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CONTROLLED BRANCHING PROCESSES WITH CONTINUOUS TIME

MIGUEL GONZÁLEZ ,* **
MANUEL MOLINA ,* *** AND
INES DEL PUERTO ,* **** *University of Extremadura*
NIKOLAY M. YANEV,***** *Bulgarian Academy of Sciences*
GEORGE P. YANEV ,****** *University of Texas RGV and Bulgarian Academy of Sciences*

Abstract

A class of controlled branching processes with continuous time is introduced and some limiting distributions are obtained in the critical case. An extension of this class as regenerative controlled branching processes with continuous time is proposed and some asymptotic properties are considered.

Keywords: Controlled branching process; renewal process; regenerative process; random time change; limit theorem

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1. Introduction

Controlled branching processes (CBPs) are integer-valued discrete-time Markov processes where the population size in every generation could be randomly regulated before reproduction by emigration of part of the population, or after reproduction by immigration of individuals. It is interesting to point out that in a CBP the evolutions of the individuals are not independent but nonetheless they reproduce independently of each other. The main models and results for this class of processes are presented in the recent monograph [11]. A general definition of a CBP with continuous time (CT) does not exist at present. The main motivation behind the present paper is to introduce a new class of continuous-time controlled branching processes.

Let $\{Z_n,\ n=0, 1, \ldots\}$ be a CBP (see (2.2) and (2.3) for their mathematical definitions), and let $\{N(t),\ t\geq 0\}$ be a renewal process. We study the process $\{Y(t),\ t\geq 0\}$, defined by $Y(t)=Z_{N(t)}$, which is a first attempt to introduce a CBP with CT. If $N(t)=n$ then, at time t, the population size is $Y(t)=Z_n$. We assume that the renewal period is the common lifespan of all individuals. It is clear that $\{Y(t),\ t\geq 0\}$ is not a Markov process unless $\{N(t),\ t\geq 0\}$ is a homogeneous Poisson process. In all cases, the evolutions of the individuals are not

Received 18 November 2019; revision received 19 January 2021.
* Postal address: Department of Mathematics, Faculty of Sciences, University of Extremadura, 06006 Badajoz, Spain.
** Email address: mvelasco@unex.es
*** Email address: mmolina@unex.es
**** Email address: idelpuerto@unex.es
***** Postal address: Department of Operations Research, Probability and Statistics, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria. Email address: yanev@math.bas.bg
****** Postal address: The University of Texas Rio Grande Valley, School of Mathematical & Statistical Sciences, 1201 W. University Drive, Edinburg, TX 78539, USA. Email address: george.yanev@utrgv.edu

independent. The proposed new process could also be referred to as a CBP *subordinated by a renewal process* or as a *randomly indexed* CBP.

A randomly indexed branching process was introduced by Epps [3] as an alternative to geometric Brownian motion for modelling daily stock prices. He considered a Bienaymé–Galton–Watson (BGW) branching process indexed by a Poisson process. Assuming four particular offspring distributions, he studied the asymptotics of the moments, estimates of certain parameters of the process, and model calibration based on real data from the New York Stock Exchange. Furthermore, using simulations, several estimates of the parameters of this process were compared in [2]. Utilizing this stock price model, formulas for pricing European call options and up-and-out barrier options were also derived in [15] and [17], respectively.

BGW branching processes subordinated by a general renewal process were introduced in [16] and [21] (critical case) and in [14] and [18] (non-critical cases). Large deviation problems for a Poisson random indexed BGW branching process were considered in [5], [6], and [9]. Large and moderate deviations for a class of renewal random indexed BGW branching processes were studied in [7] and [8].

The general CBP with CT proposed here is an essential generalization of the processes mentioned above. In this work we investigate $\{Y(t),\ t \geq 0\}$ in the critical case. The paper is organized as follows. In Section 2 we define and discuss the CBP with CT and with single and multitype control functions. In Section 3 we present auxiliary results concerning transfer-type limiting distributions and weighted renewal theory. In Section 4 we prove two limit theorems when the mean of the renewal periods is either finite (Theorem 4.1) or infinite (Theorem 4.2) for a CBP with CT and a single control function. Towards the goal of studying the general CBP with CT and multitype control functions, in Section 5 we investigate the particular case with three specific control functions, that is, the CBP with random migration. We obtain limiting distributions by again considering finite and infinite means of the renewal periods (Theorems 5.1 and 5.2, respectively). Finally, in Section 6 we propose an extension of the process $\{Y(t),\ t \geq 0\}$, namely the regenerative process $\{U(t),\ t \geq 0\}$. It coincides with $\{Y(t),\ t \geq 0\}$ until it hits zero, then upon staying at zero for a random time period, the process regenerates. The basic definition of alternating regenerative processes was proposed in [19], where the so-called Basic Regeneration Theorem was proved. We apply this theorem as well as the limit theorems from Sections 4 and 5, to obtain limiting distributions. At the end of the paper we provide some concluding remarks.

2. Description of models

On a certain probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we consider the following three independent sets of random variables.

(i) Define the set $X = \{X_n(i),\ n, i = 1, 2, \ldots\}$ of non-negative integer-valued independent and identically distributed (i.i.d.) random variables with probability generating function (p.g.f.) $f(s) = \mathbb{E}[s^{X_1(1)}],\ 0 \leq s \leq 1$. Further, let I_0 be a positive integer-valued random variable independent of X with p.g.f. $\Delta(s) = \mathbb{E}[s^{I_0}],\ 0 \leq s \leq 1$. Recall that the classical BGW branching process with I_0 ancestors is defined as follows:

$$Z_0 = I_0, \quad Z_n = \sum_{i=1}^{Z_{n-1}} X_n(i), \quad n = 1, 2, \ldots, \tag{2.1}$$

where $\sum_{i=1}^{0}=0$. Taking into account the independence of the individual evolutions, (2.1) implies $\mathbb{E}[s^{Z_n} \mid Z_0=I_0]=\Delta(f_n(s))$, where $f_n(s)=\mathbb{E}[s^{Z_n} \mid Z_0=1]$ and $f_n(s)=f_{n-1}(f(s))$, $n=1, 2, \ldots$. Clearly zero is an absorbing state.

(ii) Define the set $\phi=\{\phi_n(k),\ n=1, 2, \ldots; k=0, 1, \ldots\}$ of non-negative integer-valued random variables, independent of X, where for every fixed k the subset $\phi(k)=\{\phi_n(k),\ n=1, 2, \ldots\}$ consists of i.i.d. random variables with p.g.f. $g_k(s)=\mathbb{E}[s^{\phi_1(k)}]$, $0\le s\le 1$. The CBP is defined as follows:

$$Z_0=I_0, \quad Z_n=\sum_{i=1}^{\phi_n(Z_{n-1})} X_n(i), \quad n=1, 2, \ldots, \tag{2.2}$$

with ϕ referred to as the set of random control functions. Note that if $\phi_1(k)\equiv k$ a.s. for every k, then $\{Z_n,\ n=0, 1, \ldots\}$ is a BGW branching process defined by (2.1). If this is not the case, let us point out that since $h_n(s)=\mathbb{E}[s^{Z_n} \mid Z_0=I_0]\neq\Delta(\psi_n(s))$ where $\psi_n(s)=\mathbb{E}[s^{Z_n} \mid Z_0=1]$, we have that the evolutions of the individuals are not independent. It follows from (2.2) that the state zero will be absorbing if and only if $\phi_1(0)=0$ a.s. For more details see [11].

Definition (2.2) can be generalized by introducing the set of random control functions $\phi_D=\{\phi_{n,d}(k),\ n=1, 2, \ldots; k=0, 1, \ldots; d\in D\}$ and the set of random variables $X_D=\{X_{n,d}(i),\ n, i=1, 2, \ldots; d\in D\}$, where D is an index set. Then the CBP with multitype control functions is defined as follows:

$$Z_0=I_0, \quad Z_n=\left(\sum_{d\in D}\sum_{i=1}^{\phi_{n,d}(Z_{n-1})} X_{n,d}(i)\right)^+, \quad n=1, 2, \ldots, \tag{2.3}$$

where, as usual, $a^+=\max\{0, a\}$. In some branching models it is assumed that for every fixed d the random variables $\{X_{n,d}(i),\ n, i=1, 2, \ldots\}$ are integer-valued i.i.d. and, for every fixed k and d, the subset $\phi_d(k)=\{\phi_{n,d}(k),\ n=1, 2, \ldots\}$ consists of non-negative integer-valued i.i.d. random variables. Note that the random variable $X_{n,d}(i)$ can be negative, allowing individual emigration in the model. We will consider a particular case of (2.3) in Section 5, which admits a random migration component.

