

Number of Complete N -ary Subtrees on Galton-Watson Family Trees

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Received: 5 May 2005 / Revised: 4 January 2006 / Accepted: 18 January 2006
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Abstract We associate with a Bienaymé-Galton-Watson branching process a family tree rooted at the ancestor. For a positive integer N , define a complete N -ary tree to be the family tree of a deterministic branching process with offspring generating function s^N . We study the random variables $V_{N,n}$ and V_N counting the number of disjoint complete N -ary subtrees, rooted at the ancestor, and having height n and ∞ , respectively. Dekking (1991) and Pakes and Dekking (1991) find recursive relations for $P(V_{N,n} > 0)$ and $P(V_N > 0)$ involving the offspring probability generation function (pgf) and its derivatives. We extend their results determining the probability distributions of $V_{N,n}$ and V_N . It turns out that they can be expressed in terms of the offspring pgf, its derivatives, and the above probabilities. We show how the general results simplify in case of fractional linear, geometric, Poisson, and one-or-many offspring laws.

Keywords Branching process · Family tree · Binary tree · N -ary tree

AMS 2000 Subject Classification Primary 60J80 · Secondary 05C05

1 Introduction and Main Results

Consider the family tree associated with a Bienaymé-Galton-Watson process with the following simple reproduction rules. At generation zero, the process starts with single ancestor called root of the tree. Then each individual in the population has,

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independently of the others, a random number X of children distributed according to the offspring distribution with probability generating function (pgf)

$$f(s) = \sum_{k=0}^{\infty} p_k s^k,$$

satisfying $f(1) = 1$. Further on we adopt the well-known construction of a family tree generated by a simple branching process where the individuals are the nodes and the parent-child relations define the arcs of the tree in the following manner, see e.g., Harris (1963), Ch.7. Let the i th child of the ancestor be (i) and in general $(i_1 i_2 \dots i_{k-1} i_k)$ denotes the i_k th child of $(i_1 i_2 \dots i_{k-1})$. Then, a directed arc is assumed to emanate from $(i_1 i_2 \dots i_{k-1})$ to $(i_1 i_2 \dots i_{k-1} i_k)$. Since, in our case, the children appear simultaneously, we suppose that the ordering is performed by a chance device independently of the evolution in the process. This scheme produces family trees (also called rooted ordered trees) in which the nodes of height (also known as depth) n ($n \geq 0$) have labels $(i_1 i_2 \dots i_n)$, with the ancestor (root) having height 0. The height of a subtree equals the maximum height of its nodes.

For fixed integer $N \geq 1$, define a complete infinite N -ary tree to be the family tree of a deterministic branching process with offspring pgf $f(s) = s^N$. Further on we will consider rooted subtrees of a family tree. Two such subtrees are called disjoint if they do not have a common node different from the root. These kinds of trees appear, for example, in some computer algorithms; for more details see Knuth (1997).

Let $\{Z_n : n \geq 1; Z_0 = 1\}$ denote the generation size process, and let $T_N - 1$ be the height of a complete N -ary subtree rooted in the ancestor; $T_N = 0$ if $Z_1 < N$. Notice that T_1 is the extinction time of $\{Z_n\}$. The study of the probability $\tau_N = \lim_{n \rightarrow \infty} P(T_N > n)$ that a Bienaymé-Galton-Watson tree contains an infinite complete N -ary subtree was initiated by Dekking (1991) who considered complete binary ($N = 2$) subtrees. The general ($N \geq 2$) case was subsequently investigated in detail by Pakes and Dekking (1991). In particular, they encountered the following phenomenon: if $N \geq 2$, then there is a critical value m_N^c for the offspring mean $m = f'(1)$ such that $\tau_N = 0$ if $m < m_N^c$ and $\tau_N > 0$ if $m \geq m_N^c$. This is qualitatively different from what happens for $N = 1$ where the probability for non-extinction $\tau_1 = 0$ if $m = m_1^c = 1$, except for the trivial case where $f(s) = s$. Our work is motivated by the results of Pakes and Dekking (1991).

