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On characterizations based on regression of linear combinations of record values

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Abstract

We characterize the exponential distribution in terms of the regression of a record value with two non-adjacent record values as covariates. We also study characterizations based on the regression of linear combinations of record values.

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1 Introduction and results

There is a number of studies on characterizations of probability distributions by means of regression relations of one record value with one or two other record values as covariates. For a recent review paper on the subject we refer to Pakes (2004), see also Ahsanullah and Raqab (2006), Chapter 6. To formulate and discuss our results we need to introduce some notations. Let $X_1, X_2, \ldots$ be independent copies of a random variable X with absolutely continuous (with respect to the Lebesgue measure) distribution function $F(x)$. An observation X_j is called a (upper) record value if it exceeds all previous observations, i.e., X_j is a (upper) record if $X_j > X_i$ for all $i < j$. If we define the record times sequence by $T_1 = 1$ and $T_n = \min\{j : X_j > X_{T_{n-1}}, j > T_{n-1}\}$, for $n > 1$, then the corresponding record values are $R_n = X_{T_n}$, $n = 1, 2, \ldots$ Let $F(x)$ be the cumulative distribution function of an exponential distribution given by

$$F(x) = 1 - e^{-c(x-l_F)}, \qquad -\infty < l_F \le x, \tag{1.1}$$

where $c > 0$ is an arbitrary constant. Let us mention that (1.1) with $l_F > 0$ appears, for example, in reliability studies where l_F represents the guarantee time; that is, failure cannot occur before l_F units of time have elapsed (see Barlow and Proschan (1996), p.13).

Bairamov et al. (2005) study characterizations of exponential and related distributions in terms of the regression of R_n with two adjacent record values as covariates. They prove that $F(x)$ is exponential if and only if

$$E[h'(R_n)|R_{n-1} = u, R_{n+1} = v] = \frac{h(v) - h(u)}{v - u}, \qquad l_F < u < v, \tag{1.2}$$

where the function h satisfies some regularity conditions. Let us note that if (1.2) holds, then by the mean-value theorem, there exists at least one number ξ inside the interval (u, v) such that for $u > l_F$

$$E[h'(R_n)|R_{n-1} = u, R_{n+1} = v] = h'(\xi).$$

In particular, if $h'(x) = x$ then (1.2) becomes

$$E[R_n|R_{n-1} = u, R_{n+1} = v] = \frac{u + v}{2}, \qquad l_F < u < v.$$

Yanev et al. (2008) extend (1.2) to the case when at least one of the two covariates is adjacent to R_n. To formulate their result, we need to introduce some notations. Further on, for a given $h(x)$, we denote

$$M(u, v) = \frac{h(v) - h(u)}{v - u}, \ {}_iM_j(u, v) = \frac{\partial^{i+j}}{\partial u^i \partial v^j}\left(\frac{h(v) - h(u)}{v - u}\right), u \neq v,$$

as well as ${}_iM(u, v)$ and $M_j(u, v)$ for the ith and jth partial derivative of $M(u, v)$ with respect to u and v, respectively. Let k and n be integers, such that $1 \leq k \leq n - 1$. When at least one of the covariate record values is adjacent to R_n, it is shown in Yanev et al. (2008), under some regularity assumptions, that $F(x)$ is exponential if and only if

$$E[h^{(k)}(R_n)|R_{n-k} = u, R_{n+1} = v] = kM_{k-1}(u, v), \qquad l_F < u < v. \tag{1.3}$$

In particular, if $h(x) = x^{k+1}/(k + 1)!$, then $h^{(k)}(x) = x$ and (1.3) becomes

$$E[R_n|R_{n-k} = u, R_{n+1} = v] = \frac{u + kv}{k + 1}, \qquad l_F < u < v. \tag{1.4}$$

Observe that the right-hand side of (1.4) is the weighted mean of the covariates with weights equal to the number of spacings they are away from R_n. One can also see (using the arguments in Yanev et al. (2008)) that for $r \geq 1$

$$E[R_n|R_{n-1} = u, R_{n+r} = v] = \frac{ru + v}{r+1}, \qquad l_F < u < v \tag{1.5}$$

characterizes the exponential distribution too.

