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ON MAXIMUM FAMILY SIZE IN BRANCHING PROCESSES

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Abstract

The number Y_n of offspring of the most prolific individual in the nth generation of a Bienaymé–Galton–Watson process is studied. The asymptotic behaviour of Y_n as $n \to \infty$ may be viewed as an extreme value problem for i.i.d. random variables with random sample size. Limit theorems for both Y_n and $\mathrm{E}Y_n$ provided that the offspring mean is finite are obtained using some convergence results for branching processes as well as a transfer limit lemma for maxima. Subcritical, critical and supercritical branching processes are considered separately.

Keywords: Bienaymé–Galton–Watson branching process; max-stability; max-semi-stability; random sample size; transfer theorems

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1. Introduction

Let $\{Z_n\}$ be a Bienaymé–Galton–Watson (BGW) branching process defined by

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_i(n), \qquad n = 1, 2, \dots ; \qquad Z_0 \equiv 1,$$

where $\{X_i(n)\}, i, n = 1, 2, \dots$ are independent and identically distributed (i.i.d.) random variables, taking on non-negative integer values.

The process $\{Z_n\}$ can be thought of as stochastic model of an evolving population of particles or individuals. One of the main objects of investigation in the theory of branching processes is the size Z_n of the nth generation. However, there are many other characteristics describing the tree structure of the population process, which are studied as well. The number of individuals in a generation having a non-empty offspring set in a certain number of generations (so-called reduced branching process) and the number of individuals' pairs having the same number of offspring, are two examples of such random variables (see [13], Chapter IV). In the present work our attention is focused on a new random variable concerning the population: the maximum number of offspring of a particle living in the nth generation.

Denote by $f(s) = \mathrm{E}s^{X_i(n)}$ the offspring generating function and by $f_n(s)$ its nth functional iterate, i.e. $f_n(s) = f(f_{n-1}(s))$, $n = 1, 2, \dots$, $f_0(s) = s$, $0 \le s \le 1$. Additionally, let

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$F(x) = P(X_i(n) \leq x)$ be the distribution function of the 'offspring variable' which has mean $0 < m < \infty$ and variance $0 < \sigma^2 \leq \infty$. Define

$$Y_n = \max\{X_1(n), X_2(n), \dots, X_{Z_{n-1}}(n)\}$$

or equivalently

$$P(Y_n \leq x) = \sum_{k=0}^{\infty} P(Z_{n-1} = k)F^k(x) = f_{n-1}(F(x)). \tag{1.1}$$

The study of the sequence $\{Y_n\}$ might be motivated in different ways. A natural interpretation within a demographic framework, for example, may be given. Indeed, the random variable in question is the number of children in the largest family. Thus, the asymptotic behaviour of Y_n provides information about the influence of the largest families on the size of the whole population.

Alternatively, certain kinds of extremes in branching processes have been considered (see e.g. [5, 11] and references therein), and investigating Y_n is perhaps plausible as a contribution to this program. Similar questions are also considered in [10] (e.g. the application concerning 'hero mothers'). Note here [1], where the maximum of a random number of i.i.d. random variables is considered when the number of variables forms a supercritical BGW branching process.

We proceed with a transfer limit lemma for maxima with random sample size in Section 2. In Section 3, the asymptotic behaviour of Y_n is studied in the critical and non-critical cases, separately. Limit theorems for $\mathrm{E}Y_n$ are established in Section 4. A particular case of Lemma 2.1 as well as some of the results given here under stronger conditions are presented in [14].

2. A simple transfer limit lemma

Recall that a non-degenerate distribution function $H(s)$ is max-stable if and only if for a distribution function $F(x)$ there exist functions $a(n) > 0$ and $b(n)$ such that

$$\lim_{n\to\infty} F^n(a(n)x + b(n)) = H(x), \tag{2.1}$$

weakly. If (2.1) holds, then $F(x)$ is said to belong to the domain of attraction of $H(x)$, i.e. $F \in D(H)$. According to the classical Gnedenko result, $H(x) = \exp\{-h(x)\}$, say, is of the type of one of the following three classes:

$$\begin{aligned} &h(x) = (-x)^a && \text{for } x \in (-\infty, 0), && = 1 && x \in [0, \infty),\\ &h(x) = x^{-a} && \text{for } x \in (0, \infty), && = 0 && x \in (-\infty, 0],\\ &h(x) = \mathrm{e}^{-x} && \text{for } x \in (-\infty, \infty), \end{aligned}$$

where $a > 0$. Necessary and sufficient conditions for $F \in D(H)$ are well known. In particular, $F \in D(\exp\{-x^{-a}\})$, $a > 0$ if and only if for $x > 0$,

$$1 - F(x) = x^{-a}L(x), \tag{2.2}$$

where $L(x)$ is a slowly varying function at infinity (s.v.f.) (see e.g. [15], Proposition 1.11).

