(0)$, we shall split the sum in (22) into two as follows

$$\sum_{\mathcal{I}\setminus\mathcal{I}_0}c_{i_1,\ldots,i_s}\left(\prod_{j=1}^{s-1}f^{(i_j)}(0)\right)f^{(r_s-i_s-1)}(0)G_{n-s-1}^{(i_s)}(0)+f^{s-1}(0)f^{(t+1)}(0)G_{n-s-1}^{(n-s-1)}(0)\sum_{\mathcal{I}_0}c_{i_1,\ldots,i_s}=0, \tag{24}$$

where $\mathcal{I}=\{(i_1,\ldots,i_s):\ 0\le i_j\le r_j,\ 1\le j\le s-1, 1\le i_s\le r_s-1\}$ and $\mathcal{I}_0$ is the set of vectors $(i_1,\ldots,i_s)$ such that $i_s=n-s-1$ and among the first $s-1$ components: (i) all are zeros or (ii) exactly one is $t+1$ and the others are zeros. Notice that by Lemma 2

$$G_{n-s-1}^{(n-s-1)}(0)=f^{n-s}(0)H_{n-s-1,n-s-1}(n-s). \tag{25}$$

Consider the first sum in (24) (the one over $\mathcal{I}\setminus\mathcal{I}_0$). Inequality (23) along with the definition of the index set $\mathcal{I}\setminus\mathcal{I}_0$ implies that all derivatives of $f(x)$ included in the product term have order less than or equal to t. Therefore, applying the induction assumption to $f^{(i_j)}(0)$ for $i_j\ge 1$, we have

$$\prod_{j=1}^{s-1}f^{(i_j)}(0)=\begin{cases}f(0)[f'(0)]^{\sum_{k=1}^{s-1}i_k} & \text{if } (i_1,\ldots,i_{s-1})\ne(0,\ldots,0);\\ 1 & \text{if } (i_1,\ldots,i_{s-1})=(0,\ldots,0).\end{cases} \tag{26}$$

It is not difficult to see that over the index set $\mathcal{I}\setminus\mathcal{I}_0$ we have $r_s-i_s-1\le t$ and therefore, applying the induction assumption, we obtain for $n-s-1\le i_s\le r_s-1$

$$f^{(r_s-i_s-1)}(0)=\left[\frac{f'(0)}{f(0)}\right]^{r_s-i_s-2}f'(0). \tag{27}$$

It remains to study the factor $G_{n-s-1}^{(i_s)}(0)$. Since $i_s\le r_s-1\le n-s+t-\sum_{k=1}^{s-1}i_k$, we have that $i_s-(n-s-1)\le t+1-\sum_{k=1}^{s-1}i_k$. We consider two cases as follows. (i) Let $\sum_{k=1}^{s-1}i_k\ge 1$. Then $i_s-(n-s-1)\le t$ and, under the induction assumption, applying Lemma 2 with $m=n-s-1$ and $d=i_s-(n-s-1)\le t$, we have

$$G_{n-s-1}^{(i_s)}(0)=\left[\frac{f'(0)}{f(0)}\right]^{i_s-n+s+1}f^{n-s}(0)H_{n-s-1,i_s}(n-s). \tag{28}$$

(ii) Let $\sum_{k=1}^{s-1}i_k=0$. If $i_s\le n-s+t-1$, then (28) holds. If $i_s=n-s+t$, then we see that

$$c_{0,\ldots,0,n-s+t}=0. \tag{29}$$

Combining (25)–(29), it is not difficult to obtain that, under the induction assumption, (24) implies

$$\left[\frac{f'(0)}{f(0)}\right]^t f'(0)\sum_{\mathcal{I}\setminus\mathcal{I}_0} c_{i_1,\ldots,i_s}H_{n-s-1,i_s}(n-s)+f^{(t+1)}(0)\sum_{\mathcal{I}_0} c_{i_1,\ldots,i_s}H_{n-s-1,n-s-1}(n-s)=0.$$

Thus, to prove (10) for $k=t+1$, it is sufficient to prove

$$\sum_{\mathcal{I}\setminus\mathcal{I}_0} c_{i_1,\ldots,i_s}H_{n-s-1,i_s}(n-s)+\sum_{\mathcal{I}_0} c_{i_1,\ldots,i_s}H_{n-s-1,n-s-1}(n-s)=0$$

or, equivalently,

$$\sum_{\mathcal{I}} a_{i_1,\ldots,i_s}H_{n-s-1,i_s}(n-s)=\sum_{\mathcal{I}} b_{i_1,\ldots,i_s}H_{n-s-1,i_s}(n-s).$$

This is equivalent to (13) proven to be true in Step 3. Therefore, the proof of Step 4 is complete.