If both covariates are non-adjacent to R_n the situation is more complex. Let k, r, and n be integers, such that $1 \leq k \leq n-1$ and $r \geq 1$. Yanev et al (2008) obtain a necessary condition for exponentiality of $F(x)$. Namely, they prove, under some regularity assumptions, that if $F(x)$ is exponential, then for $l_F < u < v$

$$E[h^{(k+r+1)}(R_n)|R_{n-k} = u, R_{n+r} = v] = \frac{(k+r-1)!}{(k-1)!(r-1)!}\, {}_{r-1}M_{k-1}(u, v).$$

However, no sufficient condition for $F(x)$ to be exponential that involves only single regression of R_n on two non-adjacent covariates is known yet. For example, in Yanev et al. (2008) the necessary and sufficient condition for $F(x)$ to be exponential is that both

$$E[R_n|R_{n-k} = u, R_{n+r} = v] = \frac{ru + kv}{k+r}, \qquad l_F < u < v$$

and

$$E[R_n|R_{n-k+1} = s, R_{n+r} = v] = \frac{rs + (k-1)v}{k+r-1}, \qquad l_F < s < v$$

hold. Our first result provides new sufficient and necessary conditions for (1.1) when both covariates are non-adjacent to R_n. The conditions given below are written in a form which extends (1.3). They are alternative to and more compact than the results in Theorem 1B of Yanev et al. (2008). We have the following theorem.

Theorem 1. *Let k, r, and n be integers, such that $1 \leq k \leq n-1$ and $r \geq 1$. Assume that $F(x)$ is absolutely continuous. Suppose $h(x)$ satisfies*

(i) $h(x)$ is continuous in $[l_F, \infty)$ and $h^{(k+r-1)}(x)$ is continuous in (l_F, ∞);

(ii) ${}_{r-1}M_k(l_F, v) \neq 0$ for $v > l_F$.

Then $F(x)$ is the exponential cdf (1.1) if and only if for $l_F < u < s < v$

$$\begin{aligned} \frac{k-1}{{}_{r-1}M_{k-1}(u,v)} E\left[h^{(k+r-1)}(R_n) \mid R_{n-k} = u, R_{n+r} = v\right] \qquad (1.6) \\ = \frac{k+r-1}{{}_{r-1}M'_{k-2}(s,v)} E\left[h^{(k+r-1)}(R_n) \mid R_{n-k+1} = s, R_{n+r} = v\right], \end{aligned}$$

where $M'(u,v) = [h'(v) - h'(u)]/(v-u)$.

Notice that, setting $r = 1$ and $k = 2$ and letting $s \to v^-$, one can see that (1.6) reduces to (1.3) for the case $k = 2$. We illustrate the applicability of Theorem 1 with one corollary below. Let $h(x) = x^{k+r}/(k+r)!$ and thus $h^{(k+r-1)}(x) = x$. It is not difficult to see that with this choice of $h(x)$

$$M(u,v) = \frac{v^{k+r-1} + \ldots + v^k u^{r-1} + v^{k-1}u^r + \ldots + u^{k+r-1}}{(k+r)!}$$

and

$$\frac{(k+r-1)!}{(k-1)!(r-1)!}\ {}_{r-1}M_{k-1}(u,v) = \frac{ru+kv}{k+r}.$$

Now, Theorem 1 implies the following corollary.