We introduce the random variable V_N to be the number of disjoint complete N -ary subtrees with infinite height, rooted at the ancestor of a Bienaymé-Galton-Watson family tree. Clearly $\tau_N = P(V_N > 0)$. As usual, we assume for the offspring distribution $\{p_k\}_{k=0}^{\infty}$ that $p_k < 1$ for all k and $p_k > 0$ for some $k > N$. Let \mathcal{N} be the set of all positive integers and denote for $x, y \geq 0$ and any $j = 0, 1, \dots$

$$G_N(x, y; j) = \sum_{k=jN}^{jN+N-1} \frac{x^k}{k!} f^{(k)}(y).$$

Pakes and Dekking (1991) showed that $P(V_N = 0) = 1 - \tau_N$, where $1 - \tau_N$ is the smallest solution in $[0, 1]$ of the equation

$$x = G_N(1 - x, x; 0). \tag{1}$$

Our goal is to study the distribution of V_N . As the following result shows, the probability mass function (pmf) of V_N can be obtained using the Taylor expansion of $f(1)$ about the point $1 - \tau_N$.

THEOREM 1 *If $N \in \mathcal{N}$ then for any $j = 0, 1, \dots$*

$$P(V_N = j) = G_N(\tau_N, 1 - \tau_N; j) \tag{2}$$

and $P(V_N = 0) = 1 - \tau_N$ is the smallest solution in $[0, 1]$ of (1).

REMARKS

- (i) If $N = 1$, then obviously $P(V_1 = 0) = 1 - \tau_1 = q$ is the extinction probability of the Galton-Watson process. Now, (2) becomes

$$P(V_1 = j) = \frac{(1 - q)^j}{j!} f^{(j)}(q), \quad j = 0, 1, \dots,$$

which in turn implies that $E(s^{V_1}) = f(q + (1 - q)s)$. This identity follows directly observing that the number of distinct infinite unary trees is equal to the number of first generation nodes having infinite line of descent.¹

- (ii) Also note that a sufficient condition for $P(V_N = 0) < 1$ is given in Pakes and Dekking (1991), Theorem 3. In particular, they show that $P(V_N = 0) < 1$ ($N \geq 2$) if

$$2N \sum_{j \geq N} \frac{p_j}{j + 1} \leq (1 - \sum_{j=0}^{N-1} p_j)^2.$$

The number of complete N -ary subtrees is a measure for the rate of growth (or fertility) of the branching process. In fact, as was pointed out in Dekking (1991), if $P(V_2 > 0) > 0$ then we can say that the branching process grows faster than binary splitting. In the study of the tree structure of branching processes, an important role is played by the process' total progeny. Denote by ν_n the number of individuals who existed in the first $n + 1$ generations, i.e., $\nu_n = 1 + Z_1 + \dots + Z_n$, $n = 1, 2, \dots$. Obviously, ν_n equals the total number of nodes having height less than or equal to n . Let us also define the random variable $V_{N,n}$ to be the number of disjoint complete N -ary subtrees of height at least n rooted at the ancestor of a Bienaymé-Galton-Watson family tree. Let

$$\psi_{N,n}(s) = E(s^{\nu_n}; V_{N,n} > 0) \quad \text{and} \quad \phi_{N,n}(s) = E(s^{\nu_n}; V_{N,n} = 0). \tag{3}$$

The following result presents a recursive relation for the joint distribution of $V_{N,n}$ and ν_n .

THEOREM 2 *If $N \in \mathcal{N}$ then for $|s| \leq 1$ and any $j = 0, 1, \dots$*

$$E(s^{\nu_{n+1}}; V_{N,n+1} = j) = sG_N(\psi_{N,n}(s), \phi_{N,n}(s); j). \tag{4}$$

¹ The authors are indebted to the referee who pointed out this argument. It implies immediately the result of Theorem 1 for unary trees.

