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Extremes of geometric variables with applications to branching processes

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Abstract

We obtain limit theorems for the row extrema of a triangular array of zero-modified geometric random variables. Some of this is used to obtain limit theorems for the maximum family size within a generation of a simple branching process with varying geometric offspring laws.

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1. Introduction

It is well-known (Anderson, 1970) that the geometric law is not attracted to any max-stable law and hence maxima of independent geometric variates cannot be approximated by a max-stable law. Considering triangular arrays of zero-modified geometric laws allows adjustment of the zero-class probability independent of the success probability parameter, thus opening the possibility of approximating row maxima and minima by simple explicit laws. This general idea is implicit in the derivation by Kolchin et al. (1978, Section 2.6) of a limit theorem for maximal occupancy in an urn scheme. Our motivation is closer to that of Anderson et al. (1997), who studied the Poisson and other laws. They exploit the normal approximation to the Poisson law with large mean to show that

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row maxima are approximated by the Gumbel law under certain conditions. We obtain corresponding results which emanate from the exponential approximation to the geometric law when its mean is large.

For $n = 1, 2, \ldots$, let v_n be a positive integer and $\{X_i(n): i = 1, \ldots, v_n\}$ be independent random variables with the same zero-modified geometric law

$$P(X_i(n) = 0) = 1 - a_n \quad \text{and} \quad P(X_i(n) = j) = a_n p_n (1 - p_n)^{j-1} \quad (j = 1, 2, \ldots,), \tag{1.1}$$

where $0 < a_n \leqslant 1$ and $0 < p_n < 1$. The mean for row n is a_n/p_n and the distribution function is

$$F_n(x) = \begin{cases} 1 - a_n(1 - p_n)^{[x]} & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0, \end{cases} \tag{1.2}$$

where $[x]$ denotes the integer part of x. The standard geometric law corresponds to $a_n = 1 - p_n$.

In the next section we prove limit theorems as $v_n \to \infty$ for the row extrema and range

$$\mathscr{M}_n = \max_{1 \leqslant i \leqslant v_n} X_i(n), \quad \mu_n = \min_{1 \leqslant i \leqslant v_n} X_i(n), \quad \text{and} \quad \mathscr{R}_n = \mathscr{M}_n - \mu_n$$

and we give examples showing our hypotheses can be satisfied. More specifically, we find conditions which ensure the row extrema converge in probability to infinity, and show in Theorems 1 and 3 under a further condition that normalized versions have non-defective limit laws. Theorem 5 demonstrates their joint weak convergence, and Theorem 6 shows that the range is asymptotically proportional to the maximum. Lemmas 1–4 provide context by exhibiting possible behaviours of the extrema under differing assumptions.

Our results for row maxima are used in Section 3 to obtain corresponding limit theorems for the maximum family size, again denoted $\mathscr{M}_n$, in the nth generation of the simple branching process where the offspring law for individuals in generation $n-1$ is the geometric law (1.1). Of course this is precisely the case of a varying fractional linear offspring law which has previously been studied by Agresti (1975), Keiding and Nielsen (1975), and Fujimagari (1980). Maxima of random variables defined on the classical Galton–Watson tree have been studied by Arnold and Villasẽnor (1996), Pakes (1998), and Rahimov and Yanev (1997,1999). We show in Theorems 9–11 that results from Section 2 transfer to the branching process setting through conditional limit theorems (Theorem 7) for the generation sizes Z_n. These latter results seem to be new, and they are the strongest possible assertions which can be made within our restricted class of offspring laws.

2. Behaviour of row extrema

To set our main result for $\mathscr{M}_n$ in context we begin with some elementary descriptions of its behaviour. Observe that the distribution function of $\mathscr{M}_n$ is $H_n(x) := F_n^{v_n}(x)$.

Lemma 1. *Let* $\lim_{n\to\infty} v_n = \infty$. (i) $\mathscr{M}_n \xrightarrow{\text{p}} 0$ *if* $v_n a_n \to 0$.

(ii) *If* $\sum v_n a_n < \infty$ *then* $P(\mathscr{M}_n > 0 \text{ i.o.}) = 0$, *and if the rows are independent and* $\sum v_n a_n = \infty$ *then* $P(\mathscr{M}_n > 0 \text{ i.o.}) = 1$.