(iii) Finally, define the set $J=\{J_n,\ n=1, 2, \ldots\}$ of positive i.i.d. random variables, independent of X and ϕ, with cumulative distribution function (c.d.f.) $G(x)=\mathbb{P}(J_1\le x)$, $x>0$; $G(0)=0$. Define the renewal process $\{N(t),\ t\ge 0\}$ by

$$N(t)=\max\{n\ge 0\colon S_n\le t\}, \quad t\ge 0, \tag{2.4}$$

where $S_0=0$, $S_n=\sum_{i=1}^{n} J_i$, $n=1, 2, \ldots$. This yields the renewal function

$$\mathbb{E}[N(t)]=\sum_{n=0}^{\infty} G^{*n}(t), \quad t\ge 0,$$

where G^{*n} is the n-fold convolution of G with $G^{*0}=1$.

Definition 2.1. Let $\{Z_n,\ n=0, 1, \ldots\}$ be a CBP defined by (2.2) or (2.3) and let $\{N(t),\ t\ge 0\}$ be a renewal process given by (2.4). Then the continuous-time process $\{Y(t),\ t\ge 0\}$, defined by $Y(t) = Z_{N(t)}$, is called a controlled branching process with continuous time (CBP with CT) or CBP with CT and with multitype control functions, respectively.

Alternatively, the process $\{Y(t),\ t \geq 0\}$ can be called a *randomly indexed* CBP with CT or CBP *subordinated by a renewal process*.

Remark 2.1. If $S_n \leq t < S_{n+1}$, then $N(t) = n$ and $Y(t) = Z_n$, $n = 0, 1, \ldots$. Therefore $\{Y(t),\ t \geq 0\}$ could also be considered as an age-dependent branching process, in which all individuals in a given generation have the same lifetime and they give birth to their offspring simultaneously. More precisely, if $Y(t) = Z_n$ then the lifespan of the individuals from the n th generation is equal to J_n. In general, $\{Y(t),\ t \geq 0\}$ is not a Markov process. It is a Markov process with exponentially distributed individual lifespans in the particular case when $\{N(t),\ t \geq 0\}$ is a homogeneous Poisson process. Note that $\{Y(t),\ t \geq 0\}$ is an example of branching process with dependent evolutions of the individuals, in contrast to Bellman–Harris or Markov branching processes, where the evolutions of the individuals are independent. Finally, let us point out that the discrete-time CBP $\{Z_n,\ n = 0, 1, \ldots\}$ is embedded in the continuous-time CBP $\{Y(t),\ t \geq 0\}$. In fact $Z_n = Y(S_n)$, $n = 0, 1, \ldots$. Moreover, if $J_1 \equiv 1$ a.s., then $Y(n) = Z_n$, $n = 0, 1, \ldots$.

3. Preliminaries

We now provide auxiliary results for investigating limiting distributions of the CBPs with CT $\{Y(t),\ t \geq 0\}$ defined in the previous section. The main tools for further investigations are the p.g.f.

$$h(t;s) = \mathbb{E}[s^{Y(t)} \mid Y(0) = I_0] = \sum_{n=0}^{\infty} \mathbb{P}(N(t) = n) h_n(s), \qquad t \geq 0, \qquad 0 \leq s \leq 1, \tag{3.1}$$

and the conditional distribution function $\mathbb{P}(Y(t) \leq x \mid Y(t) > 0)$, which satisfies

$$\mathbb{P}(Y(t) \leq x \mid Y(t) > 0) = \int_0^{\infty} \mathbb{P}(Z_{\lfloor y \rfloor} \leq x \mid Z_{\lfloor y \rfloor} > 0)\, \mathrm{d}_y \mathbb{P}(N(t) \leq y \mid Z_{N(t)} > 0), \tag{3.2}$$

where $\lfloor y \rfloor$ denotes the integer part of y and $\{N(t),\ t \geq 0\}$ is the renewal process (2.4).

Let

$$\mu = \mathbb{E}[J_1] = \int_0^{\infty} (1 - G(y))\, \mathrm{d}y,$$

the mean of the renewal periods. When $\mu < \infty$, it is known (see [1, Ch. 8.6.1] or [4, Ch. XI.1]) that

$$\mathbb{P}\left(\lim_{t \to \infty} \frac{N(t)}{t/\mu} = 1\right) = 1.$$

Let $\{Z_n,\ n = 0, 1, \ldots\}$ be a non-negative discrete-time process with absorbing state zero, and denote $Q_n = \mathbb{P}(Z_n > 0)$, $n = 0, 1, \ldots$. It is shown using equation (7.17) of [21] that if $Q_n \downarrow 0$ and $Q_n \sim L(n) n^{-\gamma}$ as $n \to \infty$, where $0 < \gamma < 1$ and $L(x)$ is a slowly varying function (s.v.f.), then

$$\lim_{t \to \infty} \mathbb{P}\left(\frac{N(t)}{t/\mu} \leq x \,\middle|\, Z_{N(t)} > 0\right) = V_1(x), \qquad x \geq 0, \tag{3.3}$$

where $V_1(x) = \mathbf{1}_{\{x \geq 1\}}$, with $\mathbf{1}_A$ the indicator function of the set A.

We will also consider the case when $\mu = \infty$ and

$$1 - G(t) \sim \frac{t^{-\rho} \mathcal{L}(t)}{\Gamma(1 - \rho)} \qquad \text{as } t \to \infty, \quad 0 < \rho < 1, \tag{3.4}$$

where $\mathcal{L}(t)$ is an s.v.f. Let $a(t)=(\Gamma(1-\rho)(1-G(t)))^{-1}$. It is known (see [1, Ch. 8.6.2] or [4, Ch. XIV.3]) that

$$\lim_{t\to\infty}\mathbb{P}\left(\frac{N(t)}{a(t)}\le x\right)=G_\rho(x),\qquad x\ge 0,$$

where $G_\rho(x)=\mathbb{P}(\Lambda_\rho\le x)$, with $\Lambda_\rho\overset{d}{=}\xi_\rho^{-\rho}$ and ξ_ρ, has a one-sided ρ-stable distribution, i.e. $\mathbb{E}[\mathrm{e}^{-\lambda\xi_\rho}]=\mathrm{e}^{-\lambda^\rho}$, $\lambda>0$. Note that (3.4) yields $a(t)\sim t^\rho/\mathcal{L}(t)$ as $t\to\infty$. Hence, referring to equation (7.18) in [21] and the conditions on $\{Z_n : n=0, 1, \ldots\}$, we have

$$\lim_{t\to\infty}\mathbb{P}\left(\frac{N(t)}{a(t)}\le x\;\middle|\; Z_{N(t)}>0\right)=V_{\rho,\gamma}(x),\qquad x\ge 0, \tag{3.5}$$

where $V_{\rho,\gamma}(x)=\mathbb{E}[\Lambda_\rho^{-\gamma}\mathbf{1}_{\{\Lambda_\rho\le x\}}]/\mathbb{E}[\Lambda_\rho^{-\gamma}]$ and $\mathbb{E}[\Lambda_\rho^{-\gamma}]=\Gamma(1-\gamma)/\Gamma(1-\gamma\rho)$. Note that $G_\rho(x)$ is known as the c.d.f. of the Mittag–Leffler distribution of order ρ, with Laplace transform $\varphi_\rho(\lambda)=(1+\lambda^\rho)^{-1}$ (see [12] for more details).

Further on we will need some results from weighted renewal theory as follows. For any sequence of real numbers $\{w_n,\ n=0, 1, \ldots\}$, consider the function $W(x)=\sum_{k=0}^{\lfloor x\rfloor} w_k$, $x\ge 0$. The weighted renewal function $H_w(t)$, $t\ge 0$, is defined as follows:

$$H_w(t)=\sum_{n=0}^{\infty} w_n G^{*n}(t)=\sum_{k=0}^{\infty}\mathbb{P}(N(t)=k)W(k)=\mathbb{E}[W(N(t))].$$

Lemma 3.1. *Let* $0<\mu<\infty$ *and let* $L(t)$ *be an s.v.f.*

(i) *If* $w_n\ge 0$, $n=0, 1, \ldots$, $W(t)\uparrow\infty$ *as* $t\to\infty$, *and* $\alpha\ge 0$, *then, as* $t\to\infty$,

$$W(t)\sim L(t)t^\alpha \quad \textit{if and only if} \quad H_w(t)\sim\mu^{-\alpha}L(t)t^\alpha.$$

(ii) *If* $w_n\ge 0$, $n=0, 1, \ldots$, $W(t)\uparrow 1$ *as* $t\to\infty$, *and* $0<\alpha<1$, *then, as* $t\to\infty$,

$$1-W(t)\sim L(t)t^{-\alpha} \quad \textit{if and only if} \quad 1-H_w(t)\sim\mu^\alpha L(t)t^{-\alpha}.$$

Lemma 3.2. *Let* $\mu=\infty$. *Assume* (3.4) *and* $W(t)\sim L_w(t)t^\beta$ *as* $t\to\infty$, *where* $0<\beta<1$ *and* $L_w(x)$ *is an s.v.f. Then* $H_w(t)\sim CW(t/m(t))$ *as* $t\to\infty$, *where* $C=\Gamma(1+\beta)/(\Gamma(1+\rho\beta)\Gamma^\beta(2-\rho))$ *and*

$$m(t)=\int_0^t (1-G(x))\,\mathrm{d}x.$$

Lemma 3.1 is proved in [13, Ch. 2, Theorem 38(a) and Theorem 46(i)]. Lemma 3.2 follows from [13, Theorem 43].