Notice that, if $N = 1$ and $j = 0$, then the above recurrence reduces to the well-known $E(s^{v_{n+1}}; Z_{n+1} = 0) = sf(E(s^{v_n}; Z_n = 0))$, see e.g., Kolchin (1986), p. 120.

Applications of complete N -ary trees can be found in the analysis of algorithms, see Knuth (1997). Problems of this nature appear also in percolation theory. For instance, Pakes and Dekking (1991) point out a relationship between the model of N -ary complete and infinite subtrees and a construction employed by Chayes et al. (1988) in their study of Mandelbrot’s percolation processes. The existence of N -ary subtrees is also used by Pemantle (1988) in introducing the concept of a N -infinite branching process. Let us also mention potential connections with problems of percolation of binary words on the nodes of locally finite graphs with countably infinite node-sets, see Benjamini and Kesten (1995).

We organize our paper as follows. In Section 2 we prove the main results. Sections 3–5 contain some illustrations. In Section 3 we consider the family tree generated by the fractional linear $f(s)$ as well as the special case of geometric offspring. In the latter case, V_N itself follows a geometric distribution. It turns out that in the Poisson offspring case, given in Section 4, the pmf of V_N can be expressed in terms of certain Poisson probabilities. Note that the critical values m_N^c ($N \geq 2$) in the Poisson case are less than those in the geometric one. Finally, in Section 5 we consider the one-or-many (i.e., concentrated on two points only) offspring distribution. In this case V_N has a pmf given in terms of binomial probabilities.

2 Proofs of the Theorems

Proof of Theorem 1: Let us consider $P(V_N = j)$ where $j = 1, 2, \dots$. Recall that the random variable $V_{N,n}$ equals the number of disjoint complete N -ary subtrees of height n rooted at the ancestor of a Bienaymé-Galton-Watson family tree. First, we will find the pmf of $V_{N,n+1}$ using the total probability formula. Indeed, to have j disjoint complete N -ary subtrees rooted at the ancestor node there must be $jN + k$ ($k \geq 0$) nodes in the first generation. Each of these nodes can be considered as an ancestor of a family tree rooted at the first generation. Consider the event $A_N(l) = \{jN + l \text{ of the } Z_1 \text{ first generation nodes are ancestors of at least one complete } N\text{-ary tree of height } n\}$, where $l = 0, 1, \dots, \min\{k, N - 1\}$. If $Z_1 = jN + k$ then for fixed l the event $A_N(l)$ has conditional probability

$$P(A_N(l)|Z_1 = jN + k) = \binom{jN + k}{jN + l} (\tau_{N,n})^{jN+l} (1 - \tau_{N,n})^{k-l} \quad (0 \leq l \leq \min\{k, N - 1\}),$$

where $\tau_{N,n} = 1 - P(V_{N,n} = 0)$ and by convention let $\tau_{N,0} = 1$. We have

$$P\left(\bigcup_{l=0}^{\min\{k, N-1\}} A_N(l) | Z_1 = jN + k\right) = \sum_{l=0}^{\min\{k, N-1\}} \binom{jN + k}{jN + l} (\tau_{N,n})^{jN+l} (1 - \tau_{N,n})^{k-l}.$$

Applying the total probability formula and changing the order of summation, we obtain

$$\begin{aligned}
 P(V_{N,n+1} = j) &= \sum_{k=0}^{\infty} P(Z_1 = jN + k) P\left(\bigcup_{l=0}^{\min\{k, N-1\}} A_N(l) \mid Z_1 = jN + k\right) \\
 &= \sum_{k=0}^{\infty} p_{jN+k} \left\{ \sum_{l=0}^{\min\{k, N-1\}} \binom{jN+k}{jN+l} (\tau_{N,n})^{jN+l} (1 - \tau_{N,n})^{k-l} \right\} \\
 &= \sum_{l=0}^{N-1} \frac{\tau_{N,n}^{jN+l}}{(jN+l)!} \sum_{k=l}^{\infty} p_{jN+k} (jN+k)(jN+k-1)\dots(k-l+1)(1 - \tau_{N,n})^{k-l} \\
 &= \sum_{l=0}^{N-1} \frac{\tau_{N,n}^{jN+l}}{(jN+l)!} f^{(jN+l)}(1 - \tau_{N,n}) \\
 &= G_N(\tau_{N,n}, 1 - \tau_{N,n}; j)
 \end{aligned}$$