Corollary 1 *Let k, r and n be integers, such that $1 \le k \le n-1$ and $r \ge 1$. Suppose $F(x)$ is absolutely continuous. Then $F(x)$ is the exponential cdf (1.1) if and only if for $l_F < u < s < v$*

$$\frac{k+r}{ru+kv} E\left[R_n | R_{n-k} = u, R_{n+r} = v\right] = \frac{k+r-1}{rs+(k-1)v} E\left[R_n | R_{n-k+1} = s, R_{n+r} = v\right].$$

Corollary 1 gives a characterization of (1.1), which is alternative to that in Theorem 2B of Yanev et al. (2008), mentioned before Theorem 1 above.

Next we turn our attention to characterizations based on regressions of differences (spacings) of two record values. Consider the Weibull distribution given for $\alpha > 0$ by its cdf

$$F(x) = 1 - e^{-cx^\alpha}, \qquad x \ge 0, \tag{1.7}$$

where $c > 0$ is an arbitrary constant. Akhundov and Nevzorov (2008) study the regression of spacings of record values as follows

$$E[R_3 - R_2 | R_1 = u, R_4 = v] = \frac{v-u}{3}, \qquad u < v. \tag{1.8}$$

If $F(x)$ is the exponential (1.1) then it is clear that (1.4) and (1.5) lead to (1.8). Since (1.8) is a weaker condition than (1.4) (or (1.5)), it is not a sufficient condition for $F(x)$ to be exponential. Akhundov and Nevzorov (2008) prove the interesting fact that there is only one more family of distributions, other than the exponential, that satisfies (1.8). It turns out that (1.8) holds if and only if $F(x)$ satisfies (1.7) with either $\alpha = 1$ or $\alpha = 1/2$. Making use of the findings in Yanev et al. (2008) and Theorem 1 above, we generalize this result in two directions: (i) considering $R_m - R_n$ for any $2 \leq m \leq n-1$; and (ii) in the case of non-adjacent covariates. The following characterization result holds.

Theorem 2. *Let k, r, m and n be integers, such that $1 \leq k \leq m-1$, $r \geq 1$, and $2 \leq m \leq n-1$. Suppose $F(x)$ is absolutely continuous. Then $F(x)$ is given by (1.7) with $\alpha = 1$ or $\alpha = 1/2$ if and only if for $0 < u < s < t < v < \infty$*

$$\begin{aligned}\frac{d+2}{v-u}\ &E\left[R_n - R_m \right| R_{m-k} = u, R_{n+r} = v] \\ &= \frac{d}{t-s}\ E\left[R_n - R_m | R_{m-k+1} = s, R_{n+r-1} = t\right],\end{aligned} \tag{1.9}$$

where $d = n - m + k + r - 2$.

Setting $n = m + 1$ in (1.9) we obtain the following result.

Corollary 2. *Let k, r and m be integers, such that $1 \leq k \leq m-1$, $r \geq 1$. Suppose $F(x)$ is absolutely continuous. Then $F(x)$ is given by (1.7) with $\alpha = 1$ or $\alpha = 1/2$ if and only if for $0 < u < v < \infty$*

$$E\left[R_{m+1} - R_m \mid R_{m-k} = u, R_{m+r+1} = v\right] = \frac{k+r-1}{k+r+1}(v-u).$$

Setting $k = r = 1$ in (1.9) we obtain a corollary for adjacent covariates.

Corollary 3. *Let m and n be integers, such that $2 \leq m \leq n-1$. Suppose $F(x)$ is absolutely continuous. Then $F(x)$ is given by (1.7) with $\alpha = 1$ or $\alpha = 1/2$ if and only if for $0 < u < v < \infty$*

$$E\left[R_n - R_m \mid R_{m-1} = u, R_{n+1} = v\right] = \frac{n-m}{n-m+2}(v-u).$$