4. Critical controlled branching processes subordinated by a renewal process

In this section we will consider the process $\{Y(t)=Z_{N(t)}, t\ge 0\}$, where $\{Z_n, n=0, 1, \ldots\}$ is the CBP (2.2) with zero as an absorbing state and $\{N(t),\ t\ge 0\}$ is the renewal process (2.4). Let us introduce the following notation for $k=0, 1, \ldots$:

$$\varepsilon(k)=\mathbb{E}[\phi_1(k)],\qquad \nu^2(k)=\mathrm{Var}\,[\phi_1(k)],$$

$$\tau_m(k)=k^{-1}\mathbb{E}[Z_{n+1}|Z_n=k]=k^{-1}\varepsilon(k)m,\qquad \text{where } m=\mathbb{E}[X_1(1)],$$

$$l_2(k)=\mathrm{Var}\,[Z_{n+1}|Z_n=k]=m^2\nu^2(k)+\sigma^2\varepsilon(k),\qquad \text{where } \sigma^2=\mathrm{Var}\,[X_1(1)].$$

Later on the following condition is assumed to hold.

Condition A.

(a) $\tau_m(k) = 1 + c/k$, $k = 1, 2, \ldots, 0 < c < \infty$,

(b) $l_2(k) = \nu k + O(1)$, $k \to \infty$, $0 < \nu < \infty$,

(c) $\sup_{k\geq 1} (g_k^{1/k})'''(1) < \infty$, *where we recall that* $g_k(s) = \mathbb{E}[s^{\phi_1(k)}]$,

(d) $\{\phi_1(k),\ k = 1, 2, \ldots\}$ *have infinite divisible distributions.*

It is worth pointing out that for a CBP, the threshold parameter determining the criticality of the process is the asymptotic mean growth rate, i.e. $\lim_{k\to\infty} \tau_m(k)$, when it exists (note that the asymptotic mean growth rate for a BGW branching process coincides with m). Thus, under Condition A we consider a critical CBP with additional hypotheses.

Finally, denote $Q_n = \mathbb{P}(Z_n > 0)$, $n = 0, 1, \ldots,$ and $Q(x) = \sum_{k=0}^{\lfloor x\rfloor -1} Q_k$, $x \geq 1$, and $Q(x) = 0$ for $0 \leq x < 1$.

Lemma 4.1. *Let* $\{Z_n,\ n = 0, 1, \ldots\}$ *be the CBP* (2.2). *Assume Condition A holds and* $\delta = 2c/\nu$, $0 < \delta < 1$. *Then we have the following.*

(i) $\mathbb{P}(Z_n > 0) \sim Kn^{-(1-\delta)}$ *as* $n \to \infty$, *with* $0 < K < \infty$.

(ii)
$$\lim_{n\to\infty} \mathbb{P}\left(\frac{Z_n}{n} \leq x \,\middle|\, Z_n > 0\right) = \Gamma_{\nu/2,1}(x), \qquad x \geq 0,$$

with $\Gamma_{\nu/2,1}(x)$ *the c.d.f. of a gamma distribution with parameters* $\nu/2$ *and* 1.

(iii) $\mathbb{E}[Z_n] \sim (K\nu/2)n^{\delta}$ *and* $\mathrm{Var}\,[Z_n] \sim (K\nu^2/2(\delta+1))n^{\delta+1}$ *as* $n \to \infty$.

Proof. Assertions (i) and (ii) follow from Theorems 3 and 4 in [10] by using Condition A. We will prove (iii). Let $m_n = \mathbb{E}[Z_n]$. Using Condition A(a), we obtain

$$\begin{aligned} m_{n+1} &= \sum_{k=1}^{\infty} \mathbb{P}(Z_n = k)\mathbb{E}[Z_{n+1} | Z_n = k] \\ &= \sum_{k=1}^{\infty} \mathbb{P}(Z_n = k)(k + c) \\ &= \mathbb{E}[Z_n] + cQ_n \\ &= m_0 + c\sum_{k=0}^{n} Q_k, \end{aligned} \tag{4.1}$$

where $\{Q_n : n = 0, 1, \ldots\}$ is monotone decreasing. Applying Theorem 5 from [4, Ch. XIII.5] and using (i) of this theorem, we obtain $\sum_{k=0}^{n} Q_k \sim (K/\delta)n^{\delta}$ and the first statement in (iii) follows.

Similarly, by Condition A(b), we have

$$\begin{aligned} \mathrm{Var}\,[Z_{n+1}] &= \mathbb{E}[\,\mathrm{Var}\,[Z_{n+1} \mid Z_n]] + \mathrm{Var}\,[\mathbb{E}[Z_{n+1} \mid Z_n]] \\ &= \sum_{k=1}^{\infty} \mathbb{P}(Z_n = k)\,\mathrm{Var}\,[Z_{n+1} | Z_n = k] + \mathrm{Var}\,[m\varepsilon(Z_n)] \end{aligned}$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(Z_n = k)(\nu k + \mathrm{O}(1)) + \mathrm{Var}\,[Z_n + c]$$

$$= \mathrm{Var}\,[Z_0] + \nu \sum_{k=0}^{n} \mathbb{E}[Z_k] + \mathrm{O}(n).$$

Since $m_k = \mathbb{E}[Z_k] \sim (K\nu/2)k^{\delta}$ as $k \to \infty$, using well-known properties of the regular varying functions (r.v.f.s) (e.g. [4, Ch. VIII.9]), we obtain

$$\sum_{k=0}^{n} \mathbb{E}[Z_k] \sim \frac{K\nu}{2(\delta+1)} n^{\delta+1} \quad \text{as } n \to \infty.$$

Consequently

$$\mathrm{Var}\,[Z_n] \sim \frac{K\nu^2}{2(\delta+1)} n^{\delta+1} \quad \text{as } n \to \infty. \qquad \square$$

We will investigate the continuous-time process $\{Y(t) = Z_{N(t)},\, t \geq 0\}$ in Definition 2.1 with (2.2).

Theorem 4.1. *Let* $\{Y(t),\ t \geq 0\}$ *be a CBP with CT given in Definition 2.1 with (2.2). Assume Condition A holds,* $0 < \delta = 2c/\nu < 1$, *and* $0 < \mu < \infty$. *Then we have the following.*

(i) $\mathbb{P}(Y(t) > 0) \sim K\mu^{1-\delta} t^{-(1-\delta)}$ *as* $t \to \infty$, *with K defined in Lemma 4.1(i).*

(ii) $\mathbb{E}[Y(t)] \sim (K\nu/2\mu^{\delta})t^{\delta}$ *and* $\mathrm{Var}\,[Y(t)] \sim (K\nu^2/2(\delta+1))\mu^{-(\delta+1)} t^{\delta+1}$ *as* $t \to \infty$.

(iii)

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{Y(t)}{t/\mu} \leq x \,\Big|\, Y(t) > 0 \right) = \Gamma_{\nu/2,1}(x), \qquad x \geq 0.$$

Proof. (i) Note that from (3.1) we have that $\mathbb{P}(Y(t) = 0) = h(t;0)$ and therefore

$$\mathbb{P}(Y(t) = 0) = \sum_{n=0}^{\infty} \mathbb{P}(N(t) = n) P_n = \mathbb{E}[P_{N(t)}], \tag{4.2}$$

where $P_n = \mathbb{P}(Z_n = 0) = 1 - Q_n \uparrow 1$ as $n \to \infty$.

Introduce $w_0 = P_0 = 0$, $w_k = P_k - P_{k-1}$, $k = 1, 2, \ldots$. Note that $w_k \geq 0$ and $W(x) = \sum_{k=0}^{\lfloor x \rfloor} w_k = P_n$ for $n \leq x < n+1$, $n = 0, 1, \ldots$. Hence we can rewrite (4.2) as

$$\mathbb{P}(Y(t) = 0) = \sum_{n=0}^{\infty} \mathbb{P}(N(t) = n) W(n) = \mathbb{E}[W(N(t))]. \tag{4.3}$$

Note that in this case $W(t) \uparrow 1$ as $t \to \infty$, and by Lemma 4.1(i) we have $1 - W(t) = Q_{\lfloor t \rfloor} \sim Kt^{-(1-\delta)}$ as $t \to \infty$, where $0 < K < \infty$. Therefore, applying Lemma 3.1(ii), we obtain that $\mathbb{P}(Y(t) > 0) \sim \mu^{1-\delta}(1 - W(t))$ as $t \to \infty$, which completes the proof of (i).