Consider the following two sequences of random variables:
(a) $\{\eta_i(n)\}$: independent for any n, with a common distribution function $F(x)$;
(b) $\{\nu(n)\}$: non-negative integer-valued and independent of $\eta_i(n)$ for any n.
Let $r : \mathbb{N}^+ \to \mathbb{R}$ be a function such that $r(n)$ tends to infinity with n.

We shall prove here a transfer limit result for a maximum with random sample size (see also [6], Theorem 6.2.2 and [7]).

Lemma 2.1. *Assume that (2.1) holds and that there exists a random variable* ν *with* $\varphi(u) = \mathrm{E}\exp\{-u\nu\}$, $u > 0$ *such that*

$$\lim_{n\to\infty} P\left(\frac{\nu(n)}{r(n)} \le x \mid B_n\right) = P(\nu \le x) \tag{2.3}$$

weakly, where B_n *are some events, such that* $\{\nu(n) > 0\} \subseteq B_n$ *for every* n. *Then for* $x \in \mathbb{R}$,

$$\lim_{n\to\infty} P\left(\frac{\max_{1\le i\le \nu(n)} \eta_i(n) - b(r(n))}{a(r(n))} \le x \;\Big|\; B_n\right) = \varphi(h(x)),$$

where $a(\cdot)$ *and* $b(\cdot)$ *are defined by (2.1).*

Proof. First, note that $\{\nu(n) > 0\} \subseteq B_n$ is equivalent to $\{\nu(n) = k\} \subseteq B_n$ for every $k = 1, 2, \dots$ Now, since (2.1) implies $\lim_{n\to\infty} F^{r(n)}(a(r(n))x + b(r(n))) = H(x)$, we obtain

$$\begin{aligned}
&P\left(\max_{1\le i\le \nu(n)} \eta_i(n) \le a(r(n))x + b(r(n)) \mid B_n\right)\\
&\quad = \sum_{k=0}^{\infty} P(\nu(n) = k \mid B_n) P\left(\max_{1\le i\le \nu(n)} \eta_i(n) \le a(r(n))x + b(r(n)) \mid \nu(n) = k\right)\\
&\quad = \sum_{k=0}^{\infty} P\left(\frac{\nu(n)}{r(n)} = \frac{k}{r(n)} \mid B_n\right) F^{r(n)k/r(n)}(a(r(n))x + b(r(n)))\\
&\quad \to \int_0^{\infty} H^y(x)\,\mathrm{d}P(\nu \le y), \qquad \text{as } n \to \infty.
\end{aligned}$$

Note that there are no restrictions on the dependence of the events B_n and $\{\eta_i(k), k = 1, 2, \dots\}$ in the conditions of the lemma. In what follows we use Lemma 2.1 with $\eta_i(n) = X_i(n)$, $\nu(n) = Z_{n-1}$, and $B_n = \{Z_{n-1} > 0\}$ or $B_n = \{Z_{n-1} \ge 0\}$.

3. Limit theorems for Y_n

3.1. Critical process ($m = 1$)

Let the offspring generating function satisfy

$$f(s) = s + (1-s)^{1+\alpha}\mathcal{L}(1/(1-s)) \tag{3.1}$$

for $0 < \alpha \le 1$, where $0 \le s \le 1$ and $\mathcal{L}(x)$ is a s.v.f. It was proved by Slack (see e.g. [3], Theorem 8.12.3) that (3.1) is a necessary and sufficient condition for

$$\lim_{n\to\infty} P(Q_n Z_n > y \mid Z_n > 0) = P(Z > y), \qquad y \ge 0, \tag{3.2}$$

where $Q_n = P(Z_n > 0)$ and Z has Laplace transform

$$\varphi(u) = \mathrm{E}e^{-uZ} = 1 - (1 + u^{-\alpha})^{-1/\alpha}, \qquad u > 0. \tag{3.3}$$

Moreover, if (3.1) is fulfilled then

$$Q_n = n^{-1/\alpha} M(n), \tag{3.4}$$

where $M(n)$ is a s.v.f. and $\lim_{x\to\infty} M^{\alpha}(x)\mathcal{L}(x^{1/\alpha}/M(x)) = 1/\alpha$.

Remark 3.1. The case $\alpha = 1$ is particularly important; here $\varphi(u) = 1/(1+u)$, so the limit law is exponential. If $\sigma^2 < \infty$, then (3.1) holds for $\alpha = 1$ and $\mathcal{L}(x)$ is asymptotically constant. If $0 < \alpha < 1$, then (3.1) is equivalent to (2.2) for $a = 1 + \alpha$ and $L(x) \sim \alpha\mathcal{L}(x)/\Gamma(1-\alpha)$ as $x \to \infty$ (see [2], Thm A). Note that (3.1) does not necessarily imply (2.2) in the borderline case $\alpha = 1$.

