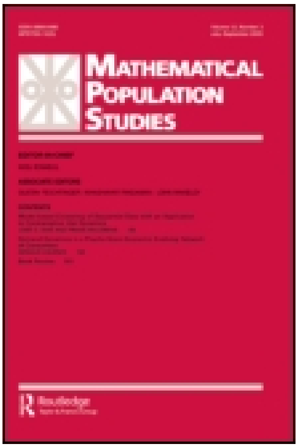

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VARIANCE ESTIMATORS IN CRITICAL BRANCHING PROCESSES WITH NON-HOMOGENEOUS IMMIGRATION

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The asymptotic normality of conditional least squares estimators for the offspring variance in critical branching processes with nonhomogeneous immigration is established, under moment assumptions on both reproduction and immigration. The proofs use martingale techniques and weak convergence results in Skorokhod spaces.

Keywords: branching processes; immigration; least squares estimators; offspring variance; Skorokhod space

1. INTRODUCTION

Kimmel and Axelrod (2002), Pakes (2003), and Haccou et al. (2005) survey applications of branching stochastic models in genetics, molecular biology, and microbiology. Yakovlev and Yanev (2006) point out that *in vivo* cell kinetics requires stochastic modeling of renewing cell populations with nonhomogeneous immigration. In this line, Hyrien and Yanev (2010) model renewing cell populations where the experimentally observable cells are supplemented by unobservable cells, for example, stem cells. They analyze the population of terminally differentiated oligodendrocytes of the central nervous system and the population of leukaemia cells. In both cases, the cell population expands through both division of existing (progenitor) cells and differentiation of stem cells. This dynamics belongs to branching processes with nonhomogeneous immigration. The population's viability (Jagers and Harding, 2009) is preserved by allowing the immigration distribution to vary in time, increasing to infinity on average. For semi-stochastic models where immigration depends on the state of the process, we refer to Cairns (2009).

In our case of time-dependent immigration (Rahimov, 1995), $\{X_{n,i}\}_{n,i\geq 1}$ and $\{\xi_n\}_{n\geq 1}$ are two families of independent, nonnegative, and integer valued random

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variables on a probability space (Ω, $\mathfrak{F}$, P). We consider discrete-time branching processes with time-dependent immigration defined recursively by:

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_{n,i} + \xi_n, \qquad n \geq 1; \quad Z_0 = 0, \tag{1}$$

where the $\{X_{n,i}\}_{n,i\geq 1}$ have a common distribution for all n and i, and the sequences $\{X_{n,i}\}_{n,i\geq 1}$ and $\{\xi_n\}_{n\geq 1}$ are independent of each other. $X_{n,i}$ is the number of offspring of the ith individual from the (n–1)th generation and ξ_n the number of immigrants (or invaders; Haccou et al., 2005) joining the population at the time of birth of the nth generation. Z_n represents the nth population size, and the independence of $\{X_{n,i}\}_{n,i\geq 1}$ from $\{\xi_n\}_{n\geq 1}$ implies the independence of reproduction and immigration. In contrast to usual branching process models with immigration, we do not assume $\{\xi_n\}_{n\geq 1}$ to be identically distributed, but that the immigration rate varies from generation to generation.

Our goal is to estimate the offspring variance $b^2 := \mathrm{Var}X_{1,1}$ assuming that the immigration mean and variance are known. We estimate b^2 based on observing a single trajectory $\{Z_1, Z_2, \ldots, Z_n\}$ as $n \to \infty$. For reproduction, we assume

$$\mathrm{E}X_{1,1} = 1 \quad \text{and} \quad \mathrm{E}X_{1,1}^4 < \infty. \tag{2}$$

For immigration, let $\alpha_n := \mathrm{E}\xi_n$, $\beta_n^2 := \mathrm{Var}\xi_n$, and $\gamma_n^4 := \mathrm{Var}(\xi_n - \alpha_n)^2$ be finite for every n and varying regularly at infinity functions of n defined by:

$$\alpha_n = n^{\alpha} L_{\alpha}(n), \quad \beta_n^2 = n^{\beta} L_{\beta}(n), \quad \text{and} \quad \gamma_n^4 = n^{\gamma} L_{\gamma}(n), \tag{3}$$

where α, β, and γ are nonnegative and $\lim_{x\to\infty} L_{(\cdot)}(cx)/L_{(\cdot)}(x) = 1$ for any $c > 0$. The immigration mean increases to infinity:

$$\lim_{n\to\infty} \alpha_n = \infty, \tag{4}$$

and the immigration moments, in addition to Eq. (3), satisfy:

$$\text{(i)} \lim_{n\to\infty} \frac{\beta_n^2}{n\alpha_n} = 0 \quad \text{or} \quad \text{(ii)} \lim_{n\to\infty} \frac{\beta_n^2}{n\alpha_n^2} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{n\alpha_n\beta_n^2}{\gamma_n^4} = 0. \tag{5}$$