(ii) Using (4.1) we obtain

$$\mathbb{E}[Y(t)] = \sum_{n=0}^{\infty} \mathbb{P}(N(t) = n) m_n = m_0 + c \sum_{n=1}^{\infty} \mathbb{P}(N(t) = n) \sum_{k=0}^{n-1} Q_k,$$

where we recall that $m_n = \mathbb{E}[Z_n]$. Then

$$\mathbb{E}[Y(t)] = m_0 + c\mathbb{E}[Q(N(t))]. \tag{4.4}$$

As it was proved in Lemma 4.1 that $Q(n) = \sum_{k=0}^{n-1} Q_k \sim Kn^\delta/\delta$ as $n \to \infty$, Lemma 3.1(i) implies $\mathbb{E}[Q(N(t))] \sim \mu^{-\delta}Kt^\delta/\delta$ as $t \to \infty$. Since $c/\delta = \nu/2$, the first statement in (ii) follows from (4.4). Note that

$$\mathrm{Var}\,[Y(t)] = \mathbb{E}[\mathbb{E}[Y(t)^2 \mid N(t)]] - \mathbb{E}[Y(t)]^2 = \sum_{n=0}^{\infty} \mathbb{E}[Z_n^2]\mathbb{P}(N(t) = n) - \mathbb{E}[Y(t)]^2.$$

Now denote $w_0 = \mathbb{E}[Z_0^2]$ and $w_n = \mathbb{E}[Z_n^2] - \mathbb{E}[Z_{n-1}^2]$, $n = 1, 2, \ldots$. Note that $w_n \geq 0$ and let us define $W(x) = \sum_{k=0}^{\lfloor x \rfloor} w_k = \mathbb{E}[Z_n^2]$ for $n \leq x < n+1$, $n = 1, 2, \ldots$, and $W(x) = 0$ for $0 \leq x < 1$. Consequently $\mathrm{Var}\,[Y(t)] = \mathbb{E}[W(N(t))] - \mathbb{E}[Y(t)]^2$. Applying Lemma 4.1(iii) with $0 < \delta < 1$, we have $W(t) \sim (K\nu^2/2(\delta+1))t^{\delta+1}$ as $t \to \infty$. Moreover, since $W(t) \uparrow \infty$ as $t \to \infty$, using Lemma 3.1(i) and the first statement in part (ii) of this theorem, we obtain $\mathrm{Var}\,[Y(t)] \sim (K\nu^2/2(\delta+1))\nu^{-(\delta+1)}t^{\delta+1}$ as $t \to \infty$. This completes the proof of (ii).

(iii) Let

$$\Phi_t(x) = \mathbb{P}\left(\frac{Y(t)}{t/\mu} \leq x \,\middle|\, Y(t) > 0\right), \quad x \geq 0.$$

Now by equation (3.2) we have

$$\Phi_t(x) = \int_0^\infty \mathbb{P}\left(Z_{\lfloor y \rfloor} \leq \frac{xt}{\mu} \,\middle|\, Z_{\lfloor y \rfloor} > 0\right) \mathrm{d}_y\mathbb{P}(N(t) \leq y \mid Z_{N(t)} > 0).$$

Then, after the substitution $y = ut/\mu$, we obtain

$$\Phi_t(x) = \int_0^\infty \mathbb{P}\left(\frac{Z_{\lfloor ut/\mu \rfloor}}{\lfloor ut/\mu \rfloor} \leq \frac{xt}{\mu\lfloor ut/\mu \rfloor} \,\middle|\, Z_{\lfloor ut/\mu \rfloor} > 0\right) \mathrm{d}_u\mathbb{P}\left(\frac{\mu N(t)}{t} \leq u \,\middle|\, Z_{N(t)} > 0\right). \tag{4.5}$$

Hence, from this latter equality, Lemma 4.1(ii), (3.3), and the generalized Lebesgue dominated convergence theorem (see Theorem 2.4 in [23]), we obtain

$$\lim_{t\to\infty} \Phi_t(x) = \int_0^\infty \Gamma_{\nu/2,1}\left(\frac{x}{u}\right) \mathrm{d}V_1(u) = \Gamma_{\nu/2,1}(x), \quad x \geq 0,$$

which proves (iii). □

Remark 4.1. By Theorem 4.1 we can conclude that in the case $0 < \delta < 1$ the asymptotic behaviour of the continuous-time CBP $\{Y(t), t \geq 0\}$ with $0 < \mu < \infty$ is similar to that of the embedded discrete-time CBP $\{Z_n, n = 0, 1, \ldots\}$. Note that the case $\delta = 1$ is an open problem, along with some other critical subclasses considered in Theorem 4 in [10].

Now we consider the case $\mu = \infty$. The appropriate normalization factor of $\{Y(t),\ t \geq 0\}$ is $a(t) = (\Gamma(1-\rho)(1-G(t)))^{-1}$ introduced in Section 3. Let $\mathcal{L}$ be the s.v.f. introduced in (3.4).

Theorem 4.2. *Let* $\{Y_t,\ t \geq 0\}$ *be a CBP with CT given in Definition 2.1 with (2.2). Assume Condition A holds,* $0 < \delta = 2c/\nu < 1$, $\mu = \infty$, *and (3.4). Then we have the following.*

(i) $\mathbb{P}(Y(t)>0)\sim L_1^*(t)t^{-(1-\delta)\rho}$ *as* $t\to\infty$, *with*

$$L_1^*(t)=K\mathcal{L}^{1-\delta}(t)\frac{\Gamma(\delta)}{\Gamma(1-\rho+\delta\rho)},$$

where K *is defined in Lemma 4.1(i).*

(ii) $\mathbb{E}[Y(t)]\sim L_2^*(t)t^{\rho\delta}$ *as* $t\to\infty$, *with*

$$L_2^*(t)=\frac{K\nu\Gamma(\delta)}{2\rho\Gamma(\delta\rho)\mathcal{L}^{\delta}(t)}.$$

(iii)

$$\lim_{t\to\infty}\mathbb{P}\bigg(\frac{Y(t)}{a(t)}\le x\;\bigg|\;Y(t)>0\bigg)=\Psi(x),\qquad x\ge 0,$$

where

$$\Psi(x)=\frac{\Gamma(1-\rho+\delta\rho)}{\Gamma(\delta)}\int_0^\infty u^{-(1-\delta)}\Gamma_{\nu/2,1}\Big(\frac{x}{u}\Big)\,\mathrm{d}G_\rho(u),\qquad x\ge 0,$$

and $G_\rho(x)$ *is the c.d.f. of the Mittag–Leffler distribution of order* ρ.

Proof. (i) From (4.2) we have

$$\mathbb{P}(Y(t)>0)=\sum_{n=0}^{\infty}\mathbb{P}(N(t)=n)Q_n=\mathbb{E}[Q_{N(t)}].\tag{4.6}$$

Moreover, since (3.4) implies $N(t)/a(t)\xrightarrow{d}\Lambda_\rho$, with $\xrightarrow{d}$ denoting the convergence in distribution (see Section 3), we can apply equation (7.10) from [21] to obtain that, as $t\to\infty$,

$$\mathbb{P}(Y(t)>0)\sim Q_{\lfloor a(t)\rfloor}\mathbb{E}[\Lambda_\rho^{-(1-\delta)}].\tag{4.7}$$

Note that from (3.4) and Lemma 4.1(i) we have, as $t\to\infty$,

$$Q_{\lfloor a(t)\rfloor}\sim Kt^{-\rho(1-\delta)}\mathcal{L}^{1-\delta}(t).$$

On the other hand (3.5) yields

$$\mathbb{E}[\Lambda_\rho^{-(1-\delta)}]=\frac{\Gamma(\delta)}{\Gamma(1-\rho+\delta\rho)},\tag{4.8}$$

which proves (i).

(ii) Under assumption (3.4) we obtain (see [4, Ch. VIII.9]) that, as $t\to\infty$,

$$m(t)=\int_0^t(1-G(x))\,\mathrm{d}x\sim\frac{\mathcal{L}(t)t^{1-\rho}}{\Gamma(2-\rho)}.\tag{4.9}$$

From (4.9) we have $t/m(t)\sim\Gamma(2-\rho)t^{\rho}/\mathcal{L}(t)$ as $t\to\infty$. Moreover, it was proved in Lemma 4.1 that $Q(n)=\sum_{k=0}^{n-1}Q_k\sim Kn^{\delta}/\delta$ as $n\to\infty$. Therefore

$$Q(t/m(t))\sim\frac{K\Gamma^{\delta}(2-\rho)}{\delta\mathcal{L}^{\delta}(t)}t^{\rho\delta}\quad\text{as } t\to\infty.$$

Then, from the latter and by Lemma 3.2, we obtain, as $t \to \infty$,

$$\mathbb{E}[Q(N(t))] \sim C_1 Q(t/m(t)), \quad \text{with } C_1 = \frac{\Gamma(1+\delta)}{\Gamma(1+\rho\delta)\Gamma^{\delta}(2-\rho)},$$

which proves (ii).