By definition $\tau_{N,0} = 1$ and $\tau_{N,n} \downarrow \tau_N$ as $n \uparrow \infty$. Letting $n \rightarrow \infty$, we obtain for $j \geq 1$

$$P(V_N = j) = \lim_{n \rightarrow \infty} P(V_{N,n+1} = j) = G_N(\tau_N, 1 - \tau_N; j).$$

Let us now consider the case $j = 0$. The above recurrence is true for $n = 0$, i.e., $P(V_{N,1} = 0) = G_N(1, 0; 0) = \sum_{k=0}^{N-1} p_k$. For $n \geq 1$, using the total probability formula and an argument similar to that for the case $j \geq 1$, we obtain

$$\begin{aligned}
 P(V_{N,n+1} = 0) &= \sum_{l=0}^{N-1} \sum_{k=l}^{\infty} p_k \binom{k}{l} (\tau_{N,n})^l (1 - \tau_{N,n})^{k-l} \tag{5} \\
 &= \sum_{l=0}^{N-1} \frac{(\tau_{N,n})^l}{l!} f^{(l)}(1 - \tau_{N,n}) \\
 &= G_N(\tau_{N,n}, 1 - \tau_{N,n}; 0).
 \end{aligned}$$

Computing the derivative of $G_N(x, 1 - x; 0)$, we get a telescoping sum which after cancelations becomes $dG_N(x, 1 - x; 0)/dx = (1 - x)^{N-1} f^{(N)}(x)/(N - 1)! \geq 0$ for $0 \leq x \leq 1$. Thus, $G_N(x, 1 - x; 0)$ is non-decreasing in $[0, 1]$, and therefore

$$1 - \tau_N = \lim_{n \rightarrow \infty} (1 - \tau_{N,n+1}) = \lim_{n \rightarrow \infty} P(V_{N,n+1} = 0) = G_N(\tau_N, 1 - \tau_N; 0)$$

is the smallest root in $[0, 1]$ of the equation $x = G_N(1 - x, x; 0)$. The proof is complete. ■

Clearly (2) implies that $\sum_{j=0}^{\infty} P(V_N = j) = \sum_{k=0}^{\infty} \tau_N^k f^{(k)}(1 - \tau_N)/k! = f(1) = 1$.

Proof of Theorem 2: Let us introduce the notation

$$\tau_{N,n}(t) = P(V_{N,n} > 0, \nu_n = t), \quad \gamma_{N,n}(t) = P(V_{N,n} = 0, \nu_n = t) = P(\nu_n = t) - \tau_{N,n}(t),$$

where $N, n,$ and t are positive integers. Proceeding as in the proof of Theorem 1, we consider the event

$$A_N(l, t) = A_N(l) \cap \{\nu_{n+1} = t\},$$

where $A_N(l)$ is defined in the proof of Theorem 1. For fixed t and l ($0 \leq l \leq \min(k, N - 1)$), using the fact that all trees rooted in the first generation grow independently, we compute the conditional probability of $A_N(l, t)$ given $Z_1 = jN + k$ to be

$$P(A_N(l, t) | Z_1 = jN + k) = \binom{jN + k}{jN + l} \sum' \prod_{u=1}^{jN+l} \tau_{N,n}(n_u) \prod_{v=jN+l+1}^{jN+k} \gamma_{N,n}(n_v),$$

where the summation in \sum' is over all nonnegative integers $\{n_i\}_{i=1}^{jN+k}$ such that $\sum_{i=1}^{jN+k} n_i = t - 1$. Then, the total probability formula implies that

$$\begin{aligned} P(V_{N,n+1} = j, \nu_{n+1} = t) &= \sum_{k=0}^{\infty} P(Z_1 = jN + k) \sum_{l=0}^{\min(k, N-1)} P(A_N(l, t) | Z_1 = jN + k) \\ &= \sum_{l=0}^{N-1} \sum_{k=l}^{\infty} p_{jN+k} \binom{jN + k}{jN + l} \sum' \prod_{u=1}^{jN+l} \tau_{N,n}(n_u) \prod_{v=jN+l+1}^{jN+k} \gamma_{N,n}(n_v). \end{aligned}$$