Eq. (5) holds true if $\beta < \alpha + 1$ or ($\beta < 2\alpha + 1$ and $\alpha + \beta + 1 < \gamma$). Finally, setting $\widetilde{\eta}_n := (\xi_n - \alpha_n)^2 - \beta_n^2$, we assume that $\{\widetilde{\eta}_n\}_{n\geq 1}$ satisfies the Lindeberg condition:

$$\lim_{n\to\infty} \sum_{k=1}^{n} E\Big(\widetilde{\eta}_k^2 \chi(|\widetilde{\eta}_k| > \varepsilon)\Big) = 0, \tag{6}$$

where $\chi(A)$ denotes the indicator of the event A.

To construct the conditional least squares estimator (CLSE) for the offspring variance, consider $\Im(k)$ the σ-algebra generated by $\{Z_j\}_{j=1}^{k}$. Denote $\mathrm{E}_k(\cdot) := \mathrm{E}(\cdot\,|\Im(k))$ and define:

$$M_k := Z_k - \mathrm{E}_{k-1} Z_k = Z_k - Z_{k-1} - \alpha_k. \tag{7}$$

Denote $V_k := M_k^2 - \mathrm{E}_{k-1} M_k^2$. The recurrence Eq. (1) yields the stochastic regression equation

$$M_k^2 = b^2 Z_{k-1} + \beta_k^2 + V_k. \tag{8}$$

The error terms V_k form a martingale difference sequence $\{V_k\}_{k\geq 1}$ with respect to the filtration $\{\Im(k)\}_{k\geq 1}$, that is, V_k is $\Im(k-1)$-measurable and $\mathrm{E}_{k-1} V_k = 0$. If α_k and β_k^2 are known and minimizing with respect to b^2 the sum of squares

$$\sum_{k=1}^{n} V_k^2 = \sum_{k=1}^{n} \left(M_k^2 - b^2 Z_{k-1} - \beta_k^2\right)^2 \tag{9}$$

the CLSE for b^2 is:

$$\widehat{b_n^2} := \frac{\sum_{k=1}^{n} (M_k^2 - \beta_k^2) Z_{k-1}}{\sum_{k=1}^{n} Z_{k-1}^2} = \frac{\sum_{k=1}^{n} \left((Z_k - Z_{k-1} - \alpha_k)^2 - \beta_k^2\right) Z_{k-1}}{\sum_{k=1}^{n} Z_{k-1}^2}. \tag{10}$$

For $\{Z_n^*\}_{n\geq 0}$, a simple branching process with homogeneous immigration, assuming that both offspring and immigration means are known, the CLSE for the offspring variance is given by

$$\widehat{b_n^{2*}} := \frac{\sum_{k=1}^{n} (M_k^*)^2 (Z_{k-1}^* - \overline{Z}_n^*)}{\sum_{k=1}^{n} (Z_{k-1}^* - \overline{Z}_n^*)^2}, \tag{11}$$

where $\overline{Z}_n^* := n^{-1} \sum_{k=1}^{n} Z_{k-1}^*$ and M_k^* is defined with the first equality of Eq. (7), replacing Z_k by Z_k^*. Yanev (1976/77) and Yanev and Tchoukova-Dantcheva (1986) pioneered the study of $\widehat{b_n^{2*}}$; Winnicki (1991) proved limit theorems for $\widehat{b_n^{2*}}$ assuming finite fourth moments for both reproduction and homogeneous immigration (surveys in Dion, 1993; Yanev, 2008). Ma and Wang (2010) studied the case when these fourth moments may be infinite.

We extend Winnicki (1991) findings in the critical case $\mathrm{E}X_{1,1} = 1$ allowing the immigration distribution to vary with time such that its mean α_n increases to infinity at a speed that could correspond to a near-critical branching population. We prove a limit theorem, which completes the results of Rahimov (2008a) in that the limiting distribution of the CLSE for the offspring mean depends on the offspring variance b^2. We establish asymptotical normality of the CLSE defined by Eq. (10).