(iii) Similarly to the proof of Theorem 4.1, by using

$$\Psi_t(x) = \mathbb{P}\bigg(\frac{Y(t)}{a(t)} \le x \,\bigg|\, Y(t) > 0\bigg),$$

Lemma 4.1(ii), and (3.5), we obtain

$$\lim_{t\to\infty} \Psi_t(x) = \int_0^\infty \Gamma_{\nu/2,1}\bigg(\frac{x}{u}\bigg)\, \mathrm{d}V_{\rho,1-\delta}(u), \tag{4.10}$$

where

$$V_{\rho,1-\delta}(x) = \frac{\mathbb{E}[\Lambda_\rho^{-(1-\delta)} I_{\{\Lambda_\rho \le x\}}]}{\mathbb{E}[\Lambda_\rho^{-(1-\delta)}]}.$$

Since

$$\mathrm{d}\mathbb{E}[\Lambda_\rho^{-(1-\delta)} \mathbf{1}_{\{\Lambda_\rho \le x\}}] = \mathrm{d}\int_0^x u^{-(1-\delta)}\, \mathrm{d}G_\rho(u) = x^{-(1-\delta)}\, \mathrm{d}G_\rho(x),$$

and from (4.8), we obtain (iii). □

Remark 4.2. Interestingly, for the limiting c.d.f. $\Psi(x)$ in (4.10), we have $\Psi(x) = \mathbb{P}(\xi\eta \le x)$, where ξ and η are independent random variables with c.d.f.s $\Gamma_{\nu/2,1}(x)$ and $V_{\rho,1-\delta}(x)$, respectively. Assuming $0 < \delta < 1$, $\mu = \infty$, and (3.4), the asymptotic behaviour of the continuous-time CBP $\{Y(t),\ t \ge 0\}$ is different from that of the embedded discrete-time process $\{Z_n, n = 0, 1, \ldots\}$ due to the heavy tail of the lifespan c.d.f. $G(x)$. As before, the case $\delta = 1$ is an open problem, along with other critical subclasses considered in Theorem 4 of [10].

5. Critical branching processes with random migration and continuous time

In this section we will consider a particular case of the CBP with multitype control functions (2.3). Let $X = \{X_n(i),\ n, i = 1, 2, \ldots\}$ be non-negative integer-valued i.i.d. random variables and let $\eta = \{(\eta_{n,1}, \eta_{n,2}),\ n = 1, 2, \ldots\}$ and $I = \{I_n,\ n = 1, 2, \ldots\}$ be two independent sets of non-negative integer-valued i.i.d. random variables, which are independent from X. Let $\{\xi_n,\ n = 1, 2, \}$ be i.i.d. random variables with $\mathbb{P}(\xi_n = -1) = p$, $\mathbb{P}(\xi_n = 0) = q$, $\mathbb{P}(\xi_n = 1) = r$, being $p + q + r = 1$, and independent from X, η, and I.

Consider $D = \{1, 2, 3\}$, $X_{n,1}(i) = X_n(i)$, $X_{n,2}(i) = -\eta_{n,2}$, $X_{n,3}(i) = I_n$, $\phi_{n,1}(k) = \min\{k, k + \xi_n \cdot \eta_{n,1}\}^+$, $\phi_{n,2}(k) = \xi_n^- \mathbf{1}_{\{k>0\}}$, $\phi_{n,3}(k) = \xi_n^+ \mathbf{1}_{\{k>0\}}$, where we recall that $a^+ = \max\{0, a\}$ and $a^- = \max\{0, -a\}$. The CBP (2.3) with the previous specifications of the offspring distributions and the control functions can be rewritten as

$$Z_0 > 0, \quad Z_n = \Bigg(\sum_{i=1}^{Z_{n-1}} X_n(i) + M_n \mathbf{1}_{\{Z_{n-1}>0\}}\Bigg)^+, \quad n = 1, 2, \ldots, \tag{5.1}$$

where

$$M_n = \begin{cases} -\sum_{j=1}^{\eta_{n,1}} X_n(j) - \eta_{n,2} & \text{with probability } p, \\ 0 & \text{with probability } q, \\ I_n & \text{with probability } r, \end{cases}$$

with $\sum_{k=1}^{x} = 0$, if $x \le 0$. This model can be interpreted as follows: in each generation a random number of families and individuals can emigrate with probability p, or new immigrants appear with probability r, or there is no migration with probability q. Zero is an absorbing state. The particular case with $\eta_{n,1} \equiv 1$ a.s. and $\eta_{n,2} \equiv 0$ a.s., for every n, is considered in detail in the monograph [11]. The CBP defined by (5.1) is studied in [24] and [25].

Denote $m = \mathbb{E}[X_1(1)]$, $2b = \mathrm{Var}\,[X_1(1)]$, $\epsilon_1 = \mathbb{E}[\eta_{1,1}]$, $\epsilon_2 = \mathbb{E}[\eta_{1,2}]$, $a = \mathbb{E}[I_1]$, and

$$\theta = \frac{2\mathbb{E}[M_1]}{\mathrm{Var}\,[X_1(1)]} = \frac{ra - p(m\epsilon_1 + \epsilon_2)}{b}.$$

Note that for this model the asymptotic mean growth rate, namely

$$\begin{aligned} \lim_{k\to\infty} k^{-1}\mathbb{E}[Z_n \mid Z_{n-1} = k] &= \lim_{k\to\infty} k^{-1} \sum_{i=1}^{3} \mathbb{E}[X_{1,i}(1)]\mathbb{E}[\phi_{1,i}(k)] \\ &= \lim_{k\to\infty} k^{-1}[(km - \epsilon_1 m - \epsilon_2)p + kmq + (km + a)r] \\ &= m. \end{aligned}$$

Thus the criticality parameter is the offspring mean, as in a BGW branching process. In this section we consider the model (5.1) in the *critical case* under the following condition.

Condition B.

(a) $m = 1$ *and* $0 < 2b < \infty$,

(b) $0 < a < \infty$, $0 \le \eta_{n,1} \le N_1 < \infty$ *a.s. and* $0 \le \eta_{n,2} \le N_2 < \infty$ *a.s., where* N_1 *and* N_2 *are some constants.*

Recall that in Section 2 we introduced $\Delta(s) = \mathbb{E}[s^{Z_0}]$. The following results take place (see [24, Theorem 2.2] and [25, Theorem 2.1, Theorem 2.2, and Section 5]).

As in the previous section, let us denote $Q_n = \mathbb{P}(Z_n > 0)$, $n = 0, 1, \ldots$.

Lemma 5.1. *Let* $\{Z_n,\ n = 0, 1, \ldots\}$ *be the CBP* (5.1). *Assume Condition B holds and* $0 < \theta < 1$.

(i) *If* $\Delta'(1) = \mathbb{E}[Z_0] < \infty$, *then*

(a) $\mathbb{P}(Z_n > 0) \sim K_1(n)n^{-(1-\theta)}$ *as* $n \to \infty$, *where* $K_1(x)$ *is an s.v.f.,*

(b) $\lim_{n\to\infty} \mathbb{P}(Z_n/bn \le x | Z_n > 0) = 1 - \mathrm{e}^{-x} = \mathcal{E}(x)$, $x \ge 0$,

(c) $m_n = \mathbb{E}[Z_n] \sim b\theta K_1(n)n^{\theta}$ *and* $b_n = \mathbb{E}[Z_n^2] \sim b^2\theta(\theta+1)K_1(n)n^{1+\theta}$ *as* $n \to \infty$, *where* m_n *and* b_n *are non-decreasing sequences (it is assumed additionally that* $\mathbb{E}[Z_0^2] < \infty$ *and* $\mathbb{E}[I_n^2] < \infty$).

(ii) *If* $\Delta(s)=1-(1-s)^{\kappa}L_0(1/(1-s))$, *where* $L_0(x)$ *is an s.v.f. and* $0<\kappa<1-\theta$, *then*

(a) $\mathbb{P}(Z_n>0)\sim K_2(n)n^{-\kappa}$ *as* $n\to\infty$, *with* $K_2(x)$ *being an s.v.f.,*

(b) $\lim_{n\to\infty}\mathbb{P}(Z_n/bn\le x|Z_n>0)=F(x), x\ge 0$, *where F(x) is the c.d.f. with Laplace transform*

$$\varphi(\lambda)=1-\frac{C\lambda^{\kappa}}{(1+\lambda)^{\theta+\kappa}}-\lambda\theta\int_0^1(1-x)^{-\kappa}(1+\lambda x)^{-\theta-1}\,\mathrm{d}x, \tag{5.2}$$

with $C=\Gamma(1-\kappa)\Gamma(1-\theta)/\Gamma(1-\theta-\kappa)$.