Recall that the de Bruijn conjugate of a s.v.f. $L(x)$ is a s.v.f. $L^{\#}(x)$, unique up to asymptotic equivalence, with

$$\lim_{x\to\infty} L(x)L^{\#}(xL(x)) = 1, \qquad \lim_{x\to\infty} L^{\#}(x)L(xL^{\#}(x)) = 1.$$

Theorem 3.1. *Assume that* $m = 1$.

(a) *Let* $\sigma^2 < \infty$ *and suppose that (2.1) holds. Then for any* $x \in \mathbb{R}$,

$$\lim_{n\to\infty} P\left(\frac{Y_n - b(n)}{a(n)} \le x \mid Z_{n-1} > 0\right) = \left(1 + \frac{\sigma^2}{2}h(x)\right)^{-1}. \tag{3.5}$$

(b) *Let* $\sigma^2 = \infty$ *and suppose that (3.1) holds.*

(i) *If* $0 < \alpha < 1$, *then for* $x \ge 0$,

$$\lim_{n\to\infty} P\left(\frac{Y_n}{Q_n^{-1/(1+\alpha)}(L_1(Q_n^{-1/(1+\alpha)}))^{1/(1+\alpha)}} \le x \mid Z_{n-1} > 0\right) = 1 - (1 + x^{\alpha(1+\alpha)})^{-1/\alpha},$$

where the s.v.f. $L_1(x)$ *is the de Bruijn conjugate of* $1/L(x)$.

(ii) *If* $\alpha = 1$ *and (2.1) holds, then for any* $x \in \mathbb{R}$,

$$\lim_{n\to\infty} P\left(\frac{Y_n - b(n/M(n))}{a(n/M(n))} \le x \mid Z_{n-1} > 0\right) = 1 - (1 + h^{-\alpha}(x))^{-1/\alpha}, \tag{3.6}$$

where $M(n)$ *is a s.v.f. defined by (3.4).*

Proof. (a) It is well known (see e.g. [9], Theorem 2.4.2) that for $x \ge 0$,

$$\lim_{n\to\infty} P\left(\frac{Z_{n-1}}{n} \le x \mid Z_{n-1} > 0\right) = P(Z \le x),$$

where $\mathrm{E}\exp\{-uZ\} = 1/(1 + \sigma^2 u/2)$. Now, (3.5) follows by Lemma 2.1.

(b) Since (2.1) and (3.2) hold, we have by Lemma 2.1

$$\lim_{n\to\infty} P\left(\frac{Y_n - b(Q_n^{-1})}{a(Q_n^{-1})} \le x\right) = \varphi(h(x)), \tag{3.7}$$

where $\varphi(x)$ is given in (3.3).

(i) Define the generalized (left continuous) inverse of a non-decreasing function U by $U^{\leftarrow}(y) = \inf\{x \in \mathbb{R} : U(x) \geq y\}$. Since $0 < \alpha < 1$ then (3.1) is equivalent to (2.2) with $a = 1 + \alpha$ (see Remark 3.1). Thus, $F \in D(\exp\{-x^{-(1+\alpha)}\})$, $x > 0$ and one can choose in (3.7) (see e.g. [15], Proposition 1.11) $b(Q_n^{-1}) = 0$ and

$$a(Q_n^{-1}) = \left(\frac{1}{1-F}\right)^{\leftarrow}(Q_n^{-1}) \sim Q_n^{-1/(1+\alpha)}(L_1(Q_n^{-1/(1+\alpha)}))^{1/(1+\alpha)}, \tag{3.8}$$

where, according to Proposition 1.5.15 in [3], $L_1(x)$ is the de Bruijn conjugate of $1/L(x)$. This completes the proof of (i).

(ii) The claim in this case follows from (3.7) by a straightforward argument.

To illustrate what the normalizing constants in Theorem 3.2 could be, let us consider two examples in which they can be calculated explicitly. In the first one, the offspring generating function satisfies (3.1) with $\mathcal{L}(x) = \log x$; in the second, it is required that the offspring distribution function has a regularly varying tail, instead.

Example 3.1. (a) Assume that $f(s) = s + (1-s)^{1+\alpha}\log(1/(1-s))$ with $0 < \alpha < 1$. That is $\mathcal{L}(x) \sim \log x$, and then (see Remark 3.1) $L(x) \sim \alpha \log x/\Gamma(1-\alpha)$. Therefore, (see [3], Corollary 2.3.4), $L_1(x) \sim L(x)$. On the other hand, from [4], p. 309, $1/Q_n \sim (n\log n)^{1/\alpha}$. Finally, by Theorem 3.1(b)(i), we obtain for $x \geq 0$,

$$\lim_{n\to\infty} P\left(\frac{Y_n}{cn^{1/(\alpha(1+\alpha))}(\log n)^{1/(1+\alpha)}} \leq x \,\middle|\, Z_{n-1} > 0\right) = 1 - (1 + x^{\alpha(1+\alpha)})^{-1/\alpha},$$

where $c = (\Gamma(1-\alpha)(1+\alpha))^{-1/(1+\alpha)}$.