We shall express $\widehat{b_n^2} - b^2$, suitably normalized, as a sum of martingale differences and functionals of $\{Z_k\}_{k=1}^n$. Then we study the asymptotic behavior of $\widehat{b_n^2} - b^2$ by applying either a limit theorem due to Rahimov (2007) giving the limit constant of some functionals of $\{Z_k\}_{k=1}^n$ (Lemma 1) or a general limit theorem, given as Lemma 2, for sequences of martingale differences in continuous time (Jacod and Shiryaev, 2003) to each term of the sum.

Define $A_n := \mathrm{E}Z_n, \tau_n^2 := \sum_{k=1}^n \gamma_k^4$, and $\theta_n := nA_n^2(nA_n^2 + \tau_n^2)^{-1}$. We have

$$\lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \frac{nA_n^2}{nA_n^2 + \tau_n^2} =: \theta \in [0, 1]. \tag{12}$$

Equality in distribution is denoted by "$\stackrel{d}{=}$".

Theorem 1. *Assume Eq. (2)–(6) hold true. Then*

$$\lim_{n\to\infty} (\theta_n n)^{1/2} \left(\widehat{b_n^2} - b^2\right) \stackrel{d}{=} N(0, \sigma^2), \tag{13}$$

where $N(0, \sigma^2)$ is a normal random variable with zero mean and variance

$$\sigma^2 = (2\alpha + 3)^2 \left(\theta \frac{2b^4}{4\alpha + 5} + (1 - \theta) \frac{\gamma + 1}{2\alpha + 3 + \gamma}\right). \tag{14}$$

For the critical process with homogeneous immigration, Winnicki (1991) established the weak limit of $\widehat{b_n^{2*}}$ with rate of convergence $n^{1/2}$. In Theorem 1 the convergence rate is $(\theta_n n)^{1/2}$, where $\lim_{n\to\infty}\theta_n = \theta \in [0, 1]$ and $\lim_{n\to\infty}\theta_n n = \infty$, provided $\lim_{n\to\infty} n^3\alpha_n^2/\gamma_n^4 = 0$.

Corollary 1. *Under the assumptions of Theorem 1,*

- *(i) If $\lim_{n\to\infty} n^2\alpha_n^2/\gamma_n^4 = 0$, then Eq. (13)–(14) hold true with $\theta = 0$.*
- *(ii) If $\lim_{n\to\infty} n^2\alpha_n^2/\gamma_n^4 = \infty$, then Eq. (13)–(14) hold true with $\theta = 1$.*

Example 1 (Poisson immigration). For $\{\xi_n\}_{n\geq 1}$ independent Poisson variables with mean $\alpha_n = n^\alpha L_\alpha(n) = o(n) \to \infty$ as $n \to \infty$, Eq. (4)–(6) and the condition (ii) in Corollary 1 satisfied, if Eq. (2) holds true, then Theorem 1 implies Eq. (13)–(14) with $\theta = 1$.

Example 2 (Neyman Type A immigration). If Eq. (2) holds true and for $\{\xi_n\}_{n\geq 1}$ independent with Neyman Type A distribution given by $Ez^{\xi_n} = \exp(\lambda_n(e^{\varphi n(z-1)} - 1))$, $|z| < 1$, if $\lambda_n = n^\lambda L_\lambda(n) \to \infty$, ($\lambda \geq 0$) and $\varphi_n = n^\varphi L_\varphi(n)$, ($\varphi \geq 0$), then (Johnson et al., 1993: 371) the rth factorial moment $\mu_n(r)$ satisfies $\mu_n(r) \sim \lambda_n^r \varphi_n^r$ for $r \geq 1$ and $n \to \infty$. Also $\{\widetilde{\eta}_n\}_{n\geq 1}$ satisfies Eq. (6). If $0 < \lambda + \varphi \leq 1/2$, then Eq. (4) and (5)(i) hold true and hence Eq. (13)–(14) with $\theta = 1$ and $\alpha = \lambda + \varphi$. If $1 < \lambda + \varphi < 3/2$, then Eq. (4) and (5)(ii) are satisfied, which yields Eq. (13)–(14) with $\theta = 0$ and $\alpha = \lambda + \varphi$.