We will investigate the continuous-time process $Y(t)=Z_{N(t)}$ in Definition 2.1 by considering (5.1). Note that in this case $\{Y(t),\ t\ge 0\}$ is a CBP with CT which admits two types of emigration (family and individual) and an immigration component in the non-zero states.

Theorem 5.1. *Let* $\{Y(t),\ t\ge 0\}$ *be a CBP with CT given in Definition 2.1 with* (5.1). *Assume Condition B holds,* $0<\theta<1$, *and* $0<\mu<\infty$.

(i) *If* $\Delta'(1)=\mathbb{E}[Y(0)]<\infty$, *then*

(a) $\mathbb{P}(Y(t)>0)\sim K_1(t)\mu^{1-\theta}t^{-(1-\theta)}$ *as* $t\to\infty$, *with* $K_1(x)$ *defined in Lemma 5.1(i)-(a),*

(b) $\mathbb{E}[Y(t)]\sim(b\theta/\mu^{\theta})K_1(t)t^{\theta}$ *and* $\mathrm{Var}\,[Y(t)]\sim(b^2\theta/\mu^{1+\theta})(\theta+1)K_1(t)t^{1+\theta}$ *as* $t\to\infty$,

(c)

$$\lim_{t\to\infty}\mathbb{P}\left(\frac{Y(t)}{bt/\mu}\le x\,\middle|\,Y(t)>0\right)=1-\mathrm{e}^{-x}=\mathcal{E}(x),\qquad x\ge 0.$$

(ii) *If* $\Delta(s)=1-(1-s)^{\kappa}L_0(1/(1-s))$, *where* $L_0(x)$ *is an s.v.f. and* $0<\kappa<1-\theta$, *then*

(a) $\mathbb{P}(Y(t)>0)\sim K_2(t)\mu^{\kappa}t^{-\kappa}$ *as* $t\to\infty$, *with* $K_2(x)$ *defined in Lemma 5.1(ii)-(a),*

(b)

$$\lim_{t\to\infty}\mathbb{P}\left(\frac{Y(t)}{bt/\mu}\le x\,\middle|\,Y(t)>0\right)=F(x),$$

where F(x) is the c.d.f. with Laplace transform (5.2).

Proof. (i)-(a) We will use (4.2) and (4.3) from the proof of Theorem 4.1, where now $W(t)\uparrow 1$ as $t\to\infty$, and by Lemma 5.1(i)-(a) we have $1-W(t)=Q_{\lfloor t\rfloor}\sim K_1(t)t^{-(1-\theta)}$ as $t\to\infty$. Therefore, applying Lemma 3.1(ii), we obtain that $\mathbb{P}(Y(t)>0)\sim\mu^{1-\theta}(1-W(t))$ as $t\to\infty$, which completes the proof of (i)-(a).

(i)-(b) Now introduce the function $W_1(x)=\mathbb{E}[Z_n]=m_n$ for $n\le x<n+1$, $n=0, 1, \ldots$. Note that in this case $W_1(n)=\sum_{k=0}^{n}w_{k,1}$, where $w_{0,1}=m_0$, $w_{k,1}=m_k-m_{k-1}$, $k=1, 2, \ldots$. Therefore $\mathbb{E}[Y(t)]=\mathbb{E}[W_1(N(t))]$. From Lemma 5.1(i)-(c) we have $W_1(t)\sim b\theta K_1(t)t^{\theta}$ as $t\to\infty$, and the conditions of Lemma 3.1(i) are satisfied; then we obtain that $\mathbb{E}[Y(t)]\sim\mu^{-\theta}W_1(t)$ as $t\to\infty$, and the first relation of (i)-(b) follows.

Similarly we can introduce the function $W_2(x)=\sum_{k=0}^{\lfloor x\rfloor}w_{k,2}=b_n$ for $n\le x<n+1$, $n=0, 1, \ldots$, where $w_{0,2}=b_0$ and $w_{k,2}=b_k-b_{k-1}$, $k=1, 2, \ldots$; recall that $b_n=\mathbb{E}[Z_n^2]$. Therefore $\mathbb{E}[Y^2(t)]=\mathbb{E}[W_2(N(t))]$. Again from Lemma 5.1(i)-(c) we have $W_2(t)\sim b^2\theta(\theta+1)K_1(t)t^{1+\theta}$ as $t\to\infty$, and by Lemma 3.1(i) we obtain $\mathbb{E}[Y^2(t)]\sim\mu^{-1-\theta}W_2(t)$ as $t\to\infty$, which proves the second relation of (i)-(b).

(i)-(c) The proof is similar to that of Theorem 4.1(iii) by considering

$$\Phi_t(x) = \mathbb{P}\left(\frac{\mu Y(t)}{bt} \le x \,\middle|\, Y(t) > 0\right), \quad x \ge 0,$$

and applying Lemma 5.1(i)-(b) and (3.3). Thus

$$\lim_{t\to\infty} \Phi_t(x) = \int_0^\infty \mathcal{E}\left(\frac{x}{u}\right) \mathrm{d}V_1(u) = \mathcal{E}(x), \quad x \ge 0,$$

which proves (i)-(c).

(ii)-(a) In this case we will use (4.2) and Lemma 5.1(ii)-(b). Then we have that, as $t \to \infty$, both $W(t) \uparrow 1$ and $1 - W(t) = Q_{\lfloor t \rfloor} \sim K_2(t)t^{-\kappa}$. Hence, by Lemma 3.1(ii), we obtain that $\mathbb{P}(Y(t) > 0) \sim \mu^\kappa (1 - W(t))$ as $t \to \infty$, which proves (ii)-(a).

(ii)-(b) It follows from equation (4.5), applying Lemma 5.1(ii)-(b) and (3.3) that

$$\lim_{t\to\infty} \Phi_t(x) = \int_0^\infty F\left(\frac{x}{u}\right) \mathrm{d}V_1(u) = F(x), \quad x \ge 0,$$

which proves (ii)-(b) in this theorem. □

Remark 5.1. Theorem 5.1 shows that in the case $0 < \theta < 1$, the asymptotic behaviour of the continuous-time CBP $\{Y(t),\ t \ge 0\}$ with $0 < \mu < \infty$ is quite similar to that of the discrete-time process $\{Z_n,\ n = 0, 1, \ldots\}$. In this case we can conclude that the limiting properties of the embedded discrete-time process are transferred to the continuous-time process. Note that investigating the asymptotic behaviour of $\{Y(t),\ t \ge 0\}$ in the cases $\theta \le 0$ and $\theta \ge 1$ is an open problem.

We now consider the case $\mu = \infty$. Recall that the appropriate normalization factor of $\{Y(t),\ t \ge 0\}$ is defined by $a(t) = (\Gamma(1-\rho)(1 - G(t)))^{-1}$ introduced in Section 3. Recall also that $\mathcal{L}$ is the s.v.f. introduced in (3.4).

Theorem 5.2. *Let* $\{Y(t),\ t \ge 0\}$ *be a CBP with CT given in Definition 2.1 by considering* (5.1). *Assume Condition B holds,* $0 < \theta < 1$, $\mu = \infty$, *and* (3.4).

(i) *If* $\Delta'(1) = \mathbb{E}[Y(0)] < \infty$, *then*

(a) $\mathbb{P}(Y(t) > 0) \sim L_3^*(t) t^{-\rho(1-\theta)}$ *as* $t \to \infty$, *with*

$$L_3^*(t) = \mathcal{L}^{1-\theta}(t) K_1\left(\frac{t^\rho}{\mathcal{L}(t)}\right) \frac{\Gamma(\theta)}{\Gamma(1 - \rho + \theta\rho)},$$

where $K_1(x)$ *is defined in Lemma 5.1(i)-(a),*

(b) $\mathbb{E}[Y(t)] \sim L_4^*(t) t^{\rho\theta}$ *as* $t \to \infty$, *with*

$$L_4^*(t) = \frac{b\theta\Gamma(1+\theta) K_1(t^\rho/\mathcal{L}(t))}{\Gamma(1+\theta\rho)\mathcal{L}^\theta(t)},$$

(c)

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{Y(t)}{ba(t)} \le x \,\middle|\, Y(t) > 0\right) = \Phi(x), \quad x \ge 0,$$

where

$$\Phi(x)=\frac{\Gamma(1-\rho+\theta\rho)}{\Gamma(\theta)}\int_0^\infty \mathcal{E}\left(\frac{x}{u}\right)u^{-(1-\theta)}\,\mathrm{d}G_\rho(u),$$

with $G_\rho(x)$ *the c.d.f. of the Mittag–Leffler distribution of order* ρ.