(b) Let $\alpha = 1$ and $1 - F(x) \sim x^{-2}\log x$. In this case (3.6) holds with $b(n/M(n)) \equiv 0$, $a(n/M(n)) = Q_n^{-1/2}(L_1(Q_n^{-1/2}))^{1/2}$ and $h(x) = x^{-2}$. By Proposition A(ii) in [2] it follows that $\mathcal{L}(x) \sim \int_1^x (\log u)/u\,du = (\log x)^2/2$. Since $\lim_{x\to\infty} M(x)\mathcal{L}(x/M(x)) = 1$, we have $1/M(n) \sim (\log n)^2/2$. Now, from (3.4), $1/Q_n \sim n(\log n)^2/2$ and, since $L_1(n) \sim \log n$ (see e.g. [3], Corollary 2.3.4), we obtain for $x \geq 0$,

$$\lim_{n\to\infty} P\left(\frac{Y_n}{n^{1/2}(\log n)^{3/2}} \leq x \mid Z_{n-1} > 0\right) = 1 - (1 + 4x^2)^{-1}.$$

Instead of using Lemma 2.1, one can directly prove the following result.

Theorem 3.2. *Assume that $m = 1$, (3.1) holds, and*

$$\lim_{n\to\infty} \frac{P(X_1(1) > n)}{P(X_1(1) > n+1)} = 1. \tag{3.9}$$

Then for $x \geq 0$,

$$\lim_{n\to\infty} P(Q_n U(Y_n) \leq x \mid Z_{n-1} > 0) = 1 - (1 + x^\alpha)^{-1/\alpha}, \tag{3.10}$$

where Q_n satisfies (3.4) and $U(x) = 1/(1 - F(x))$.

Proof. Since $\lim_{n\to\infty} Q_n = 0$ we have (cf. [12], p. 24) that (3.9) is a necessary and sufficient condition for the existence of a sequence $\{u_n\}$ such that for $x > 0$,

$$\lim_{n\to\infty} \frac{1 - F(u_n)}{Q_n} = x. \tag{3.11}$$

On the other hand, (3.1) holds if and only if

$$\lim_{n\to\infty}(1 - f_n(\exp\{-uQ_n\}))/Q_n = 1 - \varphi(u), \qquad u > 0,$$

where $\varphi(x)$ is defined by (3.3). Since $\lim_{n\to\infty}(1 - F(u_n)) = 0$, we get as $n \to \infty$,

$$\begin{aligned}
P(Y_n > u_n \mid Z_{n-1} > 0) &= \frac{1 - f_{n-1}(F(u_n))}{Q_{n-1}} = \frac{1 - f_{n-1}(\exp\{\ln F(u_n)\})}{Q_{n-1}} \\
&= \frac{1 - f_{n-1}(\exp\{-(1 - F(u_n))(1 + o(1))\})}{Q_{n-1}} \\
&= \frac{1 - f_{n-1}(\exp\{-xQ_{n-1}(1 + o(1))\})}{Q_{n-1}} \\
&\to 1 - \varphi(x).
\end{aligned} \tag{3.12}$$

Further, from (3.11), using Lemma 2.2.1 in [6], one can obtain for $x > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} P\left(\frac{1 - F(Y_n)}{Q_{n-1}} \le x \,\Big|\, Z_{n-1} > 0\right) \\
&\quad = \lim_{n\to\infty} P\left(\frac{1 - F(Y_n)}{Q_{n-1}} \le \frac{1 - F(u_n)}{Q_{n-1}} + x - \frac{1 - F(u_n)}{Q_{n-1}} \,\Big|\, Z_{n-1} > 0\right) \\
&\quad = \lim_{n\to\infty} P(Y_n > u_n \mid Z_{n-1} > 0).
\end{aligned}$$

From this, taking into account (3.12) and (3.3), we obtain (3.10).