2. PRELIMINARIES

The proof of the theorem uses auxiliary results given in this section. "D" denotes convergence or equality in the Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$, "$P$" in probability, and "$d$" in distribution. The first lemma summarizes limit results for functionals of Eq. (1). Its proof is similar to that of Corollary 2 in Rahimov (2008b) and is omitted here.

Lemma 1. *For the critical process (1) with $b^2 < \infty$, $\lim_{n\to\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}\beta_n^2(n\alpha_n^2)^{-1} = 0$, for any continuous function Φ on $\mathbb{R}_+$ and any sequence $\{c_n\}_{n\geq 0}$, varying regularly at infinity with exponent $\rho \geq 0$, for $t > 0$ we have:*

$$\lim_{n\to\infty} \frac{1}{nc_n} \sum_{k=0}^{[nt]} c_k \Phi\left(\frac{Z_k}{A_n}\right) \stackrel{P}{=} \int_0^t u^\rho \Phi(u^{\alpha+1})\, du. \tag{15}$$

A necessary and sufficient condition for weak convergence in a Skorokhod space of a sequence of martingale differences (Jacod and Shiryaev, 2003: Theorem VIII.2.29; Ispàny et al., 2006) is:

Lemma 2 (CLT for Martingales). *For a sequence of martingale differences $\{U_k^n\}_{k\geq 1}$, $n \geq 1$, with respect to a filtration $\{\Im_k^n\}_{k\geq 1}$, such that for all $\varepsilon > 0$ and $t \geq 0$ the Lindeberg condition*

$$\lim_{n\to\infty} \sum_{k=1}^{[nt]} \mathrm{E}\Big((U_k^n)^2 \chi(|U_k^n| > \varepsilon) \mid \Im_{k-1}^n\Big) \stackrel{P}{=} 0 \tag{16}$$

holds true. Then

$$\lim_{n\to\infty} \sum_{k=1}^{[nt]} U_k^n \stackrel{D}{=} U(t), \tag{17}$$

where $U(t)$ is a continuous Gaussian martingale with mean zero and covariance function $C(t)$ if and only if for every $t \geq 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{[nt]} \mathrm{E}\Big((U_k^n)^2 \mid \Im_{k-1}^n\Big) \stackrel{P}{=} C(t). \tag{18}$$

We use a tilde to indicate that a random variable ζ is centered around its mean, that is $\widetilde{\zeta} = \zeta - \mathrm{E}\zeta$. Denote $Y_{k,i} := \widetilde{X}_{k,i}^2$, $S_j := \sum_{i=1}^{j} \widetilde{X}_{k,i}$, and recall that $\eta_n := (\xi_n - \alpha_n)^2$. The following expansion of the error term V_k plays a key role in our analysis:

$$\begin{aligned} V_k &= M_k^2 - \mathrm{E}_{k-1} M_k^2 \\ &= (Z_k - Z_{k-1} - \alpha_k)^2 - b^2 Z_{k-1} - \beta_k^2 \\ &= \left(2\sum_{j=2}^{Z_{k-1}} \widetilde{X}_{k,j} S_{j-1} + \widetilde{\eta}_k\right) + 2\widetilde{\xi}_k \sum_{i=1}^{Z_{k-1}} \widetilde{X}_{k,i} + \sum_{i=1}^{Z_{k-1}} \widetilde{Y}_{k,i} \\ &=: V_k^{(1)} + V_k^{(2)} + V_k^{(3)}. \end{aligned} \tag{19}$$

Denoting $V_n^{(i)}(t) := H_n^{-1} \sum_{k=1}^{[nt]} V_k^{(i)}$, $1 \le i \le 3$, where $H_n^2 = nA_n^2 + \tau_n^2$, we decompose the normalized sum of the error terms into three parts:

$$V_n(t) := \frac{1}{H_n} \sum_{k=1}^{[nt]} V_k = V_n^{(1)}(t) + V_n^{(2)}(t) + V_n^{(3)}(t), \quad t > 0. \tag{20}$$

We shall show that the asymptotic behavior of $V_n(t)$ as $n \to \infty$ is governed by $V_n^{(1)}(t)$, while the contributions of $V_n^{(2)}(t)$ and $V_n^{(3)}(t)$ are negligible. Define

$$V(t) := W\left(\theta \frac{2b^4}{2\alpha + 3} t^{2\alpha+3} + (1-\theta) t^{\gamma+1}\right), \quad t > 0, \tag{21}$$

where $W(t)$ is a standard Wiener process and θ is the limiting constant in Eq. (12).