(ii) *If* $\Delta(s)=1-(1-s)^\kappa L_0(1/(1-s))$, *where* $0<\kappa<1-\theta$ *and* $L_0(t)$ *is an s.v.f., then*

(a) $\mathbb{P}(Y(t)>0)\sim L_5^*(t)t^{-\kappa\rho}$ *as* $t\to\infty$, *with*

$$L_5^*(t)=\mathcal{L}^\kappa(t)K_2\left(\frac{t^\rho}{\mathcal{L}(t)}\right)\frac{\Gamma(1-\kappa)}{\Gamma(1-\kappa\rho)},$$

where $K_2(x)$ *is defined in Lemma 5.1(ii)-(a),*

(b)

$$\lim_{t\to\infty}\mathbb{P}\left(\frac{Y(t)}{ba(t)}\le x\,\middle|\,Y(t)>0\right)=\tilde{\Phi}(x),\qquad x\ge 0,$$

where

$$\tilde{\Phi}(x)=\frac{\Gamma(1-\rho+\theta\rho)}{\Gamma(\theta)}\int_0^\infty F\left(\frac{x}{u}\right)u^{-(1-\theta)}\,\mathrm{d}G_\rho(u),$$

with $F(x)$ *the c.d.f. with the Laplace transform presented by* (5.2).

Proof. (i)-(a) As in the proof of Theorem 4.2(i), we obtain

$$\mathbb{P}(Y(t)>0)\sim Q_{\lfloor a(t)\rfloor}\mathbb{E}[\Lambda_\rho^{-(1-\theta)}]\quad \text{as } t\to\infty.$$

Now, from Lemma 5.1(i)-(a) and (3.4), we have

$$Q_{\lfloor a(t)\rfloor}\sim K_1(t^\rho/\mathcal{L}(t))t^{-\rho(1-\theta)}\mathcal{L}^{1-\theta}(t)\quad \text{as } t\to\infty.$$

Therefore, by the line after (3.5), we obtain

$$\mathbb{E}[\Lambda_\rho^{-(1-\theta)}]=\frac{\Gamma(\theta)}{\Gamma(1-\rho+\theta\rho)},$$

which completes the proof of (i)-(a).

(i)-(b) From (4.9) we have that $t/m(t)\sim\Gamma(2-\rho)t^\rho\mathcal{L}(t)$ as $t\to\infty$. Moreover, $W_1(t)\sim b\theta K_1(t)t^\theta$ as $t\to\infty$ (see the proof of Theorem 5.1(i)-(b)). Hence, as $t\to\infty$,

$$W_1(t/m(t))\sim b\theta K_1(t^\rho/\mathcal{L}(t))\Gamma^\theta(2-\rho)t^{\rho\theta}/\mathcal{L}^\theta(t). \tag{5.3}$$

Since all assumptions of Lemma 3.2 are fulfilled, then, as $t\to\infty$,

$$\mathbb{E}[Y(t)]\sim C_2W_1(t/m(t)),\qquad \text{with } C_2=\frac{\Gamma(1+\theta)}{\Gamma(1+\rho\theta)\Gamma^\theta(2-\rho)}. \tag{5.4}$$

Therefore, by (5.3) and (5.4), we obtain (i)-(b).

(i)-(c) As in the proof of Theorem 4.2, denoting

$$\Phi_t(x)=\mathbb{P}\left(\frac{Y(t)}{ba(t)}\le x\middle|Y(t)>0\right),\qquad x\ge 0,$$

and taking into account Lemma 5.1(i)-(b) and (3.5), we have

$$\lim_{t\to\infty} \Phi_t(x) = \int_0^\infty \mathcal{E}\left(\frac{x}{u}\right) \mathrm{d}V_{\rho,1-\theta}(u) \tag{5.5}$$

and consequently (i)-(c).

(ii)-(a) The proof is similar to that in (i)-(a), applying (4.6), (4.7), and Lemma 5.1(ii)-(a).

(ii)-(b) The proof is similar to that in (i)-(c), by using Lemma 5.1(ii)-(b) and (3.5). □

Remark 5.2. Note that in Theorems 5.1 and 5.2 the parameter θ plays a similar role to that of δ in the case of a CBP with CT and single control function. Each of these two parameters appears in the rate of convergence of $\mathbb{P}(Z_n > 0)$. This rate of convergence is needed to apply the renewal theory and obtain the behaviour of the process in continuous time.

It is also interesting to point out that for the limiting c.d.f. $\Phi(x)$ in (5.5) we have $\Phi(x) = \mathbb{P}(\xi\eta \le x)$, where ξ and η are independent random variables with c.d.f. $\mathcal{E}(x)$ and $V_{\rho,1-\theta}(x)$, respectively. Theorem 5.2 shows that for $0 < \theta < 1$, $\mu = \infty$, and (3.4), the asymptotic behaviour of the continuous-time CBP $\{Y(t),\ t \ge 0\}$ is quite different from that of the embedded discrete-time process $\{Z_n,\ n = 0, 1, \ldots\}$. One explanation is that the c.d.f. of the individual lifespan $G(x)$ has a heavy tail with $\mu = \infty$. Investigating the process $\{Y(t),\ t \ge 0\}$ in the cases $\theta \le 0$ and $\theta \ge 1$ is an open problem.

6. Regenerative controlled branching processes with continuous time

So far we have studied models of branching processes absorbed at zero. In this section we will extend these models allowing an immigration component at zero.

Let $Y = \{Y(t),\ t \ge 0\}$ be the CBP with CT investigated in Sections 4 or 5, where $\tau = \inf\{t\colon Y(t) = 0\}$ is the life period with $\mathbb{P}(\tau < \infty) = 1$ and c.d.f. $B(t) = \mathbb{P}(\tau \le t) = \mathbb{P}(Y(t) = 0)$. Assume also that $\zeta = \{\zeta_i,\ i = 1, 2, \ldots\}$ are non-negative i.i.d. random variables with c.d.f. $A(x) = \mathbb{P}(\zeta_1 \le x)$. The sets ζ and Y are assumed independent.

Let $Y_k = \{Y_k(t),\ t \ge 0\}$, $k = 1, 2, \ldots$, be the i.i.d. copies of $Y = \{Y(t),\ t \ge 0\}$ with corresponding life periods τ_k and c.d.f. $B(x)$.

We will use the sequence of the random vectors $\{(\zeta_i, \tau_i),\ i = 1, 2, \ldots\}$ to define the renewal epochs $S_0 = 0$, $S_n = S_{n-1} + \eta_n$, where $\eta_n = \zeta_n + \tau_n$, $n = 1, 2, \ldots$. Let $\varkappa(t) = \max\{n\colon S_n \le t\}$ be the corresponding renewal process. Also consider the alternating renewal epochs

$$\{(S_n,\ S^*_{n+1})\colon S^*_{n+1} = S_n + \zeta_{n+1},\ n = 0, 1, \ldots\}$$

and introduce the process $\{\sigma(t),\ t \ge 0\}$, $\sigma(t) = t - S^*_{\varkappa(t)+1}$. Then the regenerative branching process $U = \{U(t),\ t \ge 0\}$ is defined as follows:

$$U(t) = Y_{\varkappa(t)+1}(\sigma(t))1_{\{\sigma(t)\ge 0\}}, \qquad t \ge 0. \tag{6.1}$$

Note that ζ is interpreted as a set of waiting periods. If $\sigma(t) \ge 0$ then it is called a *spent lifetime*, and if $\sigma(t) < 0$ then $|\sigma(t)|$ is called a *rest waiting time*.

The process $\{U(t),\ t \ge 0\}$ develops as follows: $U(t)$ is defined as zero during the waiting periods $S_{n-1} \le t < S^*_n$, and $U(t)$ coincides with the process $Y_n(t - S^*_n)$ during the life periods $S^*_n \le t < S_n$, $n = 1, 2, \ldots$.

Recall that $\Delta(s) = \mathbb{E}[s^{I_0}] = \mathbb{E}[s^{Y(0)}]$. Then $Y_k(0)$ can be interpreted as immigration components at state zero. If $Y(n) = Z_n$, $n = 0, 1, \ldots$, and $\{Z_n,\ n = 0, 1, \ldots\}$ is a BGW branching process, then $\{U(n),\ n=0, 1, \ldots\}$ is a branching process with state-dependent immigration (i.e. immigration at zero only), well known as the Foster–Pakes model (see [19] for more details and further generalizations).

Further on we will consider the case when $A(x)$ and $B(x)$ are non-lattice c.d.f.s, $A(0)=B(0)=0$, and there exists

$$\lim_{t\to\infty} \frac{1-A(t)}{1-B(t)} = C, \quad 0 \le C \le \infty. \tag{6.2}$$

For the distribution of the waiting periods, $A(x)$ say, we will assume one of the following conditions:

$$m_A = \mathbb{E}[\zeta_1] < \infty, \tag{6.3}$$

$$m_A = \infty, \quad 1-A(x) \sim x^{-\alpha} L_A(x) \quad \text{as } x\to\infty, \tag{6.4}$$

where $1/2 < \alpha \le 1$, $L_A(x)$ is an s.v.f., and for every $h>0$ fixed, $A(t)-A(t-h)=\mathrm{O}(1/t)$ as $t\to\infty$.