Remark 3.2. It is known (cf. [6], Corollary 2.4.1) that if (3.9) does not hold then there are no functions $a(n) > 0$ and $b(n)$ such that $(Y_n - b(n))/a(n)$ converges in distribution to a non-degenerate limit. Therefore, (3.9) is a necessary condition for Theorem 3.1 to hold. One can verify that (3.9) is not true for geometric and Poisson distributions. On the other hand, some classes of offspring distributions which satisfy the conditions of Theorem 3.1 are provided by Theorems 3.48 and 3.50 in [16]. Define $F_{[X]}(x) = P([X] \le x)$, where $[a]$ stands for the greatest integer less than or equal to a. According to the theorems mentioned in [16], if the distribution function F_X of a random variable X belongs to the domain of attraction of $H(x) = \exp\{-x^{-a}\}$, $x > 0, a > 0$, then so does $F_{[X]}$. Further, $F_X \in D(\exp\{-\exp\{-x\}\})$, $x \in \mathbb{R}$ iff $F_{[X]} \in D(\exp\{-\exp\{-x\}\})$, $x \in \mathbb{R}$, provided that $\sup\{x : F_X(x) < 1\} = \infty$.

3.2. Non-critical processes ($m \neq 1$)

It is known (see e.g. [3], Theorem 8.12.5) that if $1 < m < \infty$, then there exists a sequence of constants $\{C_n\}$, $\lim_{n\to\infty} C_n = \infty$ such that $\{Z_n/C_n\}$ converges almost surely to a non-degenerate limit W. The Laplace transform $\psi(u) = \mathrm{E}\exp\{-uW\}$, $u > 0$ of the limiting random variable is the unique (up to a scale factor) solution of the functional equation

$$\psi(u) = f\left(\psi\left(\frac{u}{m}\right)\right). \tag{3.13}$$

The constants C_n take on the form $C_n = m^n/L_2(m^n)$, where (see [3], Theorem 8.12.6) $L_2(x) = \int_0^x P(W > y)\,dy$ is a s.v.f.

Instead of (2.1), let us assume that the following weaker condition holds. For a distribution function $F(x)$ there exist two functions $\bar{a}(k) > 0$ and $\bar{b}(k)$ such that

$$\lim_{k\to\infty} F^k(\bar{a}(k)x + \bar{b}(k)) = G(x) = \exp\{-g(x)\}, \qquad \text{say}, \tag{3.14}$$

weakly, where k runs over a sequence of positive integers $k(1) < k(2) < \dots$ subject to the condition

$$\lim_{n\to\infty} \frac{k(n+1)}{k(n)} = r, \qquad 1 \le r < \infty. \tag{3.15}$$

This assumption leads to the following extension of the class of max-stable distributions. A non-degenerate distribution function $G(s)$ is max-semistable (under linear transformation) if and only if (3.14) holds for a distribution function $F(x)$; $F(x)$ is said to belong to the domain of attraction of $G(x)$, in our notation, $F \in SD(G)$. The case $r = 1$ in (3.15) corresponds to max-stable laws. It was proved by Grinevich that convergence (3.14) implies that $G(x)$ belongs to one of three types corresponding to the three max-stable laws. The explicit expression for $g(x)$ in (3.14) as well as necessary and sufficient conditions for $F \in SD(G)$ were established in [8].

In the subcritical case when $0 < m < 1$, it is known (see e.g. [9], Theorem 2.6.2) that

$$\lim_{n\to\infty} P(Z_n = j \mid Z_n > 0) = p_j, \qquad j = 0, 1, \dots, \tag{3.16}$$

where $\{p_j\}$ is a probability distribution whose generating function $\gamma(s) = \sum_{j=0}^{\infty} p_j s^j$, $\mid s \mid \le 1$, is the unique generation-function solution of the functional equation

$$\gamma(f(s)) = m\gamma(s) + 1 - m, \qquad \gamma(0) = 0, \tag{3.17}$$

where $f(s)$ is the offspring generating function.

Now, we are in a position to prove the following result.

Theorem 3.3. (a) *Assume that* $1 < m < \infty$.

(i) *If (2.1) holds, then for* $x \in \mathbb{R}$,

$$\lim_{n\to\infty} P\left(\frac{Y_n - b(C_n)}{a(C_n)} \le x\right) = \psi(h(x)). \tag{3.18}$$

(ii) *Assume (3.14) with* $k = [C_n]$ *and normalizing constants* $a(C_n)$ *and* $b(C_n)$. *Then (3.18) still holds with* $h(x) \equiv g(x)$.

(b) *If* $0 < m < 1$, *then for* $x \ge 0$,

$$\lim_{n\to\infty} P(Y_n \le x \mid Z_{n-1} > 0) = \gamma(F(x)),$$

where γ *is the unique solution of (3.17) among the probability generating functions.*

Proof. (a) (i) Since $\{Z_n/C_n\}$ converges almost surely, and hence in distribution, to W the assertion follows by Lemma 2.1.

(ii) Since

$$\lim_{n\to\infty} F^{[C_n]}(a(C_n)x + b(C_n)) = G(x),$$

it is not difficult to see that

$$\begin{aligned} P(Y_n \le a(C_n)x + b(C_n)) &= \sum_{k=0}^{\infty} P\left(\frac{Z_{n-1}}{C_n} = \frac{k}{C_n}\right) F^{C_n k/C_n}(a(C_n)x + b(C_n)) \\ &\to \int_0^{\infty} G^y(x)\,\mathrm{d}P(\nu \le y), \end{aligned}$$

as $n \to \infty$, which completes the proof of part (a).