Corollary 3 can be interpreted as follows. Let us fix the integers m and n, such that $2 \leq m < n$. According to (1.4) and (1.5), the conditional expectations of the spacings $R_m - R_{m-1}$ and $R_{n+1} - R_n$ given $R_{m-1} = u$ and $R_{n+1} = v$ are equal and their sum is $2(v-u)/(n-m+2)$, that is

$$\begin{aligned}E[R_m - R_{m-1}|R_{m-1} = u, R_{n+1} = v] &= E[R_{n+1} - R_n|R_{m-1} = u, R_{n+1} = v] \\ &= \frac{v-u}{n-m+2},\end{aligned}$$

if and only if $F(x)$ is exponential. Now, assume that it is only known that the conditional expectations above have a sum $2(v-u)/(n-m+2)$, that is

$$E[R_m - R_{m-1}|R_{m-1} = u, R_{n+1} = v] + E[R_{n+1} - R_n|R_{m-1} = u, R_{n+1} = v] = \frac{2(v-u)}{n-m+2}. \tag{1.10}$$

This is equivalent to

$$\begin{aligned} &E[R_n - R_m|R_{m-1} = u, R_{n+1} = v] \\ &= E[R_{n+1} - R_{m-1}|R_{m-1} = u, R_{n+1} = v] - E[R_m - R_{m-1}|R_{m-1} = u, R_{n+1} = v] \\ &\quad -E[R_{n+1} - R_n|R_{m-1} = u, R_{n+1} = v] \\ &= v - u - \frac{2(v-u)}{n-m+2} \\ &= \frac{n-m}{n-m+2}(v-u). \end{aligned}$$

Therefore, according to Corollary 3, (1.10) holds if and only if the underlying distribution is either exponential or Weibull with $\alpha = 1/2$.

Finally, we investigate the regression

$$\frac{1}{cv - du} E\left[aR_n - bR_m \mid R_{m-k} = u, R_{n+r} = v\right], \qquad l_F < u < v,$$

where a, b, c and d with $a \neq b$ are some real numbers. What choice of these numbers characterizes the exponential distribution alone? The theorem below answers this question.

Theorem 3. *Let k, r, m and n be integers, such that $1 \leq k \leq m-1$, $r \geq 1$, and $2 \leq m \leq n-1$. Suppose $F(x)$ is absolutely continuous. Then $F(x)$ is exponential given by (1.1) if and only if for $l_F < s < u < v < t$*

$$\begin{aligned} &E\left[kR_n - (n-m+k)R_m|R_{m-k} = u, R_{n+r} = v\right] \qquad (1.11) \\ &= \frac{du}{(d+1)s - t} E\left[kR_n - (n-m+k)R_m|R_{m-k+1} = s, R_{n+r-1} = t\right], \end{aligned}$$

where $d = n-m+k+r-2$.

Setting $k = r = 1$ in (1.11) we obtain

$$E\left[R_n - (n-m+1)R_m \mid R_{m-1} = u, R_{n+1} = v\right] = (m-n)u.$$

Therefore, we have the following corollary.

Corollary 4. *Let m and n be integers, such that $2 \le m \le n-1$. Suppose $F(x)$ is absolutely continuous. Then $F(x)$ is exponential given by (1.1) if and only if for $l_F < u < v$*

$$E\left[R_n - (n-m)R_m \mid R_{m-1} = u, R_{n+1} = v\right] \qquad (1.12)$$
$$= E\left[R_m - (n-m)R_{m-1} | R_{m-1} = u, R_{n+1} = v\right].$$

It is interesting to note that setting $n = m+1$ the condition (1.12) becomes

$$E[R_{m+1} - R_m | R_{m-1} = u, R_{m+2} = v] = E[R_m - R_{m-1} | R_{m-1} = u, R_{m+2} = v].$$

We shall prove the results presented here in the next three sections.