Multiplying both sides of this equality by s^t and summing over t , we get

$$\begin{aligned} E(s^{\nu_{n+1}}; V_{N,n+1} = j) &= s \sum_{l=0}^{N-1} \frac{1}{(jN + l)!} \sum_{k=l}^{\infty} p_{jN+k} (jN + k)(jN + k - 1) \dots (k - l + 1) \\ &\quad \times \sum_{t=1}^{\infty} \sum' \prod_{u=1}^{jN+l} \tau_{N,n}(n_u) \prod_{v=jN+l+1}^{jN+k} \gamma_{N,n}(n_v) s^{t-1}. \end{aligned}$$

Observe that the coefficient of s^{t-1} in the series

$$\sum_{t=1}^{\infty} \sum' \prod_{u=1}^{jN+l} \tau_{N,n}(n_u) \prod_{v=jN+l+1}^{jN+k} \gamma_{N,n}(n_v) s^{t-1}$$

can be written as

$$\sum_{h=0}^{t-1} \sum_{n_1 + \dots + n_{jN+l} = h} \prod_{u=1}^{jN+l} \tau_{N,n}(n_u) \sum_{n_{jN+l+1} + \dots + n_{jN+k} = t-1-h} \prod_{v=jN+l+1}^{jN+k} \gamma_{N,n}(n_v).$$

The rule of multiplying power series implies that this coefficient equals the coefficient of s^{t-1} in the power series expansion of

$$\left[\sum_{i=1}^{\infty} \tau_{N,n}(i) s^i \right]^{jN+l} \left[\sum_{i=1}^{\infty} \gamma_{N,n}(i) s^i \right]^{k-l} = [\psi_{N,n}(s)]^{jN+l} [\phi_{N,n}(s)]^{k-l},$$

where $\psi_{N,n}(s)$ and $\phi_{N,n}(s)$ are defined in (3). Therefore,

$$\begin{aligned} E(s^{V_{n+1}}; V_{N,n+1} = j) &= s \sum_{l=0}^{N-1} \frac{[\psi_{N,n}(s)]^{jN+l}}{(jN+l)!} \sum_{k=l}^{\infty} p_{jN+k} (jN+k)(jN+k-1) \dots \\ &\quad (k-l+1) [\phi_{N,n}(s)]^{k-l} \\ &= s \sum_{l=0}^{N-1} \frac{[\psi_{N,n}(s)]^{jN+l}}{(jN+l)!} f^{(jN+l)}(\phi_{N,n}(s)), \end{aligned}$$

which coincides with the right-hand side of (4). This completes the proof. ■

3 Fractional Linear Offspring

Let $f(s)$ be a fractional linear pgf given by

$$f(s) = 1 - \frac{b}{1-p} + \frac{bs}{1-ps} \tag{6}$$

and the parameter space $\{(p, b): 0 < p < 1, 0 < b \leq 1 - p\}$. Then the offspring distribution is given by the geometric series $p_k = bp^{k-1}$, $k = 1, 2, \dots$; $p_0 = 1 - \sum_{k=1}^{\infty} p_k$ and the offspring mean is $m = b/(1-p)^2$. In the particular case $b = p(1-p)$ we have $p_k = (1-p)p^k$, $k \geq 0$ which is the standard geometric distribution with pgf $f(s) = (1-p)/(1-ps)$. It can be verified, see Pakes and Dekking (1991), p. 361 if $N \geq 2$ and Harris (1963), p. 9 if $N = 1$, that for $N \in \mathcal{N}$