(iii) *Let* $0 < \varepsilon < 1$. *If the rows are independent and* $v_n a_n > \log n + (1+\varepsilon)\log(\log n)$ *for all large* n *then* $P(\mathscr{M}_n = 0 \text{ i.o.}) = 1$. *If* $v_n a_n < \log n + \log(\log n)$ *then* $P(\mathscr{M}_n = 0 \text{ i.o.}) = 0$.

Proof. For (i) observe that $H_n(x) \to 1$ for all $x > 0$. The remaining assertions follow from the Borel–Cantelli lemma and elementary estimates of $P(\mathscr{M}_n > 0)$ and $P(\mathscr{M}_n = 0) = (1 - a_n)^{v_n}$. □

The quantity $\alpha_n = \log(v_n a_n)$ is important to our further considerations. The following limit theorem is easily proved.

Lemma 2. *Let* $\lim_{n\to\infty} v_n = \infty$. (i) *If*

$$\lim_{n\to\infty} \alpha_n = \alpha \quad (-\infty < \alpha < \infty) \tag{2.1}$$

and

$$\lim_{n\to\infty} p_n = p \tag{2.2}$$

then

$$\lim_{n\to\infty} H_n(x) = \begin{cases} \exp(-\mathrm{e}^{\alpha}(1-p)^{[x]}) & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{2.3}$$

(ii) *If*

$$\lim_{n\to\infty} \alpha_n = \infty \quad \text{and} \quad \limsup_{n\to\infty} p_n < 1, \tag{2.4}$$

then $\mathscr{M}_n \xrightarrow{\mathrm{p}} \infty$.

The limiting distribution function in (2.3) is non-defective if $p > 0$. It is defective if $p = 0$, and then $\mathscr{M}_n \xrightarrow{\mathrm{d}} \mathscr{M}_\infty$ where $P(\mathscr{M}_\infty = 0) = 1 - P(\mathscr{M}_\infty = \infty) = G(-\alpha)$, and $G(x) = \exp(-\mathrm{e}^{-x})$ $(-\infty < x < \infty)$ is the distribution function of the standard Gumbel law.

Theorem 1 characterizes the rate of divergence to infinity under some further conditions.

Theorem 1. *Let* $\lim_{n\to\infty} v_n = \infty$. *Assume that for some real* c

$$\lim_{n\to\infty} p_n = 0 \quad \textit{and} \quad \lim_{n\to\infty} \alpha_n p_n = 2c. \tag{2.5}$$

(i) *If* $\lim_{n\to\infty} \alpha_n = \infty$, *then* $c \geqslant 0$ *and* $p_n \mathscr{M}_n - \alpha_n \xrightarrow{\mathrm{d}} \Lambda - c$, *where* Λ *has a standard Gumbel law.*

(ii) *If* $\lim_{n\to\infty} \alpha_n = \alpha$ $(-\infty < \alpha < \infty)$, *then*

$$p_n \mathscr{M}_n \xrightarrow{\mathrm{d}} (\Lambda + \alpha)^+.$$

Proof. Define $x_n = (x + \alpha_n)/p_n$ and $\eta_n(x) = v_n a_n (1 - p_n)^{[x_n]}$, $(-\infty < x < \infty)$. By using the expansion $[x_n] = (x + \alpha_n)/p_n - \delta_n$, where $0 \leqslant \delta_n < 1$ is the fractional part of x_n, it is easily seen that

$$\begin{aligned} \log \eta_n(x) = \alpha_n + [x_n]\log(1 - p_n) &= \alpha_n + ((x + \alpha_n)/p_n)(-p_n - \tfrac{1}{2} p_n^2 + \mathrm{O}(p_n^3)) \\ &= -x + \mathrm{O}(p_n) - \tfrac{1}{2} \alpha_n p_n (1 + \mathrm{o}(1)). \end{aligned}$$

The right-hand side converges iff (2.5) holds, and the limit is $-x - c$. Thus (2.5) is equivalent to

$$\lim_{n\to\infty} \eta_n(x) = \mathrm{e}^{-x-c} \quad (-\infty < x < \infty). \tag{2.6}$$