Proposition 1. *If Eq. (2)–(5), then for every* $t > 0$

$$\lim_{n\to\infty} V_n^{(1)}(t) \stackrel{D}{=} V(t). \tag{22}$$

Proof. Denote $T_k := 2\sum_{j=2}^{Z_{k-1}} \widetilde{X}_{k,j} S_{j-1}$. The independence of $\{X_{n,i}\}_{n,i\ge1}$ and Lemma 1 yield

$$\begin{aligned} \sum_{k=1}^{[nt]} \mathrm{E}_{k-1}\left(\frac{V_k^{(1)}}{H_n}\right)^2 &= \frac{1}{H_n^2}\sum_{k=1}^{[nt]} \mathrm{E}_{k-1} T_k^2 + \frac{1}{H_n^2}\sum_{k=1}^{[nt]} \mathrm{E}_{k-1}\widetilde{\eta}_k^2 \\ &= \frac{2b^4\theta_n}{nA_n^2}\sum_{k=1}^{[nt]} Z_{k-1}(Z_{k-1}-1) + \frac{1-\theta_n}{\tau_n^2}\sum_{k=1}^{[nt]} \gamma_k^4 \\ &\xrightarrow{P} \frac{2b^4\theta}{2\alpha+3} t^{2\alpha+3} + (1-\theta)t^{\gamma+1}. \end{aligned} \tag{23}$$

Then $V_k^{(1)}/H_n$ satisfies Eq. (18) with $C(t) = 2b^4\theta t^{2\alpha+3}(2\alpha+3)^{-1} + (1-\theta)t^{\gamma+1}$. We shall verify Eq. (16). Indeed,

$$\begin{aligned} &\sum_{k=1}^{[nt]} \mathrm{E}_{k-1}\left(\left(\frac{V_k^{(1)}}{H_n}\right)^2 \chi\left(\left|\frac{V_k^{(1)}}{H_n}\right| > \varepsilon\right)\right) \\ &\quad\le \frac{1}{H_n^2}\sum_{k=1}^{[nt]} \mathrm{E}_{k-1}\Big((T_k^2 + \widetilde{\eta}_k^2)\chi(|T_k + \widetilde{\eta}_k| > \varepsilon H_n)\Big) \\ &\quad\le \frac{1}{H_n^2}\sum_{k=1}^{[nt]} \mathrm{E}_{k-1}\left(T_k^2\,\chi\left(|\widetilde{\eta}_k| > \frac{\varepsilon H_n}{2}\right) + T_k^2\,\chi\left(|T_k| > \frac{\varepsilon H_n}{2}\right)\right) \\ &\qquad + \frac{1}{H_n^2}\sum_{k=1}^{[nt]} \mathrm{E}_{k-1}\left(\widetilde{\eta}_k^2\,\chi\left(|\widetilde{\eta}_k| > \frac{\varepsilon H_n}{2}\right) + \widetilde{\eta}_k^2\,\chi\left(|T_k| > \frac{\varepsilon H_n}{2}\right)\right) \\ &\qquad =: I_1(n) + I_2(n) + I_3(n) + I_4(n). \end{aligned} \tag{24}$$

The independence assumption, Chebyshev inequality, and Lemma 1 imply

$$\begin{aligned}
I_1(n)+I_4(n) &= \frac{1}{H_n^2}\sum_{k=1}^{[nt]}\left(\mathrm{E}_{k-1}(T_k^2)P\left(|\widetilde{\eta}_k|>\frac{\varepsilon H_n}{2}\right)+\mathrm{E}_{k-1}(\widetilde{\eta}_k^2)P\left(|T_k|>\frac{\varepsilon H_n}{2}\right)\right)\\
&\le \frac{4}{\varepsilon^2 H_n^4}\sum_{k=1}^{[nt]}(\mathrm{Var}_{k-1}T_k\,\mathrm{Var}\widetilde{\eta}_k+\mathrm{Var}\widetilde{\eta}_k\,\mathrm{Var}_{k-1}T_k)\\
&= \frac{8b^4\theta_n(1-\theta_n)}{\varepsilon^2\tau_n^2}\frac{1}{n}\sum_{k=1}^{[nt]}\frac{Z_{k-1}(Z_{k-1}-1)}{A_n^2}\gamma_k^4\\
&\xrightarrow{P} 0.
\end{aligned} \tag{25}$$