2 Proof of Theorem 1

Sufficiency. Denote the cumulative hazard function of the cdf $F(x)$ by $H_x = -\ln(1 - F(x))$. Also, for simplicity, we will sometimes write $W_{x,y} = H_y - H_x$. Referring to the Markov dependence of the record values, one can show (e.g., Ahsanullah (2008)) that the conditional density of R_n given $R_{n-i} = u$ and $R_{n+j} = v$ is for $1 \le i \le n-1$ and $j \ge 1$

$$f(t) = \frac{(i+j-1)!}{(i-1)!(j-1)!} \frac{W_{u,t}^{i-1} W_{t,v}^{j-1}}{W_{u,v}^{i+j-1}} H'_t. \qquad (2.1)$$

Assuming (1.6), we will show that $F(x)$ satisfies (1.1). Denote $d = k + r - 2$. Referring to (2.1), it is not difficult to obtain

$$\begin{aligned} &E\left[h^{(d+1)}(R_n) | R_{n-k} = u, R_{n+r} = v\right] \\ &= \frac{(d+1)!}{(k-1)!(r-1)! W_{u,v}^{d+1}} \int_u^v h^{(d+1)}(x) W_{u,x}^{k-1} W_{x,v}^{r-1} dH_x \\ &= \frac{(d+1)!}{(k-1)!(r-1)! W_{u,v}^{d+1}} I(u, v; k, r) \quad \text{say,} \end{aligned}$$

and

$$\begin{aligned} &E\left[h^{(d+1)}(R_n) | R_{n-k+1} = s, R_{n+r} = v\right] \\ &= \frac{d!}{(k-2)!(r-1)! W_{s,t}^{d}} \int_s^v h^{(d+1)}(x) W_{s,x}^{k-2} W_{x,v}^{r-1} dH_x \\ &= \frac{d!}{(k-2)!(r-1)! W_{s,t}^{d}} I(s, v; k-1, r) \quad \text{say.} \end{aligned}$$

Now, one can see that (1.6) is equivalent to

$$\frac{I(u,v;k,r)}{I(s,v;k-1,r)}\ {}_{r-1}M'_{k-2}(s,v)W^{d}_{s,v} = {}_{r-1}M_{k-1}(u,v)W^{d+1}_{u,v}. \tag{2.2}$$

Differentiating (2.2) with respect to u and letting $s \to u^+$, we have

$$\begin{aligned} &-(k-1)\ {}_{r-1}M'_{k-2}(u,v)W^{d}_{u,v}H'_u \\ &\quad = {}_{r}M_{k-1}(u,v)W^{d+1}_{u,v} - (k+r-1)\ {}_{r-1}M_{k-1}(u,v)W^{d}_{u,v}H'_u. \end{aligned}$$

Dividing by $W^{d+1}_{u,v}$ and grouping, we arrive at the equation

$$\frac{H'_u}{H_u - H_v} = \frac{{}_{r}M_{k-1}(u,v)}{(k-1)\ {}_{r-1}M'_{k-2}(u,v) - (k+r-1)\ {}_{r-1}M_{k-1}(u,v)}\ , \tag{2.3}$$

provided that the denominator in the right-hand side is not zero. (This is equivalent to the assumption ${}_{r}M_{k-1}(u,v) \neq 0$, as we will see below.) Since

$$\begin{aligned} {}_{r-1}M'_{k-2}(u,v) &= \frac{\partial^{k+r-3}}{\partial u^{r-1}\partial v^{k-2}}\left[\frac{h'(v)-h'(u)}{v-u}\right] \\ &= \frac{\partial^{k+r-3}}{\partial u^{r-1}\partial v^{k-2}}\left[M_1(u,v) + {}_{1}M(u,v)\right] \\ &= {}_{r-1}M_{k-1}(u,v) + {}_{r}M_{k-2}(u,v), \end{aligned}$$

for the denominator in the right-hand side of (2.3) we have

$$\begin{aligned} &(k-1)\ {}_{r-1}M'_{k-2}(u,v) - (k+r-1)\ {}_{r-1}M_{k-1}(u,v) \qquad (2.4) \\ &\quad = (k-1)[\ {}_{r-1}M_{k-1}(u,v) + {}_{r}M_{k-2}(u,v)] - (k+r-1)\ {}_{r-1}M_{k-1}(u,v) \\ &\quad = (k-1)\ {}_{r}M_{k-2}(u,v) - r\ {}_{r-1}M_{k-1}(u,v) \\ &\quad = {}_{r}M_{k-1}(u,v)(u-v). \end{aligned}$$