The distribution function $G_n(x)$ of $p_n\mathscr{M}_n - \alpha_n$ equals $[1 - v_n^{-1}\eta_n(x)]^{v_n}$ if $x_n \geqslant 0$, and equals zero otherwise. It follows that

$$\lim_{n\to\infty} G_n(x) = \begin{cases} G(x+c) & \text{if } \liminf_{n\to\infty} x_n \geqslant 0, \\ 0 & \text{if } \limsup_{n\to\infty} x_n < 0. \end{cases} \tag{2.7}$$

But $x_n \to \infty$ for all real x iff $\alpha_n \to \infty$, and assertion (i) follows. If (2.1) and (2.5) hold then $c = 0$ and $x_n \to \pm\infty$ according as $x > -\alpha$ or $x < -\alpha$, respectively. It follows from (2.7) that

$$\lim_{n\to\infty} G_n(x) = \begin{cases} G(x) & \text{if } x > -\alpha, \\ 0 & \text{if } x < -\alpha, \end{cases}$$

and hence $p_n\mathscr{M}_n - \alpha \xrightarrow{\mathrm{d}} \max(-\alpha, \Lambda)$, and (ii) follows. □

Observe that the first member of (2.5) implies that if $n \gg 1$ then $F_n(x/p_n) \approx 1 - a_n + a_n(1 - \mathrm{e}^{-x})$, the exponential approximation mentioned in Section 1. Also note that Lemma 2(ii) holds under (2.5).

The assumptions of Theorem 1 can be realized. Let $a_n = \mathrm{e}^{-\alpha} v_n^{-\delta}$ $(\delta \geqslant 0)$ and $p_n = \gamma v_n^{-\zeta}$ $(\zeta > 0)$. Then (2.1) holds iff $0 \leqslant \delta < 1$, and then (2.5) holds with $c = 0$. Condition (2.4) holds if $\delta = 1$, and then we can admit any $p_n \to 0$. Now let $\delta < 1$ and choose $p_n \sim A(\log v_n)^{-\zeta}$ where A is a positive constant. Then (2.1) still holds and

$$\alpha_n p_n = A\,\frac{(1-\delta)\log v_n - c}{(\log v_n)^{\zeta}} \to \begin{cases} 0 & \text{if } \zeta > 1, \\ (1-\delta)A & \text{if } \zeta = 1, \\ \infty & \text{if } \zeta < 1. \end{cases}$$

Then Theorem 1(i) holds with $c = 0$ if $\zeta > 1$ and with $c = (1-\delta)A/2$ if $\zeta = 1$. The case $\zeta < 1$ is an instance of (2.5) where $c = \infty$. In this case Theorem 1(i) suggests that the limit law is concentrated at $-\infty$, and indeed this is true. In fact there is no affine transformation of $\mathscr{M}_n$ which has a non-degenerate limit law. However we have the following large deviation estimate,

$$\lim_{n\to\infty} \frac{\log P(\mathscr{M}_n > x_n)}{\alpha_n p_n} = -\frac{1}{2}.$$

Further consideration of $\eta_n(x)$, defined in the proof of Theorem 1, shows that if (2.1) holds and if (2.2) holds with $0 < p < 1$ then no affine transformation of $\mathscr{M}_n$ has a non-defective limit law. The following result, generalizing the direct assertion of Theorem 2 in Anderson (1970) and with more explicit centering constants, shows that it is possible to stabilize the law of $\mathscr{M}_n$. The proof is similar to that for (2.6).

Theorem 2. *Set* $C_n = -\alpha_n/\log(1 - p_n)$. *If* $\lim_{n\to\infty} \alpha_n = \infty$ *and* (2.2) *holds with* $0 < p < 1$, *then*

$$G(\gamma(x-1)) = \liminf_{n\to\infty} P(\mathscr{M}_n - C_n \leqslant x) \leqslant \limsup_{n\to\infty} P(\mathscr{M}_n - C_n \leqslant x) = G(\gamma x) \quad (-\infty < x < \infty),$$

where $\gamma = -\log(1 - p)$.