(b) Using (1.1) and (3.16) we obtain as $n \to \infty$,

$$\begin{aligned} P(Y_n > x \mid Z_{n-1} > 0) &= \frac{1 - f_{n-1}(F(x))}{1 - f_{n-1}(0)} = 1 - \frac{f_{n-1}(F(x)) - f_{n-1}(0)}{1 - f_{n-1}(0)} \\ &= 1 - \mathrm{E}(F^{Z_{n-1}}(x) \mid Z_{n-1} > 0) \\ &\to 1 - \gamma(F(x)). \end{aligned}$$

Example 3.2. Let us consider the case when the offspring distribution is geometric, i.e. $f(s) = p/(1 - qs)$, where $1/2 < p = 1 - q < 1$. Then $m = q/p < 1$ and it is not difficult to see that the solution of (3.17) is $\gamma(s) = (1 - m)s/(1 - ms)$. Hence

$$\lim_{n\to\infty} P(Y_n \le k \mid Z_{n-1} > 0) = \frac{(p-q)(1-q^{k+1})}{p - q(1-q^{k+1})}.$$

4. Convergence of $\mathrm{E}Y_n$

Let $\{V_n\}$ be a sequence of non-negative random variables. Using an argument from [15], p. 77, it is not difficult to prove the following technical lemma.

Lemma 4.1. *Assume that there exists a random variable V with $\mathrm{E}V < \infty$, such that*

$$\lim_{n\to\infty} P(V_n \le x \mid B_n) = P(V \le x),$$

weakly, for a sequence of events $\{B_n\}$. If for any $N > 0$

$$\lim_{N\to\infty} \limsup_{n\to\infty} \sum_{j=N+1}^{\infty} P(V_n > j \mid B_n) = 0, \tag{4.1}$$

then

$$\lim_{n\to\infty} \mathrm{E}(V_n \mid B_n) = \mathrm{E}(V). \tag{4.2}$$

In the subcritical case the following theorem holds.

Theorem 4.1. *If $0 < m < 1$ and $\mathrm{E}X_1(1)\log(1 + X_1(1)) < \infty$, then*

$$\lim_{n\to\infty} \mathrm{E}(Y_n \mid Z_{n-1} > 0) = \sum_{k=0}^{\infty}(1 - \gamma(F(k)) < \infty,$$

where γ is the unique solution of (3.17) among the probability generating functions.

Proof. Since $\gamma'(1) < \infty$ if and only if $\mathrm{E}X_1(1)\log(1 + X_1(1)) < \infty$ (cf. [9], Theorem 2.6.2), we have by Theorem 3.3(b)

$$\mathrm{E}Y = \sum_{k=0}^{\infty}(1 - \gamma(F(k)) \le m\gamma'(1) < \infty, \tag{4.3}$$

where $P(Y \le x) = \lim_{n\to\infty} P(Y_n \le x \mid Z_{n-1} > 0)$, weakly. Further, for $j = 0, 1, \ldots$

$$\begin{aligned} P(Y_n > j \mid Z_{n-1} > 0) &= \sum_{k=1}^{\infty} P(Z_{n-1} = k \mid Z_{n-1} > 0) P(\max_{1\le i\le k} X_i(n) > j) \\ &= \sum_{k=1}^{\infty} P(Z_{n-1} = k \mid Z_{n-1} > 0)(1 - F^k(j)) \\ &\le \sum_{k=1}^{\infty} P(Z_{n-1} = k \mid Z_{n-1} > 0) k (1 - F(j)) \\ &= (1 - F(j)) \mathrm{E}(Z_{n-1} \mid Z_{n-1} > 0). \end{aligned} \tag{4.4}$$

Therefore, since $\lim_{n\to\infty} \mathrm{E}(Z_{n-1} \mid Z_{n-1} > 0) = \gamma'(1) < \infty$ (cf. [9], Theorem 2.6.1),

$$\lim_{N\to\infty} \limsup_{n\to\infty} \sum_{j=N+1}^{\infty} P(Y_n > j \mid Z_{n-1} > 0) = \lim_{N\to\infty} \gamma'(1) \sum_{j=N+1}^{\infty} (1 - F(j)) = 0. \tag{4.5}$$

The claim of the theorem follows from (4.3) and (4.5), appealing to Lemma 4.1.