$$1 - p(1 - \tau_N) = [b/(1-p)]^{1/N} [p\tau_N]^{1-1/N}. \tag{7}$$

PROPOSITION 1 *If the offspring distribution has the fractional linear pgf (6), then V_N follows a zero-modified geometric (i.e., fractional linear) distribution given by*

$$P(V_N = j) = \frac{b}{p(1-p)} (1 - \theta_N) \theta_N^j \quad (j \geq 1), \quad P(V_N = 0) = 1 - \frac{b}{p(1-p)} \theta_N \tag{8}$$

and

$$EV_N = \frac{b}{p(1-p)} \frac{\theta_N}{1 - \theta_N}, \tag{9}$$

where

$$\theta_N = \left(\frac{p\tau_N}{1 - p(1 - \tau_N)} \right)^N$$

and τ_N is the largest solution in $[0, 1]$ of (7).

Proof: Since $f^{(i)}(s) = i! bp^{i-1}/(1 - ps)^{i+1}$ ($i \geq 1$), we have from (2) for $j \geq 1$

$$\begin{aligned}
 P(V_N = j) &= \sum_{k=0}^{N-1} \frac{\tau_N^{jN+k}}{(jN+k)!} \frac{b(jN+k)! p^{jN+k-1}}{(1 - p(1 - \tau_N))^{jN+k+1}} \\
 &= \frac{bp^{jN-1} \tau_N^{jN}}{(1 - p(1 - \tau_N))^{jN+1}} \sum_{k=0}^{N-1} \frac{(p\tau_N)^k}{(1 - p(1 - \tau_N))^k}.
 \end{aligned}$$

Now, setting $(\theta_N)^{1/N} = p\tau_N/(1 - p(1 - \tau_N))$ one can obtain the first formula in (8), which in turn leads to (8) and (9). ■

COROLLARY *If the offspring distribution is geometric, i.e., $p_k = (1 - p)p^k$, $k \geq 0$, then V_N is geometric as well, $P(V_N = j) = (1 - \tau_N)\tau_N^j$ ($j \geq 0$) and $EV_N = \tau_N (1 - \tau_N)^{-1}$, where τ_N is the largest solution in $[0, 1]$ of $(\tau_N + 1/m)^N = \tau_N^{N-1}$ ($N \geq 1$).*

Proof: In the case of geometric offspring (6) holds with $b = p(1 - p)$ and $m = p/(1 - p)$. The equation for τ_N follows by inspection from (7). It is also given in Pakes and Dekking (1991), p.361 if $N \geq 2$. Simple algebraic manipulations show that this equation simplifies to $\theta_N = \tau_N$. Now, the rest of the statement follows from (8) and (9). ■

REMARK For geometric offspring with mean $m > 1$ we have $P(V_1 = j) = (1/m)(1 - 1/m)^j$ and $EV_1 = m - 1$. In particular, $P(V_1 = 0) = 1/m$ which equals the probability of extinction, see Harris (1963), p. 9.

Table 1 lists the probabilities $P(V_N = j)$, $j = 0, 1, 2, \dots, 9$ as well as EV_N for $1 \leq N \leq 5$. The critical mean values (see Section 1) are as follows: $m_1^c = 1$, $m_2^c = 4$, $m_3^c = 6.75$, $m_4^c = 9.481$, $m_5^c = 12.207$. The expected values in the last column provide a measure of how many N -ary subtrees ($1 \leq N \leq 5$) are supported by the geometric family tree with offspring mean fixed to be $m = 13$. See also Table 2 below for a comparison with the Poisson offspring case.