Parallel to Lemmas 1 and 2 we have the following results for the row minimum μ_n, and they are easy consequences of its distribution function

$$K_n(y) = \begin{cases} 1 - (a_n(1-p_n)^{[y]})^{v_n} & \text{if } y \geqslant 0, \\ 0 & \text{if } y < 0. \end{cases}$$

Lemma 3. *Suppose that* $v_n \to \infty$. (i) $\mu_n \xrightarrow{\mathrm{p}} 0$ iff $a_n^{v_n} \to 0$.

(ii) *If* $\sum a_n^{v_n} < \infty$ *then* $P(\mu_n > 0 \text{ i.o.}) = 0$, *and if the rows are independent and* $\sum a_n^{v_n} = \infty$ *then* $P(\mu_n > 0 \text{ i.o.}) = 1$.

(iii) *If the rows are independent and* $\sum(1-a_n^{v_n}) = \infty$ *then* $P(\mu_n = 0 \text{ i.o.}) = 1$, *and if* $\sum(1-a_n^{v_n}) < \infty$ *then* $P(\mu_n = 0 \text{ i.o.}) = 0$.

Lemma 4. *Let* $\lim_{n\to\infty} v_n = \infty$. *Assume that*

$$\lim_{n\to\infty} v_n(1-a_n) = \beta \quad \text{and} \quad \lim_{n\to\infty} v_n p_n = \rho. \tag{2.8}$$

(i) *If* $\beta + \rho < \infty$, *then*

$$\lim_{n\to\infty} K_n(y) = \begin{cases} 1 - \mathrm{e}^{-\beta-\rho[y]} & \text{if } y \geqslant 0, \\ 0 & \text{if } y < 0. \end{cases}$$

(ii) *If* $\beta = \rho = 0$, *then* $\mu_n \xrightarrow{\mathrm{p}} \infty$.

The limit law in Lemma 4(i) is non-defective if $\rho > 0$ and we see that it defines a zero-modified geometric law. If $\rho = 0$ then $\mu_n \xrightarrow{\mathrm{d}} \mu_\infty$ where $P(\mu_\infty = \infty) = 1 - P(\mu_\infty = 0) = \mathrm{e}^{-\beta}$.

Our principle result shows that if (2.8) holds with $\rho = 0$ then μ_n can be centered and scaled to give a non-degenerate limit law. Set $\beta_n = -v_n \log a_n$.

Theorem 3. *Let* $\lim_{n\to\infty} v_n = \infty$ *and assume that for some* $b \in [0,\infty)$

$$\lim_{n\to\infty} v_n p_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \beta_n p_n = 2b. \tag{2.9}$$

If

$$\lim_{n\to\infty} \beta_n = \beta \in [0,\infty), \tag{2.10}$$

then

$$v_n p_n \mu_n \xrightarrow{\mathrm{d}} (\mathscr{E} - \beta)^+ \tag{2.11}$$

where $\mathscr{E}$ *has a standard exponential law.*

Proof. Let $y_n = (y - \beta_n)/v_n p_n$ and $\psi_n(y) = (a_n(1-p_n)^{[y_n]})^{v_n}$, $(-\infty < y < \infty)$. Observing that

$$\log \psi_n(y) = -\beta_n + v_n\left(\frac{y-\beta_n}{v_n p_n} - \delta_n\right)(-p_n - \tfrac{1}{2}p_n^2 + \mathrm{O}(p_n^3)) = -y + \tfrac{1}{2}\beta_n p_n + v_n \delta_n p_n(1+\mathrm{o}(1)),$$

where δ_n is the fractional part of y_n, it is clear that (2.9) is equivalent to

$$\lim_{n\to\infty} \psi_n(y) = \mathrm{e}^{b-y} \quad (-\infty < y < \infty). \tag{2.12}$$

If (2.10) holds then $b=0$ and $y_n \to \pm\infty$ according as $y > \beta$ or $y < \beta$, respectively. Since $K_n(y_n) = 1 - \psi_n(y)$ if $y_n > 0$, $=0$ otherwise, it follows that $v_n p_n \mu_n + \beta_n \xrightarrow{\mathrm{d}} \max(\beta, \mathscr{E})$, and (2.11) follows. □

The proof shows that $\beta < \infty$ is a necessary condition for a non-defective limit law. The limit assertion (2.11) extends for $\lim_{n\to\infty} v_n p_n > 0$ in a manner similar to Theorem 2 as follows.