For $I_3(n)$, referring to the Lindeberg condition in Eq. (6), we have

$$\begin{aligned}
I_3(n) &= \frac{1}{H_n^2}\sum_{k=1}^{[nt]}\mathrm{E}_{k-1}\left(\widetilde{\eta}_k^2\,\chi\left(|\widetilde{\eta}_k|>\frac{\varepsilon H_n}{2}\right)\right)\\
&\le \frac{1-\theta_n}{\tau_n^2}\sum_{k=1}^{[nt]}\mathrm{E}_{k-1}\left(\widetilde{\eta}_k^2\,\chi\left(|\widetilde{\eta}_k|>\frac{\varepsilon\tau_n}{2}\right)\right)\\
&\xrightarrow{P} 0.
\end{aligned} \tag{26}$$

It remains to show $\lim_{n\to\infty} I_2(n)\stackrel{P}{=}0$. From Burkholder inequality, for $\delta>0$ and $p>1$,

$$\mathrm{E}\left|\sum_{j=2}^{r}\widetilde{X}_{k,j}S_{j-1}\right|^{2p}\le D_{1,p}\mathrm{E}\left|\sum_{j=2}^{r}\widetilde{X}_{k,j}^2S_{j-1}^2\right|^{p}, \tag{27}$$

where $D_{1,p}>0$ depends on p only. Using Minkovski inequality,

$$\left(\mathrm{E}\left|\sum_{j=2}^{r}\widetilde{X}_{k,j}^2S_{j-1}^2\right|^{p}\right)^{\frac{1}{p}}\le\sum_{j=2}^{r}\left(\mathrm{E}\left(|\widetilde{X}_{k,j}|^{2p}|S_{j-1}|^{2p}\right)\right)^{\frac{1}{p}}=b_{2p}^{\frac{1}{p}}\sum_{j=2}^{r}\left(\mathrm{E}|S_{j-1}|^{2p}\right)^{\frac{1}{p}}, \tag{28}$$

where, by assumption, $b_{2p}:=\mathrm{E}|\widetilde{X}_{1,1}|^{2p}<\infty$ for $0<p\le 2$. Similarly, we obtain

$$\mathrm{E}|S_{j-1}|^{2p}\le D_{2,p}\mathrm{E}\left|\sum_{i=1}^{j-1}\widetilde{X}_{k,i}^2\right|^{p}\le D_{2,p}b_{2p}(j-1)^p, \tag{29}$$

where $D_{2,p}>0$ depends on p only. From Eq. (27)–(29),

$$\mathrm{E}\left|\sum_{j=2}^{r}\widetilde{X}_{k,j}S_{j-1}\right|^{2p}\le D_{1,p}D_{2,p}b_{2p}\left[\sum_{j=2}^{r}(j-1)\right]^{p}\le D_pb_{2p}r^{2p}, \tag{30}$$

where $D_p > 0$ depends on p only. Applying Eq. (30) with $p = (2+\delta)/2$ to $\mathrm{E}_{k-1}|T_k|^{2+\delta}$, for every $\varepsilon > 0$ and $\delta \geq 0$:

$$\begin{aligned} I_2(n) &\leq \frac{1}{\varepsilon^{\delta} H_n^{2+\delta}} \sum_{k=1}^{[nt]} \mathrm{E}_{k-1}|T_k|^{2+\delta} \\ &\leq \frac{D_\delta b_{2+\delta}}{\varepsilon^{\delta} H_n^{2+\delta}} \sum_{k=1}^{[nt]} Z_{k-1}^{2+\delta} \\ &= \frac{D_\delta b_{2+\delta} \theta_n}{\varepsilon^{\delta}} \left(\frac{A_n}{H_n}\right)^{\delta} \frac{1}{n} \sum_{k=1}^{[nt]} \left(\frac{Z_{k-1}}{A_n}\right)^{2+\delta} \\ &\xrightarrow{P} 0, \end{aligned} \tag{31}$$

where we have used Lemma 1 and $\lim_{n\to\infty} A_n/H_n = 0$. Hence, $V_k^{(1)}/H_n$ satisfies Eq. (16) and because $V_k^{(1)}$ is a martingale difference with respect to $\{\Im_k\}_{k\geq 1}$, all assumptions of Lemma 2 are verified. Lemma 2 implies Eq. (22).