5. Proofs of Lemmas 1 and 2

It is known (cf. Ruiz, 1996) that for any non-negative integer n and real x

$$H_{n,i}(x)=\begin{cases} n! & \text{if } i=n;\\ 0 & \text{if } 0\le i<n.\end{cases}$$

This information will be useful in the proofs of the lemmas that follow.

Proof of Lemma 1. (i) By the definition of $H_{n,i}(x)$ in (4), we obtain

$$\begin{aligned}\sum_{j=0}^{r-1}\binom{r}{j}H_{s-1,j}(s) &= \sum_{i=0}^{s-1}(-1)^i\binom{s-1}{i}\sum_{j=0}^{r-1}\binom{r}{j}(s-i)^j\\ &= \sum_{i=0}^{s-1}(-1)^i\binom{s-1}{i}\left[(s+1-i)^r-(s-i)^r\right]\\ &= (s+1)^r-\left[s^r+\binom{s-1}{1}s^r\right]+\cdots+(-1)^{s-1}\left[\binom{s-1}{s-2}2^s+2^s\right]+(-1)^s\\ &= (s+1)^r-\binom{s}{1}s^r+\cdots+(-1)^{s-1}\binom{s}{s-1}2^r+(-1)^s\\ &= \sum_{j=0}^{s}(-1)^j\binom{s}{j}(s+1-j)^r\\ &= H_{s,r}(s+1).\end{aligned}$$

(ii) Indeed, using the definition of $H_{s,j}(x)$ in (4), we have

$$\begin{aligned}\sum_{j=0}^{r-1}\binom{r}{j+1}H_{s,j}(s+1) &= \sum_{j=0}^{r-1}\binom{r}{j+1}\sum_{i=0}^{s}(-1)^i\binom{s}{i}(s+1-i)^j\\ &= \sum_{i=0}^{s}(-1)^i\binom{s}{i}\sum_{k=1}^{r}\binom{r}{k}(s+1-i)^{k-1}\\ &= \sum_{i=0}^{s}(-1)^i\binom{s}{i}\frac{1}{s+1-i}\left[\sum_{k=0}^{r}\binom{r}{k}(s+1-i)^k-1\right]\\ &= \frac{1}{s+1}\sum_{i=0}^{s}(-1)^i\binom{s+1}{i}\left[(s+2-i)^r-1\right]\\ &= \frac{1}{s+1}\sum_{i=0}^{s+1}(-1)^i\binom{s+1}{i}(s+2-i)^r\\ &= \frac{1}{s+1}H_{s+1,r}(s+2).\end{aligned}$$

(iii) We have

$$\begin{aligned}\sum_{j=0}^{r-1}(s+2)^{r-1-j}H_{s,j}(s+1) &= \sum_{j=0}^{r-1}(s+2)^{r-1-j}\sum_{i=0}^{s}(-1)^i\binom{s}{i}(s+1-i)^j \\ &= \sum_{i=0}^{s}(-1)^i\binom{s}{i}(s+2)^{r-1}\sum_{j=0}^{r-1}\left(\frac{s+1-i}{s+2}\right)^j \\ &= \sum_{i=0}^{s}(-1)^i\binom{s}{i}\frac{1}{i+1}\left[(s+2)^r-(s+1-i)^r\right] \\ &= \frac{1}{s+1}\sum_{i=0}^{s}(-1)^i\binom{s+1}{i+1}\left[(s+2)^r-(s+1-i)^r\right] \\ &= \frac{1}{s+1}\sum_{j=0}^{s+1}(-1)^j\binom{s+1}{j}(s+2-j)^r \\ &= \frac{1}{s+1}H_{s+1,r}(s+2).\end{aligned}$$

Proof of Lemma 2. (i) If $-m \le d < 0$, then $G_m^{(m+d)}(0) = 0$ because all the terms in the expansion of $G_m^{(m+d)}(0)$ have a factor $F(0) = 0$.