Theorem 4. *If* $\lim_{n\to\infty} v_n p_n = \xi \in (0,\infty)$ *and* (2.10) *holds then*

$$1 - \mathrm{e}^{-y-\beta-\xi} \leqslant \liminf_{n\to\infty} P(v_n p_n \mu_n \leqslant y) \leqslant \limsup_{n\to\infty} P(v_n p_n \mu_n \leqslant y) \leqslant 1 - \mathrm{e}^{-y-\beta}.$$

Observe again that (2.10) implies the second member of (2.9) with $b=0$, and that $a_n \to 1$. Hence our assumptions for (2.10) are precisely (2.8) with $\rho = 0$. In addition, $p_n \log v_n \to 0$ if $\rho = 0$ and $\alpha_n = \log v_n + \mathrm{o}(1)$. Thus, the assumptions for Theorem 1(i) are satisfied with $c=0$ and hence we have $p_n \mathscr{M}_n - \log v_n \xrightarrow{\mathrm{d}} \Lambda$ and $v_n p_n \mu_n \xrightarrow{\mathrm{d}} (\mathscr{E} - \beta)^+$. Our next result extends this pair of weak limit statements to joint convergence, showing in particular that μ_n and $\mathscr{M}_n$ are asymptotically independent.

Theorem 5. *If* $\lim_{n\to\infty} v_n p_n = 0$ *and* (2.10) *hold, then* $(p_n \mathscr{M}_n - \log v_n, v_n p_n \mu_n) \xrightarrow{\mathrm{d}} (\Lambda, (\mathscr{E} - \beta)^+)$.

Proof. If $x_n = (x + \log v_n)/p_n$ and $y_n = y/v_n p_n$, where x, y are real, then $x_n - y_n = (v_n \log v_n + v_n x - y)/v_n p_n \to \infty$. Consequently for any real x, y and n large enough we have

$$\begin{aligned} \Delta_n(x,y) &:= P(\mathscr{M}_n \leqslant x_n, \mu_n \leqslant y_n) = H_n(x_n) - P(y_n < \mu_n, \mathscr{M}_n \leqslant x_n) \\ &= H_n(x_n) \left[1 - \left(1 - \frac{F_n(y_n)}{F_n(x_n)} \right)^{v_n} \right]. \end{aligned}$$

The proofs of Theorems 1 and 3 show that $H_n(x_n) \to G(x)$, and hence $F_n(x_n) \to 1$, and $v_n F_n(y_n) \to y + \beta$ if $y > 0$, $\to 0$ otherwise. It follows that $\Delta_n(x,y) \to G(x)(1 - \mathrm{e}^{-y-\beta})$ if $y > 0$, $\to 0$ if $y < 0$. □

Our last result shows that the row ranges $\mathscr{R}_n = \mathscr{M}_n - \mu_n$ are determined by the row maxima.

Theorem 6. *If the conditions of Theorem* 1 *hold, then* $\mathscr{R}_n$ *has the same limit behaviour as* $\mathscr{M}_n$.

Proof. The assumptions imply that $p_n \to 0$. For any $y > 0$ we have

$$-\log P(p_n \mu_n > y) = -\beta_n - v_n y + \mathrm{O}(v_n p_n) \to -\infty,$$

i.e., $p_n \mu_n \xrightarrow{\mathrm{p}} 0$. The assertion follows from Slutsky's lemma. □

3. Maximum family sizes for the simple branching process

Let $(Z_n : n \geqslant 0)$ denote the generation sizes of the simple branching process with varying geometric environments, $Z_n = \sum_{i=1}^{Z_{n-1}} X_i(n)$ $(n = 1, 2, \ldots)$ where $Z_0 = 1$ and the $X_i(n)$ $(i, n \geqslant 1)$ have the same geometric laws as in Section 1, and they are mutually independent. Thus $X_1(n)$ is a generic family size of a parent in generation $n-1$, and its probability generating function (pgf) is

$$f_n(s) = \frac{(1-s)R_n + s}{(1-s)r_n + 1}, \tag{3.1}$$

where $r_n = p_n^{-1} - 1$, $R_n = p_n^{-1} - m_n$ and $m_n = f_n'(1) = a_n/p_n$ is the mean nth generation family size.