The last equality follows from Lemma 1 in Yanev et al. (2008). Now, (2.3) and (2.4) imply

$$\frac{H'_u}{H_u - H_v} = \frac{1}{u-v}.$$

Integrating both sides with respect to u from l_F to v, we obtain

$$\ln(H_v - H_{l_F}) = \ln(v - l_F) + \ln c, \qquad c > 0$$

and thus $H_v = c(v - l_F)$ which implies (1.1).

Necessity. According to Theorem 1B in Yanev et al. (2008), if $F(x)$ satisfies (1.1), then

$$E[h^{(k+r-1)}(R_n)|R_{n-k}=u, R_{n+r}=v] = \frac{(k+r-1)!}{(k-1)!(r-1)!}\ {}_{r-1}M_{k-1}(u,v)$$

and

$$E[h^{(k+r-1)}(R_n)|R_{n-k+1}=s, R_{n+r}=v] = \frac{(k+r-2)!}{(k-2)!(r-1)!}\ {}_{r-1}M'_{k-2}(s,v).$$

These two equalities imply (1.6). This completes the proof of the theorem.

3 Proof of Theorem 2

To prove Theorem 2 we will need the following three lemmas.

Lemma 1 (Akhundov and Nevzorov (2008)) *Let $F(x)$ be absolutely continuous. The equation*

$$H_v - H_u = \frac{2H'_uH'_v}{H'_u+H'_v}(v-u), \qquad 0<u<v<\infty$$

has exactly two solutions given by $F(x)=1-\exp\{-cx^\alpha\}$ for $\alpha=1$ or $\alpha=1/2$, where $c>0$ is an arbitrary constant.

The following lemma is a straightforward corollary of Lemma 2 in Yanev et al. (2008).

Lemma 2 *Let k, r and n be integers such that $1\le k\le n-1$ and $r\ge 1$. If $F(x)=1-\exp\{-c(x-l_F)\}$, $(l_F<x<\infty)$, where $c>0$ is an arbitrary constant, then*

$$E[R_n|R_{n-k}=u, R_{n+r}=v] = \frac{ru+kv}{k+r}, \qquad l_F<u<v<\infty.$$

Lemma 3 *Let a and $b>a$ be real numbers and i and j be positive integers. Then*

$$I=\int_a^b[(y-a)^j(b-y)^i-(y-a)^i(b-y)^j]y^2\,dy = \frac{i!j!(j-i)}{(i+j+2)!}(b-a)^{i+j+1}(b^2-a^2).$$

Proof. We have

$$\begin{aligned} I &= \int_a^b(y-a)^j(b-y)^iy^2\,dy - \int_a^b(y-a)^i(b-y)^jy^2\,dy \qquad (3.1)\\ &= I_1 - I_2. \end{aligned}$$

Making in I_1 the change of variables $w = (y-a)/(b-a)$, we obtain

$$I_1 = (b-a)^{i+j+1} \int_0^1 w^j (1-w)^i [(b-a)w + a]^2 dw. \tag{3.2}$$

Similarly, making in I_2 the change of variables $w = (b-y)/(b-a)$, we have

$$I_2 = (b-a)^{i+j+1} \int_0^1 w^j (1-w)^i [b - (b-a)w]^2 dw. \tag{3.3}$$

From (3.2) and (3.3), we have

$$\begin{aligned} I &= (b-a)^{i+j+1} \int_0^1 w^j (1-w)^i \{[(b-a)w+a]^2 - [b-(b-a)w]^2\} dw \\ &= (b-a)^{i+j+2}(b+a)[2B(j+2, i+1) - B(j+1, i+1)] \\ &= (b-a)^{i+j+1}(b^2-a^2)B(j+1, i+1)\left(2\frac{j+1}{i+j+2} - 1\right) \\ &= (b-a)^{i+j+1}(b^2-a^2)\frac{i!j!(j-i)}{(i+j+2)!}, \end{aligned}$$

which proves the lemma.