We derive the limit of the normalized sum in Eq. (20).

Proposition 2. *If Eq. (2)–(6), then for every $t > 0$*

$$\lim_{n\to\infty} V_n(t) \stackrel{D}{=} V(t). \tag{32}$$

Proof. From Eq. (20) and Proposition 1, in order to prove Eq. (32), it remains to establish that both $V_n^{(2)}(t)$ and $V_n^{(3)}(t)$ are asymptotically negligible. First we show

$$\lim_{n\to\infty} V_n^{(3)}(t) \stackrel{P}{=} 0. \tag{33}$$

Observing that $\lim_{n\to\infty} nA_n/H_n^2 = 0$ and applying Lemma 1, we obtain

$$\begin{aligned} \sum_{k=1}^{[nt]} \mathrm{E}_{k-1}\left(\left(\frac{V_k^{(3)}}{H_n}\right)^2 \chi\left(\left|\frac{V_k^{(3)}}{H_n}\right| > \varepsilon\right)\right) &\leq \frac{1}{H_n^2} \sum_{k=1}^{[nt]} \mathrm{E}_{k-1}\left(V_k^{(3)}\right)^2 \\ &= \frac{\theta_n}{A_n} \frac{\mathrm{E}\widetilde{Y}_{1,1}^2}{n} \sum_{k=1}^{[nt]} \frac{Z_{k-1}}{A_n} \\ &\xrightarrow{P} 0, \end{aligned} \tag{34}$$

where $\mathrm{E}\widetilde{Y}_{1,1}^2 = \mathrm{E}X_{1,1}^4 - 4\mathrm{E}X_{1,1}^3 - 4b^4 + 3b^2 + 3 < \infty$ (due to Eq. (2)). Then, Eq. (16) and Eq. (18) with $C(t) \equiv 0$ hold true. Lemma 2 yields $V_n^{(3)}(t) \xrightarrow{D} 0$ and hence Eq. (33).

As for $V_n^{(2)}(t)$, applying Lemma 1, we obtain

$$\begin{aligned}\sum_{k=1}^{[nt]} \mathrm{E}_{k-1}\left(\frac{V_k^{(2)}}{H_n}\right)^2 \chi\left(\left|\frac{V_k^{(2)}}{H_n}\right| > \varepsilon\right) &\le \frac{1}{H_n^2}\sum_{k=1}^{[nt]} \mathrm{E}_{k-1}\left(V_k^{(2)}\right)^2 \\ &= \frac{\beta_n^2\theta_n}{A_n}\frac{4b^2}{n\beta_n^2}\sum_{k=1}^{[nt]} \beta_k^2 \frac{Z_{k-1}}{A_n} \\ &\xrightarrow{P} 0,\end{aligned} \tag{35}$$

where, according to Eq. (5), $\lim_{n\to\infty} \beta_n^2\theta_n/A_n = 0$. Therefore, Eq. (16) and (18) with $C(t) \equiv 0$ are satisfied. From Lemma 2 again, we have $\lim_{n\to\infty} V_n^{(2)}(t) \stackrel{D}{=} 0$ and hence

$$\lim_{n\to\infty} V_n^{(2)}(t) \stackrel{P}{=} 0. \tag{36}$$

Eq. (32) follows from Proposition 1, Eq. (33), Eq. (36), and Slutsky theorem.

3. PROOF OF THEOREM 1

Recalling Eq. (10) and the definition of V_k, we have

$$\widehat{b_n^2} - b^2 = \frac{\sum_{k=1}^{n}\left(M_k^2 - \beta_k^2 - b^2 Z_{k-1}\right)Z_{k-1}}{\sum_{k=1}^{n} Z_{k-1}^2} = \frac{\sum_{k=1}^{n} V_k Z_{k-1}}{\sum_{k=1}^{n} Z_{k-1}^2}. \tag{37}$$