Notice that, as could be expected, under the conditions of Theorem 4.1, we obtain

$$\lim_{n\to\infty} \mathrm{E}(Y_n \mid Z_{n-1} > 0) \le m \lim_{n\to\infty} \mathrm{E}(Z_n \mid Z_{n-1} > 0).$$

Example 4.1. Let us come back to Example 3.2, where the offspring distribution is geometric. Then, by Theorem 4.1,

$$\lim_{n\to\infty} \mathrm{E}(Y_n \mid Z_{n-1} > 0) = \sum_{k=0}^{\infty} \frac{q^{k+1}}{mq^{k+1} + 1 - m} = \frac{1}{1-m} \sum_{k=0}^{\infty} \frac{1}{q^{-(k+1)} + c}, \tag{4.6}$$

where $c = m/(1-m)$. Denote by S_n the partial sum of $S = \sum_{k=0}^{\infty} 1/(q^{-(k+1)} + c)$. Then $S_n < \sum_{k=0}^{n} q^{k+1}$ and $S \le m$. Further,

$$S_n < \sum_{k=0}^{n} \frac{1}{q^{-(k+1)} + qc} = q^{-1} \sum_{k=0}^{n} \frac{1}{q^{-(k+2)} + c} = q^{-1} \left(S_n - \frac{1}{q^{-1} + c} + \frac{1}{q^{-(n+2)} + c} \right).$$

Hence $S \ge m(1-m)/(1-pm)$. Finally,

$$\frac{m}{1-pm} \le \lim_{n\to\infty} \mathrm{E}(Y_n \mid Z_{n-1} > 0) \le \frac{m}{1-m}.$$

Further on we assume that (2.2) holds, i.e. $F \in D(\exp\{-x^{-a}\})$, $x > 0$. Let us consider the critical process.

Theorem 4.2. *Assume that $m = 1$ and (2.2) holds with $a > 1$.*
(i) *If $\sigma^2 = \infty$, then*

$$\lim_{n\to\infty} \frac{1}{n^{1/(a(a-1))} L_2(n)} \mathrm{E}(Y_n \mid Z_{n-1} > 0) = \frac{1}{a(a-1)} B\left(\frac{1}{a}, \frac{1}{a(a-1)}\right). \tag{4.7}$$

where $B(u, v)$ is the beta function and $L_2^a(n)M(n) \sim L_1(n^{1/a(a-1)}/M^{1/a}(n))$, $n \to \infty$ is a s.v.f., where $L_1(x)$ is the de Bruijn conjugate of $1/L(x)$ and $M(n)$ is defined by (3.4).

(ii) *If $\sigma^2 < \infty$, then*

$$\lim_{n\to\infty} \frac{1}{n^{1/a}(L_1(n^{1/a}))^{1/a}} \mathrm{E}(Y_n \mid Z_{n-1} > 0) = \left(\frac{\sigma^2}{2}\right)^{1/a} \frac{\pi/a}{\sin(\pi/a)},$$

where the s.v.f. $L_1(x)$ is the de Bruijn conjugate of $1/L(x)$.

Proof. (i) Since (2.2) holds, we have $F \in D(\exp\{-x^{-a}\})$, $x > 0$ and by (3.7)

$$\lim_{n\to\infty} P(Y_n \le a(Q_n^{-1})x \mid Z_{n-1} > 0) = 1 - (1 + x^{a(a-1)})^{-1/(a-1)}, \tag{4.8}$$

where $a(Q_n^{-1}) = (1/(1-F))^{\leftarrow}(Q_n^{-1})$. The right-hand side in (4.7) is equal to the expectation of the limiting distribution in (4.8).

On the other hand, by (4.4) we obtain

$$\begin{aligned} P(Y_n > ja(Q_n^{-1}) \mid Z_{n-1} > 0) &\le (1 - F(ja(Q_n^{-1})))\,\mathrm{E}(Z_{n-1} \mid Z_{n-1} > 0) \\ &= \frac{(1 - F(ja(Q_n^{-1})))(1 - F(a(Q_n^{-1})))}{(1 - F(a(Q_n^{-1})))Q_{n-1}}. \end{aligned} \tag{4.9}$$

By the properties of regularly varying functions (see e.g. [15], Proposition 0.8(ii)), for a given $\varepsilon > 0$ and large n

$$\frac{1 - F(ja(Q_n^{-1}))}{1 - F(a(Q_n^{-1}))} \le (1+\varepsilon) j^{-a+\varepsilon}. \tag{4.10}$$

In addition, by Theorem 1.5.12 in [3],

$$\limsup_{n\to\infty} \frac{1 - F(a(Q_n^{-1}))}{Q_n} = 1. \tag{4.11}$$

By (4.9)–(4.11) we obtain

$$\lim_{N\to\infty} \limsup_{n\to\infty} \sum_{j=N+1}^{\infty} P\left(\frac{Y_n}{a(Q_n^{-1})} > j \mid Z_{n-1} > 0\right) = \lim_{N\to\infty} \sum_{j=N+1}^{\infty} \frac{1+\varepsilon}{j^{a-\varepsilon}} = 0. \tag{4.12}$$

Now, by (4.8) and (4.12), applying Lemma 4.1 and using (3.8), one can obtain (4.7).