If $Z_0 = 1$ then (Harris, 1963) the pgf $\phi_n(s)$ of Z_n is obtained by functional composition, $\phi_n(s) = \phi_{n-1}(f_n(s))$. The group structure of Mobius transformations (3.1) yields

$$\phi_n(s) = \frac{(1-s)A_n + s}{(1-s)B_n + 1},$$

where $A_n = M_n \sum_{j=1}^n R_j/M_j$, $B_n = M_n \sum_{j=1}^n r_j/M_j$, $M_0 = 1$ and $M_n = E(Z_n \mid Z_0 = 1) = \prod_{j=1}^n m_j$, $(n \geqslant 1)$. The proof is (barely) indicated by Agresti (1975), and with differing notation.

Proofs of the following two results are omitted.

Theorem 7. *The conditioned process $(Z_n \mid Z_n > 0)$ has a limit law iff $\lim_{n\to\infty} B_n = B \in [0, \infty]$. Suppose this condition holds.* (i) *If $0 \leqslant B < \infty$, then $(Z_n \mid Z_n > 0) \xrightarrow{d} \Xi$ where $E(s^{\Xi}) = s/(1 + B - Bs)$.*

(ii) *If $B = \infty$ then $(Z_n/B_n \mid Z_n > 0) \xrightarrow{d} \mathscr{E}$, where $\mathscr{E}$ has a standard exponential law.*

Theorem 8. (i) *$Q := \lim_{n\to\infty} P(Z_n = 0) = 1$ iff $\lim_{n\to\infty} M_n = 0$ and/or $\sum_{j\geqslant 1} r_j/M_j = \infty$.*

(ii) *If Q exists and $B = \infty$, then*

$$Z_n/M_n \xrightarrow{\text{a.s.}} I + (1-I)\mathscr{E} \tag{3.2}$$

where $P(I = 1) = Q = 1 - P(I = 0)$, and I and $\mathscr{E}$ are independent.

(iii) *If $Q = 1$, then $P(T < \infty) = 1$, where $T = \inf\{n : Z_n = 0\}$ is the time to extinction.*

Definition 1. The environments are weakly varying if $M = \lim_{n\to\infty} M_n$ exists, and then they are supercritical, critical or subcritical according as $M = \infty$, $\in (0, \infty)$ or $=0$, i.e. if the sum $\sum_n (m_n - 1)$ diverges to $+\infty$, converges, or diverges to $-\infty$.

Keiding and Nielsen (1975, Theorem 2.2) prove (3.2) in the supercritical case. Theorem 7(ii) can hold for any mode of criticality, and (3.2) can hold in the critical or supercritical modes. Examples are constructed by setting $a_n = m p_n$ and $r_n = m^n b_n$, where $b_n > 0$, giving the classical form $M_n = m^n$ and $B_n = m^n \sum_{j=1}^n b_j$. So if $m \geqslant 1$ then $Q = 1$ iff $\sum_{n\geqslant 1} b_n = \infty$. Constructing $m_n = (1+n)^\delta$ (δ real) gives $M_n = n^\delta$, and the mode of criticality is determined by the sign of δ. Also $B_n = n^\delta \sum_{j=1}^n j^{-\delta} r_j$, and choosing $r_n = n^\alpha l(n)$ where α is real and $l(x)$ is slowly varying, allows combinations of α and δ which realize the outcomes of Theorems 7 and 8.

Consider now the maximum nth generation family size $\mathscr{M}_n := \max_{1\leqslant i\leqslant Z_{n-1}} X_i(n)$. Since $\mathscr{M}_n = 0$ if $Z_{n-1} = 0$ we consider the conditional distribution function

$$\mathscr{H}_n(x) := P(\mathscr{M}_n \leqslant x \mid Z_{n-1} > 0) = E[F_n^{Z_{n-1}}(x) \mid Z_{n-1} > 0], \tag{3.3}$$

where F_n is defined at (1.2). Analogues of both Lemmas 1 and 2 can be given, but here we consider only the analogue of Lemma 2.