(ii) Let $d = 0$. We shall prove (9) by induction on m. One can verify directly the case $m = 1$. Assuming (9) for $m = k$, we shall prove it for $m = k + 1$. Since $G_{k+1}(x) = F(x)G_k(x)$, applying (i), we see that

$$\begin{aligned}G_{k+1}^{(k+1)}(0) &= \sum_{j=0}^{k+1}\binom{k+1}{j}F^{(j)}(0)G_k^{(k+1-j)}(0) \\ &= F(0)G_k^{(k+1)}(0) + (k+1)F'(0)G_k^{(k)}(0) + \sum_{j=2}^{k+1}\binom{k+1}{j}F^{(j)}(0)G_k^{(k+1-j)}(0) \\ &= (k+1)!f^{k+2}(0),\end{aligned}$$

which completes the proof of (ii).

(iii) Let $d > 0$ and m be any positive integer. For simplicity, we will write $f^{(j)} := f^{(j)}(0)$ below.

(a) Let $m = 1$. If $d = 1$, then we have $G_1^{(2)}(0) = 3f'f = f'fH_{1,2}(2)$ since $H_{1,2}(2) = 3$. Thus, (9) is true for $d = 1$. Next, assuming (9) for $G_1^{(k)}(0)$, we shall prove it for $G_1^{(k+1)}(0)$. Since $G_1(x) = F(x)f(x)$, using (8) we obtain

$$\begin{aligned}G_1^{(k+1)}(0) &= \sum_{j=1}^{k+1}\binom{k+1}{j}f^{(j-1)}f^{(k+1-j)} \\ &= \sum_{j=1}^{k+1}\binom{k+1}{j}\left(\frac{f'}{f}\right)^{j-2}f'\left(\frac{f'}{f}\right)^{k-j}f' \\ &= \left(\frac{f'}{f}\right)^{k-2}(f')^2\sum_{j=1}^{k+1}\binom{k+1}{j} \\ &= \left(\frac{f'}{f}\right)^{k-2}(f')^2(2^{k+1}-1) \\ &= \left(\frac{f'}{f}\right)^{k}f^2H_{1,1+k}(2).\end{aligned}$$

This completes the proof for the case (a) $m = 1$ and any $d > 0$.

(b) Assuming (9) for $m = 1, 2, \ldots, k$ and any $d > 0$ we shall prove it for $m = k + 1$ and any $d > 0$. Since $G_{k+1}(x) = F(x)G_k(x)$, by (8) and the induction assumption, we obtain

$$\begin{aligned}G_{k+1}^{(k+1+d)}(0) &= \sum_{j=1}^{k+1+d}\binom{k+1+d}{j}f^{(j-1)}G_k^{(k+1+d-j)}(0) \\ &= \sum_{j=1}^{d+1}\binom{k+1+d}{j}f^{(j-1)}G_k^{(k+1+d-j)}(0)\end{aligned}$$

$$= \sum_{j=1}^{d+1} \binom{k+1+d}{j} \left(\frac{f'}{f}\right)^{j-2} f' \left(\frac{f'}{f}\right)^{1+d-j} f^{k+1} H_{k,k+1+d-j}(m)$$

$$= \left(\frac{f'}{f}\right)^{d} f^{k+2} \sum_{j=1}^{k+1+d} \binom{k+1+d}{j} H_{k,k+1+d-j}(k+1)$$

$$= \left(\frac{f'}{f}\right)^{d} f^{k+2} \sum_{i=0}^{k+d} \binom{k+1+d}{i} H_{k,i}(k+1)$$

$$= \left(\frac{f'}{f}\right)^{d} f^{k+2} H_{k+1,k+1+d}(k+2),$$

where the last equality follows from (5) with $s = k+1$ and $r = k+1+d$. This proves the induction step (b). Now (iii) follows from (a) and (b).

6. Concluding remarks

We study the distributional equation $X + T \overset{d}{=} Y$, where the shift (translator) T is a sum of i.i.d. random variables without a specified distribution. The main result in this paper is a characterization of the exponential distribution via a relationship involving a pair of maxima of i.i.d. continuous random variables. As a corollary, we prove that the Sukhatme–Rényi decomposition of maxima is also a characterization property for the exponential distribution.

The proof of the main theorem uses a new technique based on an argument from Arnold and Villaseñor (2013), which requires analyticity of the density function. It is an open question if this assumption can be weakened.

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