Proof of Theorem 2. Sufficiency. We shall prove that (1.9) implies (1.7) with either $\alpha = 1$ or $\alpha = 1/2$. First, assume that $1 \leq k \leq m-1$ and $r \geq 2$. Referring to (2.1), one can obtain (recall that $d = n - m + k + r - 2$)

$$\begin{aligned} &E[R_n - R_m | R_{m-k} = u, R_{n+r} = v] \qquad (3.4) \\ &= \frac{(d+1)!}{W_{u,v}^{d+1}} \int_u^v \left[\frac{W_{u,x}^{d-r+1} W_{x,v}^{r-1}}{(d-r+1)!(r-1)!} - \frac{W_{u,x}^{k-1} W_{x,v}^{d-k+1}}{(k-1)!(d-k+1)!} \right] x dH_x \\ &= \frac{(d+1)!}{W_{u,v}^{d+1}} \, I(u, v; k, r), \quad \text{say,} \end{aligned}$$

and

$$\begin{aligned} &E[R_n - R_m | R_{m-k+1} = u, R_{n+r-1} = v] \qquad (3.5) \\ &= \frac{(d-1)!}{W_{s,t}^{d-1}} \int_s^t \left[\frac{W_{s,x}^{d-r} W_{x,t}^{r-2}}{(d-r)!(r-2)!} - \frac{W_{s,x}^{k-2} W_{x,t}^{d-k}}{(k-2)!(d-k)!} \right] x dH_x \\ &= \frac{(d-1)!}{W_{s,t}^{d-1}} \, I(s, t; k-1, r-1), \quad \text{say.} \end{aligned}$$

Now, making use of (3.4) and (3.5), we can write (1.9) as

$$\frac{(d+1)!I(u,v;k,r)}{(d-1)!I(s,t;k-1,r-1)}(t-s)W_{s,t}^{d-1} = (v-u)W_{u,v}^{d+1}. \qquad (3.6)$$

Let us differentiate both sides of (3.6) with respect to u and v. Then, after letting $s \to u^+$ and $t \to v^-$, (3.6) simplifies to

$$d(v-u)H'_u H'_v = (H'_u + H'_v)(H_v - H_u) + d(v-u)H'_u H'_v.$$

Therefore,

$$H_v - H_u = \frac{2H'_u H'_v}{H'_u + H'_v}(v-u). \qquad (3.7)$$

According to Lemma 1, equation (3.7) has the two solutions given by (1.7) with $\alpha = 1$ or $\alpha = 1/2$. In the case $k = 1$ and $r \geq 2$, the proof is similar and is omitted here. If $k = r = 1$, then (1.9) simplifies to

$$\frac{d+2}{d}E\left[R_n - R_m | R_{m+1} = u, R_{n-1} = v\right] = v - u.$$

Repeating the arguments for the case $k \geq 2$ above, it is not difficult to obtain equation (3.7). The sufficiency is proved.

Necessity. We need to show that both cdf's $F_1(x) = 1 - \exp\{-cx\}$ and $F_2(x) = 1 - \exp\{-cx^{1/2}\}$ satisfy (1.9). In case of $F_1(x)$, it is not difficult to obtain the relation (1.9) using Lemma 2 above. It remains to prove that $F_2(x) = 1 - \exp\{-cx^{1/2}\}$ satisfies (1.9) as well. First assume that $2 \leq k \leq m-1$ and $r \geq 2$. Since $x = H^2(x)/c^2$, for the left-hand side of (1.9) we have

$$\frac{d+2}{v-u}E\left[R_n - R_m \mid R_{m-k} = u, R_{n+r} = v\right]$$
$$= \frac{(d+2)!}{c^2 W_{u,v}^{d+1}(v-u)} \int_{H_u}^{H_v} \left[\frac{W_{u,x}^{d-r+1}W_{x,v}^{r-1}}{(d-r+1)!(r-1)!} - \frac{W_{u,x}^{k-1}W_{x,v}^{d-k+1}}{(d-k+1)!(k-1)!}\right] H_x^2 dH_x.$$