4 Poisson Offspring

Consider the case of Poisson offspring distribution with pgf given by

$$f(s) = e^{m(s-1)} \quad (m > 0). \tag{10}$$

Table 1 Probability distribution of V_N assuming geometric offspring with $m = 13$

$V_N =$	0	1	2	3	4	5	6	7	8	9	≥ 10	$E(V_N)$
$N = 1$	0.08	0.07	0.07	0.06	0.06	0.05	0.05	0.04	0.04	0.04	0.44	12
$N = 2$	0.16	0.14	0.11	0.10	0.08	0.07	0.06	0.05	0.04	0.03	0.16	5.22
$N = 3$	0.26	0.19	0.14	0.11	0.08	0.06	0.04	0.03	0.02	0.02	0.05	2.91
$N = 4$	0.37	0.23	0.15	0.09	0.06	0.04	0.02	0.01	0.01	0.01	0.01	1.71
$N = 5$	0.53	0.25	0.12	0.05	0.03	0.01	0.01	0	0	0	0	0.87

Table 2 Probability distribution of V_N assuming Poisson offspring with $m = 13$

$V_N =$	0	1	2	3	4	5	6	7	8	9	≥ 10	$E(V_N)$
$N = 2$	0	0	0.01	0.04	0.11	0.19	0.22	0.19	0.13	0.07	0.04	6.25
$N = 3$	0	0.01	0.09	0.25	0.32	0.22	0.08	0.02	0	0	0.01	4.00
$N = 4$	0	0.05	0.30	0.41	0.19	0.04	0	0	0	0	0.01	2.87
$N = 5$	0	0.17	0.51	0.28	0.04	0	0	0	0	0	0	2.19

Then, the probability τ_N is the largest solution of

$$(1 - s)e^{ms} = \sum_{j=0}^{N-1} (ms)^j / j! \tag{11}$$

(see Pakes and Dekking (1991), p. 364). Since $f^{(i)}(s) = m^i e^{m(s-1)} (i \geq 0)$, formula (2) becomes

$$P(V_N = j) = e^{-m\tau_N} \sum_{k=0}^{N-1} \frac{(m\tau_N)^{jN+k}}{(jN+k)!}, j \geq 0.$$

Therefore, we have the following

PROPOSITION 2 *If the offspring distribution has the Poisson pgf (10), then*

$$P(V_N = j) = P(jN \leq Y_N \leq jN + N - 1),$$

where Y_N has the Poisson pmf

$$P(Y_N = k) = (m\tau_N)^k e^{-m\tau_N} / k! \quad k = 0, 1, 2, \dots$$

and τ_N is the largest solution in $[0, 1]$ of equation (11).

Notice that V_1 has a Poisson distribution with parameter $m\tau_1$. To calculate the critical value m_N^c that yields a non-zero solution τ_N^c in $[0, 1]$ of equation (11) we first notice that the product $y = m_N^c \tau_N^c$ satisfies the equations

$$y^N / (N - 1)! + \sum_{j=0}^{N-1} y^j / j! = e^y; \tag{12}$$

see Pakes and Dekking (1991), p. 365. Following their way of calculation, one can find m_N^c and τ_N^c by substituting the solution of (12) into

$$my^{N-1} / (N - 1)! = e^y. \tag{13}$$

In case of binary trees, one can also use the Cayley’s tree function $y(z) = \sum_{k=1}^{\infty} k^{k-1} z^k / k!$ (see e.g., Odlyzko (1995), Section 6.2) evaluated at $z = 1/m_N^c$ for the solution of (12). Inserting it into (12), we obtain $m_2^c = 3.3509$ and $\tau_2^c = 0.5352$.

Our final remark concerns the case $m \rightarrow \infty$. It is easily seen that Proposition 2 and the normal approximation of the Poisson distribution imply a local limit theorem for V_N . Moreover, Pakes and Dekking (1991) showed that in this case $\tau_N \rightarrow 1$. This enables one to centralize and scale the limiting variable V_N in terms of the single parameter m only.

Table 2 gives the probabilities $P(V_N = j)$, $j = 0, 1, 2, \dots, 9$ as well as EV_N for $2 \leq N \leq 5$. The critical mean values are as follows: $m_2^c = 3.3509$, $m_3^c = 5.1494$, $m_4^c = 6.7993$, $m_5^c = 8.3653$.