(ii) The proof is similar to those in (i), using (3.5) instead of (3.7).

It is worth noting that Theorem 4.2 and (3.4) (with $\alpha = a - 1$) imply

$$\mathrm{E}(Y_n \mid Z_{n-1} > 0) \sim n^{-(1-1/a)} L_3(n) \mathrm{E}(Z_n \mid Z_{n-1} > 0) \qquad \text{for } \sigma^2 < \infty,$$

$$\mathrm{E}(Y_n \mid Z_{n-1} > 0) \sim n^{-1/a} L_4(n) \mathrm{E}(Z_n \mid Z_{n-1} > 0) \qquad \text{for } \sigma^2 = \infty.$$

where $a > 1$ and $L_3(n)$ and $L_4(n)$ are certain s.v.f.s.

Example 4.2. As a continuation of Example 3.1b, from (4.7) with $a = 2$ we obtain

$$\lim_{n\to\infty} \frac{1}{n^{1/2}(\log n)^{3/2}} \mathrm{E}(Y_n \mid Z_{n-1} > 0) = \frac{\pi}{4}.$$

In the supercritical case we shall prove the following result.

Theorem 4.3. *Assume that* $m > 1$ *and* $\mathrm{E}X_1(1)\log(1+X_1(1)) < \infty$. *If (2.2) holds, then*

$$\lim_{n\to\infty} \frac{\mathrm{E}Y_n}{m^{n/a}(L_1(m^{n/a}))^{1/a}} = \int_0^\infty (1-\psi(x^{-a}))\,\mathrm{d}x, \tag{4.13}$$

where the s.v.f. $L_1(x)$ *is the de Bruijn conjugate of* $1/L(x)$ *and* $\psi(x)$ *is the unique (up to a scale factor) Laplace transform solution of the functional equation (3.13).*

Proof. Since $F \in D(\exp\{-x^{-a}\})$, $a > 0$ and $\mathrm{E}X_1(1)\log(1+X_1(1)) < \infty$, we have by (3.18)

$$\lim_{n\to\infty} P(Y_n > a(m^n)x \mid Z_{n-1} > 0) = \frac{1-\psi(x^{-a})}{1-q},$$

where $a(m^n) = (1/(1-F))^{\leftarrow}(m^n)$ and q is the extinction probability. Note that

$$\int_0^\infty (1-\psi(x^{-a}))\,\mathrm{d}x < \infty \quad \text{iff} \quad \mathrm{E}X_1(1)\log(1+X_1(1)) < \infty$$

(see e.g. [9], Theorem 2.7.2).

Further, by (4.4) we obtain

$$\begin{aligned} P(Y_n > ja(m^n) \mid Z_{n-1} > 0) &\le (1-F(ja(m^n)))\,\mathrm{E}(Z_{n-1} \mid Z_{n-1} > 0) \\ &= \frac{(1-F(ja(m^n)))\,(1-F(a(m^n)))m^n}{(1-F(a(m^n)))Q_{n-1}}. \end{aligned} \tag{4.14}$$

By Theorem 1.5.12 in [3], we obtain

$$\limsup_{n\to\infty} m^n(1-F(a(m^n))) = 1.$$

Therefore, by (4.14), appealing to (4.10) with $Q_n^{-1} = m^n$, we have for $\varepsilon > 0$,

$$\lim_{N\to\infty}\limsup_{n\to\infty} \sum_{j=N+1}^{\infty} P\left(\frac{Y_n}{a(m^n)} > j \,\Big|\, Z_{n-1} > 0\right) = \lim_{N\to\infty} \sum_{j=N+1}^{\infty} \frac{1+\varepsilon}{j^{a-\varepsilon}} = 0.$$

Applying Lemma 4.1, we obtain

$$\lim_{n\to\infty} \mathrm{E}\left(\frac{Y_n}{a(m^n)} \,\Big|\, Z_{n-1} > 0\right) = \frac{1}{1-q}\int_0^\infty (1-\psi(x^{-a}))\,\mathrm{d}x.$$

Now, one can complete the proof, using (3.8) with $Q_n^{-1} = m^n$ and $1-\alpha = a$, and the equality $\mathrm{E}Y_n = \mathrm{E}(Y_n \mid Z_{n-1} > 0)P(Z_{n-1} > 0)$.

Notice that, under the conditions of Theorem 4.3, we have

$$\mathrm{E}Y_n \sim m^{-n(1-1/a)} L_5(n)\mathrm{E}Z_n,$$

where $a > 1$ and $L_5(n)$ is a certain s.v.f.

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