Theorem 9. *Suppose that* $\lim_{n\to\infty} p_n = p$ *with* $0 < p < 1$.

(i) *If* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} B_n = B < \infty$, *then*

$$\lim_{n\to\infty} \mathscr{H}_n(x) = \frac{1 - a(1-p)^{[x]}}{1 + aB(1-p)^{[x]}} \quad (x \geqslant 0).$$

(ii) *If* $\lim_{n\to\infty} B_n = \infty$ *and* $\lim_{n\to\infty} \log(a_n B_n) = \alpha$, *then*

$$\lim_{n\to\infty} \mathscr{H}_n(x) = [1 + \mathrm{e}^{\alpha}(1-p)^{[x]}]^{-1} \quad (x \geqslant 0).$$

Proof. For (i) observe that $F_n(x) \to 1 - a(1-p)^{[x]}$. The assertion follows from Theorem 7(i), (3.3), and the uniform convergence property of the continuity theorem for probability generating functions. For (ii), observe that the assertion of Theorem 7(ii) is equivalent to the limit statement

$$\lim_{n\to\infty} E(s^{Z_{n-1}/B_{n-1}} \mid Z_{n-1} > 0) = [1 + \log s^{-1}]^{-1},$$

and the convergence is uniform with respect to s in the interval $[0 < s' \leqslant s \leqslant 1]$. The assertion follows by setting $s = s_n := F_n^{[B_{n-1}]}(x)$ and seeing that $\{s_n\}$ has the limit (2.3). □

Our next result extends Theorem 1 to the branching process setting. The proof shows that the role played by v_n in Section 2 is here played by $[B_{n-1}]$ and indeed that its fractional part can be ignored. Accordingly, we define $\alpha_n^* = \log(a_n B_{n-1})$, and $\mathscr{V}$ denotes a random variable having the standard logistic distribution function $L(x) = (1 + \mathrm{e}^{-x})^{-1}$, all real x.

Theorem 10. *Suppose that* $\lim_{n\to\infty} B_n = \infty$ *and for some real* c

$$\lim_{n\to\infty} p_n = 0 \quad \textit{and} \quad \lim_{n\to\infty} \alpha_n^* p_n = 2c. \tag{3.4}$$

(i) *If* $\lim_{n\to\infty} \alpha_n^* = \infty$, *then*

$$(p_n \mathscr{M}_n - \alpha_n^* \mid Z_{n-1} > 0) \xrightarrow{\mathrm{d}} \mathscr{V} - c.$$

(ii) *If* $\lim_{n\to\infty} \alpha_n^* = \alpha$ $(-\infty < \alpha < \infty)$, *then*

$$(p_n \mathscr{M}_n \mid Z_{n-1} > 0) \xrightarrow{\mathrm{d}} (\mathscr{V} + \alpha)^+.$$

Proof. The proof is essentially the same as for Theorem 9(ii), letting $s_n := F_n^{[B_{n-1}]}(x_n)$ where $x_n = (x + \alpha_n^*)/p_n$, just as for Theorem 1. The limit of $\{s_n\}$ is given by (2.7) for (i), and for (ii) by $G(x)$ if $x > -\alpha$, $=0$ otherwise. □

Consideration of the proofs of Theorems 1 and 9 shows that the conditions listed in Theorem 10 are necessary for its conclusions. The simple approach we use for Theorems 9(ii) and 10 give an obvious analogue for Theorem 2.

Theorem 11. *If the notation and assumptions of Theorem 2 stand with α_n replaced by α_n^*, then*

$$L(\gamma(x-1)) = \liminf_{n\to\infty} P(\mathscr{M}_n - C_n \leqslant x) \leqslant \limsup_{n\to\infty} P(\mathscr{M}_n - C_n \leqslant x) = L(\gamma x), \quad (-\infty < x < \infty).$$