5 One-or-many Offspring

In this section we consider a two-parameter family of 1-or- r offspring distributions defined for some $p \in (0, 1)$ by $p_1 = 1 - p$ and $p_r = p$, where $r > N > 1$. Its pgf is $f(s) = (1 - p)s + ps^r$ and thus $f'(s) = 1 - p + prs^{r-1}$ and $f^{(k)}(s) = pr(r - 1) \dots (r - k + 1)s^{r-k}$ ($2 \leq k \leq r$). The probability τ_N is the largest solution in $[0, 1]$ of

$$s = p \sum_{k=N}^r \binom{r}{k} s^k (1 - s)^{r-k} \tag{14}$$

(see again Pakes and Dekking (1991), p.366). Applying (2) it is not difficult to obtain

$$P(V_N = 0) = 1 - p + p \sum_{k=0}^{N-1} \binom{r}{k} \tau_N^k (1 - \tau_N)^{r-k}$$

and for $j = 1, 2, \dots$ and $r \geq jN$

$$P(V_N = j) = p \sum_{k=jN}^{jN+U} \binom{r}{k} \tau_N^k (1 - \tau_N)^{r-k},$$

where $U = \min\{N - 1, r - jN\}$. Let $B_r(\tau_N)$ denote a binomial (r, τ_N) random variable.

PROPOSITION 3 *If the offspring pgf is $f(s) = (1 - p)s + ps^r$ ($1 \leq N < r$) and τ_N is the largest solution in $[0, 1]$ of (14), then $P(V_N = 0) = 1 - p + pP(B_r(\tau_N) \leq N - 1)$ and for $j = 1, 2, \dots$*

$$P(V_N = j) = pP(jN \leq B_r(\tau_N) \leq jN + U) \quad \text{if } jN \leq r, \tag{15}$$

where $U = \min\{N - 1, r - jN\}$ and $P(V_N = j) = 0$ if $jN > r$. The expected value of V_N is

$$EV_N = p \sum_{j=1}^{\lfloor r/N \rfloor} jP(jN \leq B_r(\tau_N) \leq jN + U),$$

where $\lfloor x \rfloor$ is the integer part of x .

Table 3 Probability distribution of V_N assuming 1-or-14 offspring with $p = 0.93$ ($m = 13.09$)

$V_N =$	0	1	2	3	4	5	6	7	$E(V_N)$
$N = 2$	0.07	0	0	0	0	0.06	0.53	0.34	5.86
$N = 3$	0.07	0	0	0.06	0.87	0	0	0	5.05
$N = 4$	0.07	0	0.06	0.87	0	0	0	0	2.73
$N = 5$	0.07	0	0.93	0	0	0	0	0	1.86

In particular, if $r = N + 1$ or $r = N + 2$ and $N > 2$, then (15) implies that V_N takes on values 0 or 1; if $N = 2$ and $r = 4$, then V_N takes on values 0, 1, or 2. Table 3 provides some numerical illustrations. Note that the offspring mean $m = 13.09$ enables comparisons with Tables 1 and 2.

It is interesting to point out the following relationship between the 1-or- r and Poisson offspring cases. There exists (see Pakes and Dekking (1991)) a critical value p_N^c such that for $p = p_N^c$ equation (14) has a single solution τ_N^c in $(0, 1)$. Suppose that $\lim_{r \rightarrow \infty} (r\tau_N^c) \rightarrow y$, where y satisfies (13) and (12). Then, applying Theorem 7, Pakes and Dekking (1991), one can obtain that $V_N(r)$ converges in distribution to $V_N(y)$, where $V_N(r)$ and $V_N(y)$ are copies of V_N assuming one-or-many and Poisson offspring with mean m_N^c , respectively.

Acknowledgments We thank the referee for his valuable comments and suggestions and especially for his help to eliminate some defects in Proposition 1. This work was done during L. Mutafchiev's visit at the Mathematics Department of the University of South Florida in 2004-05 academic year. He thanks for the hospitality and support. G. Yanev is partially supported by NFSI-Bulgaria, MM-1101/2001.

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