First, we examine the asymptotic behavior as $n \to \infty$ of the numerator in Eq. (37). Theorem 3.1 in Rahimov (2009) and Proposition 2 for $t > 0$ imply:

$$\lim_{n\to\infty} Z_n(t) = \lim_{n\to\infty} \frac{Z_{[nt]}}{A_n} \stackrel{D}{=} t^{\alpha+1} \quad \text{and} \quad \lim_{n\to\infty} V_n(t) = \lim_{n\to\infty} \frac{1}{H_n}\sum_{k=1}^{[nt]} V_k \stackrel{D}{=} V(t), \tag{38}$$

where the convergence is on $D(\mathbb{R}_+, \mathbb{R}_+)$ and $D(\mathbb{R}_+, \mathbb{R})$, respectively. Since both limiting processes are continuous, referring to Theorem 2.2 in Kurtz and Protter (1991),

$$\lim_{n\to\infty}\left(Z_n(t),\ V_n(t),\ \int_0^1 Z_n(u)\, dV_n(u)\right) \stackrel{D}{=} \left(t^{\alpha+1},\ V(t),\ \int_0^1 u^{\alpha+1}\, dV(u)\right), \tag{39}$$

on $D(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R})$. Eq. (39) and the continuous mapping theorem (Billingsley, 1968: Theorem 5.5) yield:

$$\begin{aligned}\frac{1}{H_n A_n}\sum_{k=1}^{n} V_k Z_{k-1} &= \sum_{k=1}^{n-1} Z_n\left(\frac{k}{n}\right)\left(V_n\left(\frac{k+1}{n}\right) - V_n\left(\frac{k}{n}\right)\right) \\ &= \int_0^1 Z_n(u)\, dV_n(u) \\ &\xrightarrow{d} \int_0^1 u^{\alpha+1}\, dV(u).\end{aligned} \tag{40}$$

For the denominator in Eq. (37), applying Lemma 1 with $t = 1$, we have

$$\lim_{n\to\infty} \frac{1}{nA_n^2} \sum_{k=1}^{n} Z_{k-1}^2 \stackrel{P}{=} \frac{1}{2\alpha+3}. \tag{41}$$

Finally, Eq. (40) and (41) and Slutsky theorem yield:

$$\lim_{n\to\infty} \frac{nA_n}{H_n}\left(\widehat{b_n^2} - b^2\right) = \lim_{n\to\infty} \frac{(H_n A_n)^{-1} \sum_{k=1}^{n} V_k Z_{k-1}}{(nA_n^2)^{-1} \sum_{k=1}^{n} Z_{k-1}^2} \stackrel{d}{=} (2\alpha+3) \int_0^1 u^{\alpha+1}\, dV(u), \tag{42}$$

which implies the normality of the limit in Eq. (13). Ito's formula yields:

$$\int_0^1 u^{\alpha+1} dV(u) = V(1) - (\alpha+1) \int_0^1 u^{\alpha} V(u) du = (\alpha+1) \int_0^1 (V(1) - V(u)) u^{\alpha}\, du. \tag{43}$$

The variable $\zeta := \int_0^1 (V(1) - V(u)) u^{\alpha}\, du$ is normally distributed with zero mean and

$$\mathrm{E}\zeta^2 = \int_0^1 \int_0^1 s^{\alpha} t^{\alpha} \mathrm{E}((V(1) - V(t))(V(1) - V(s)))\, ds\, dt. \tag{44}$$

For $0 \le s \le t \le 1$,

$$\begin{aligned} &\mathrm{E}((V(1) - V(t))(V(1) - V(s))) \\ &\quad = \mathrm{E}((V(1) - V(t))^2 + (V(1) - V(t))(V(t) - V(s))) \\ &\quad = \theta \frac{2b^4}{2\alpha+3}\left(1 - t^{2\alpha+3}\right) + (1-\theta)\left(1 - t^{\gamma+1}\right), \end{aligned} \tag{45}$$

we calculate

$$\mathrm{E}\zeta^2 = \frac{1}{(\alpha+1)^2}\left(\theta \frac{2b^4}{4\alpha+5} + (1-\theta)\frac{\gamma+1}{2\alpha+\gamma+3}\right), \tag{46}$$

which, taking into account Eq. (42)–(43), implies Eq. (14) and completes the proof.

4. CONCLUSION

Branching processes with time-dependent immigration are encountered in a variety of applications to population biology. We study conditional least-squares estimators for the offspring variance in the critical case, assuming that the immigration mean increases to infinity over time. Theorem 1 establishes the asymptotic normality of the proposed estimators with convergence rate $(\theta_n n)^{1/2}$, where $\lim_{n\to\infty} \theta_n = \theta \in [0, 1]$. A next question concerns the conditional consistency of the estimators with different weights.

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