Using Lemma 3 (twice) with $a = H_u$, $b = H_v$, after some algebra, we obtain

$$\frac{d+2}{v-u}E\left[R_n - R_m \mid R_{m-k} = u, R_{n+r} = v\right] = 2(n-m). \qquad (3.8)$$

Similarly, using Lemma 3 with $a = H_s$ and $b = H_t$ for the right-hand side of (1.9) we have

$$\frac{d}{t-s}E\left[R_n - R_m \mid R_{m-k+1} = s, R_{n+r-1} = t\right] = 2(n-m). \qquad (3.9)$$

It follows from (3.8) and (3.9) that $F_2(x)$ satisfies (1.9). When $k = 1$ and $r \geq 2$ or $k = r = 1$ the proof is similar and is omitted here.

4 Proof of Theorem 3

Sufficiency. We shall prove that (1.11) implies (1.1). First, assume that $2 \leq k \leq m-1$ and $r \geq 2$. Referring to (2.1) we obtain

$$\begin{aligned} & E\left[kR_n - (n-m+k)R_m \mid R_{m-k} = u, R_{n+r} = v\right] \\ &= \frac{(d+1)!}{W_{u,v}^{d+1}} \int_u^v \left[\frac{kW_{u,x}^{d-r+1} W_{x,v}^{r-1}}{(d-r+1)!(r-1)!} - \frac{(d-r+2)W_{u,x}^{k-1} W_{x,v}^{d-k+1}}{(k-1)!(d-k+1)!} \right] x dH_x \\ &= \frac{(d+1)!}{W_{u,v}^{d+1}} J(u,v;k,r), \quad \text{say,} \end{aligned}$$

and

$$\begin{aligned} & E\left[kR_n - (n-m+k)R_m \mid R_{m-k+1} = s, R_{n+r-1} = t\right] \\ &= \frac{(d-1)!}{W_{s,t}^{d-1}} \int_s^t \left[\frac{kW_{s,x}^{d-r} W_{x,t}^{r-2}}{(d-r)!(r-2)!} - \frac{(d-r+2)W_{s,x}^{k-2} W_{x,t}^{d-k}}{(k-2)!(d-k)!} \right] x dH_x \\ &= \frac{(d-1)!}{W_{s,t}^{d-1}} J(s,t;k-1,r-1), \quad \text{say.} \end{aligned}$$

Now, we can write (1.11) as

$$\frac{(d+1)! J(u,v;k,r)}{(d-1)! J(s,t;k-1,r-1)} ((d+1)s - t) W_{s,t}^{d-1} = du W_{u,v}^{d-1}. \tag{4.1}$$

Differentiating both sides of (4.1) with respect to u and v and letting $s \to u^+$ and $t \to v^-$, after some algebra, we obtain

$$\frac{H_u'}{H_v - H_u} = \frac{1}{v-u}. \tag{4.2}$$

Equation (4.2) has the only solution given by (1.1). If $k = 1$ and $r \geq 2$ the proof is similar and is omitted here. If $k = r = 1$, then (1.11) simplifies to

$$E\left[R_n - (d+1)R_m | R_{m-1} = u, R_{n+1} = v\right] = -du.$$

Repeating the arguments for the case $k \geq 2$ above, it is not difficult to obtain equation (4.2) with only solution (1.1). The sufficiency is proved.

Necessity. Using Lemma 2, one can verify that the distribution function $F(x) = 1 - \exp\{-c(x - l_F)\}$ satisfies (1.11). The theorem is proved.

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