The various conditions in Theorems 9–11 can be satisfied, but we will show that all but one set is satisfied by the branching process obtained from sampling the linear birth and death process $(\mathscr{B}_t)$ at irregular times, an example mentioned by Keiding and Nielsen (1975). Let $Z_n = \mathscr{B}_{t_n}$ where $0 < t_n < t_{n+1} \to t_\infty \leqslant \infty$. If λ and μ are the birth and death rates, respectively, and $d_n = t_n - t_{n-1}$, then $a_n = m_n p_n$,

$$p_n = \begin{cases} \dfrac{\lambda-\mu}{\lambda m_n - \mu} & \text{if } \lambda \neq \mu, \\ (1+\lambda d_n)^{-1} & \text{if } \lambda = \mu, \end{cases} \quad \text{and} \quad m_n = \mathrm{e}^{(\lambda-\mu)d_n}.$$

and

$$B_n = \begin{cases} \dfrac{\lambda(M_n-1)}{\lambda-\mu} & \text{if } \lambda \neq \mu, \\ \lambda t_n & \text{if } \lambda = \mu, \end{cases} \quad \text{and} \quad M_n = \mathrm{e}^{(\lambda-\mu)t_n}.$$

The environments are weakly varying in this case, and our criticality classification coincides with the standard one if $t_\infty = \infty$, and they are critical if $t_\infty < \infty$. The latter case is vacuous as far as Theorems 9–11 are concerned, since $B < \infty$ and $p=1$, i.e., there exists no non-degenerate conditional limit law for $\mathscr{M}_n$. So we assume that $t_\infty = \infty$ and that $d_n \to d \leqslant \infty$.

If the environments are subcritical, $\lambda < \mu$, then $B < \infty$, $p_n \to p$ with

$$0 < p = \frac{\mu-\lambda}{\mu - \lambda m_\infty} < 1 \quad \text{and} \quad 0 \leqslant m_\infty = \mathrm{e}^{-(\mu-\lambda)d} < 1$$

and $a_n \to a = p m_\infty$. Hence Theorem 9(i) holds and the limit law is degenerate at the origin if $d = \infty$, but not otherwise.

If $\lambda \geqslant \mu$ then $B=\infty$, but Theorem 9(ii) cannot hold because $a=(1-\lambda/\mu)/(1-\mu/m_\infty) > 0$, whence $\alpha_n^* \to \alpha < \infty$ is violated. We show that the conditions of Theorems 10 and 11 can be satisfied. First, suppose that $d = \infty$, in which case $p = 0$. In the supercritical case $\lambda > \mu$ we further suppose that the second member of (3.4) is satisfied, i.e.,

$$\lim_{n\to\infty} (t_n/m_n) = \frac{2c}{\lambda-\mu} \in [0,\infty). \tag{3.5}$$

For example, this condition is satisfied with $c=0$ if $d_n > \ell(n) \to \infty$ where $\ell(x)$ is a slowly varying function. If (3.5) holds then

$$((\mathscr{M}_n/m_n) - (\lambda-\mu)t_n \,|\, Z_{n-1} > 0) \xrightarrow{\mathrm{d}} \mathscr{V} - c.$$

Now suppose that $\lambda = \mu$ and $t_n = n^\delta \ell(n)$ where $\delta \geqslant 1$, $\ell(x)$ is slowly varying and that $t_n/n \to \infty$. Then (3.4) holds with $c = 0$, $a_n B_{n-1} \sim n$, and again Theorem 10(i) holds in the form

$$\left(\frac{\mathscr{M}_n}{\lambda \delta n^{\delta-1} \ell(n)} - \log n \,|\, Z_{n-1} > 0 \right) \xrightarrow{\mathrm{d}} \mathscr{V}.$$

On the other hand, if $t_{n-1}/t_n \to \tau \in (0,1)$ then $anB_{n-1} \to \omega := \tau/(1-\tau) > 1$ and $\alpha_n^* \to \alpha = \log \omega$. Hence Theorem 10(b) holds in the form

$$\left(\frac{\mathscr{M}_n}{(1-\tau)\lambda t_n} \mid Z_{n-1} > 0 \right) \xrightarrow{\mathrm{d}} (\mathscr{V} + \alpha)^+.$$

If $\lambda \geqslant \mu$, $t_\infty = \infty$ but $d < \infty$ then $0 < p < 1$ and Theorem 11 holds.

We end by observing that analogues of results in Section 2 for the minimum and range can be taken into the branching process context, but we leave this as an exercise for the reader.

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