The asymptotic behaviour of $U(t)$ is related to the asymptotic behaviour of the regeneration period. Assume that the latter is described by the following condition:

$$\lim_{t\to\infty} \mathbb{P}\biggl(\frac{Y_k(t)}{M(t)} \le x \biggm| \tau_k > t\biggr) = D(x), \quad x \ge 0, \tag{6.5}$$

where $M(t)$ is a positive r.v.f. with exponent $\varsigma \ge 0$ and $D(x)$ is a proper c.d.f.

Lemma 6.1. *Let $\{U(t)\colon t>0\}$ be the regenerative branching process defined in (6.1). Assume (6.2), (6.5), and $\mathbb{P}(Y(t)>0) \sim L_R(t)t^{-\beta}$ as $t\to\infty$, where $L_R(t)$ is an s.v.f. and $1/2<\beta<1$.*

(i) *If (6.3) or (6.4) holds true and $0\le C<\infty$, then for $x\ge 0$*

$$\lim_{t\to\infty} \mathbb{P}\biggl(\frac{U(t)}{M(t)} \le x\biggr) = \frac{\digamma_1(x)+C}{1+C}, \tag{6.6}$$

where

$$\digamma_1(x) = \pi^{-1}\sin\pi\beta \int_0^1 D(xu^{-\varsigma})u^{-\beta}(1-u)^{\beta-1}\,\mathrm{d}u.$$

(ii) *If (6.4) holds true and $C=\infty$, then for $x\ge 0$*

$$\lim_{t\to\infty} \mathbb{P}\biggl(\frac{U(t)}{M(t)} \le x \biggm| U(t)>0\biggr) = \digamma_2(x), \tag{6.7}$$

where

$$\digamma_2(x) = \frac{1}{B(1-\beta,\alpha)} \int_0^1 D(xu^{-\varsigma})u^{-\beta}(1-u)^{\alpha-1}\,\mathrm{d}u.$$

Note that Lemma 6.1 follows by Theorem 2.1 (Basic Regeneration Theorem) in [19].

Referring to the limiting results in Theorems 4.1–4.2, we are now in a position to apply Lemma 6.1. Let $\{U(t),\ t\ge 0\}$ be the regenerative branching process defined by (6.1), where $\{Y(t),\ t\ge 0\}$ is the CBP with CT investigated in Section 4. The following result holds.

Theorem 6.1. *Let $\{U(t),\ t\ge 0\}$ be the regenerative branching process defined by (6.1), where $\{Y(t),\ t\ge 0\}$ is a CBP with CT given by Definition 2.1 with (2.2). Assume Condition A holds. Let $0<\mu<\infty$ and $0<\delta=2c/\nu<1/2$.*

(i) *If, additionally,* (6.3) *or* (6.4) *holds and* $0 \le C < \infty$*, then for* $x \ge 0$

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{U(t)}{t/\mu} \le x\right) = \frac{\digamma_1(x) + C}{1 + C}, \tag{6.8}$$

where

$$\digamma_1(x) = \pi^{-1} \sin \pi(1-\delta) \int_0^1 \Gamma_{\nu/2,1}\left(\frac{x}{u}\right)(1-u)^{-\delta} u^{-(1-\delta)}\, \mathrm{d}u.$$

(ii) *If* (6.4) *holds and* $C = \infty$*, then for* $x \ge 0$

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{U(t)}{t/\mu} \le x \,\middle|\, U(t) > 0\right) = \digamma_2(x), \tag{6.9}$$

where

$$\digamma_2(x) = \frac{1}{B(\delta, \alpha)} \int_0^1 \Gamma_{\nu/2,1}\left(\frac{x}{u}\right)(1-u)^{\alpha-1} u^{-(1-\delta)}\, \mathrm{d}u.$$

Proof. Note that by Theorem 4.1(i), by denoting $L_\delta(t) = K\mu^{1-\delta}$, and (6.4), we have in (6.2) that

$$\frac{1-A(t)}{1-B(t)} = (L_A(t)/L_\delta(t))t^{1-\delta-\alpha}.$$

Therefore $C = 0$ if $1-\delta < \alpha$ and $C = \infty$ if $1-\delta > \alpha$. In the case $1-\delta = \alpha$ we have that $C = 0$ if $L_A(t)/L_\delta(t) \to 0$, $C = \infty$ if $L_A(t)/L_\delta(t) \to \infty$, and $0 < C < \infty$ if $L_A(t)/L_\delta(t)$ converges to a positive constant. One has $1 - B(t) = \mathrm{o}(1 - A(t))$ and hence $C = \infty$ in this case. Note that under the conditions of the theorem, the limiting results (i) and (iii) in Theorem 4.1 are valid. Therefore we can use the limiting distributions (6.6) and (6.7) of Lemma 6.1, where upon substituting $\beta = 1-\delta$, $M(t) = t/\mu$, $D(x) = \Gamma_{\nu/2,1}(x)$, and $\varsigma = 1$, we establish (6.8) and (6.9). □

Remark 6.1. We proved Theorem 4.1 under the condition $0 < \delta < 1$, whereas Lemma 6.1 requires $1/2 < \beta < 1$, which in turn implies $0 < \delta < 1/2$. Moreover, note that $\digamma_1(x)$ is a c.d.f. of a product of two independent random variables with c.d.f.s $\Gamma_{n/2,1}(x)$ and that of a beta distribution of parameters $1-\delta$ and δ. The analogous conclusion is valid for $\digamma_2(x)$, with c.d.f.s $\Gamma_{n/2,1}(x)$ and that of a beta distribution with parameters α and δ.

Theorem 6.2. *Let* $\{U(t),\ t \ge 0\}$ *be the regenerative branching process defined by* (6.1)*, where* $\{Y(t),\ t \ge 0\}$ *is a CBP with CT given by Definition* 2.1 *with* (2.2)*. Assume Condition A holds. Let* $\mu = \infty$ *and* (3.4) *holds with* $1/2 < (1-\delta)\rho < 1$.

(i) *If, additionally,* (6.3) *or* (6.4) *holds and* $0 \le C < \infty$*, then for* $x > 0$

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{U(t)}{a(t)} \le x\right) = \frac{G_1(x) + C}{1 + C},$$

where

$$G_1(x) = \pi^{-1} \sin \pi(1-\delta)\rho \int_0^1 \Psi(x/u^\rho)(1-u)^{(1-\delta)\rho-1} u^{-(1-\delta)\rho}\, \mathrm{d}u,$$

with $\Psi(x)$ *defined in Theorem* 4.2*.*

(ii) *If* (6.4) *holds and* $C=\infty$, *then for* $x\geq 0$

$$\lim_{t\to\infty}\mathbb{P}\left(\frac{U(t)}{a(t)}\leq x\,\middle|\,U(t)>0\right)=G_2(x),$$

where

$$G_2(x)=\frac{1}{B(1-(1-\delta)\rho,\ \alpha)}\int_0^1\Psi(x/u^\rho)(1-u)^{\alpha-1}u^{-(1-\delta)\rho}\,\mathrm{d}u.$$

Proof. The proof is similar to that of Theorem 6.1. Indeed, in the limiting distributions (6.6) and (6.7) of Lemma 6.1, we have to make the substitutions $\beta=(1-\delta)\rho$, $M(t)=a(t)$, $\varsigma=\rho$, and $D(x)=\Psi(x)$. □

We can also consider the process $\{U(t),\ t\geq 0\}$ defined by (6.1), where $\{Y(t),\ t\geq 0\}$ is the CBP with CT given by Definition 2.1 with (5.1), investigated in Section 5. Then, applying limiting distributions (6.6) and (6.7) of Lemma 6.1 together with Theorems 5.1 and 5.2, we can obtain counterparts of Theorems 6.1 and 6.2.

More precisely, Theorem 5.1 has to be combined with Lemma 6.1, where we have to set $\beta=1-\theta$ under the condition $0<\theta<1/2$, $M(t)=bt/\mu$, $D(x)=\mathcal{E}(x)$, and $\varsigma=1$. Similarly, Theorem 5.2 has to be combined with Lemma 6.1, where we have to set $\beta=(1-\delta)\rho$ under the condition $1/2<(1-\delta)\rho<1$, $M(t)=a(t)$, $D(x)=\Phi(x)$, and $\varsigma=\rho$.

Finally, note that some other applications of the Basic Regeneration Theorem from [19] in the theory of branching processes are given in [21] and references therein.

7. Concluding remarks

We have introduced a new class of CBPs with CT and investigated its asymptotic behaviour in some critical cases, using the limiting distributions of the CBP with discrete time and certain transfer-type limit theorems for renewal and regenerative processes. If the mean μ of the renewal periods is finite, then the limiting behaviour of the processes with discrete and continuous time is similar. However, if μ is infinite, then the normalization of the processes is different as well as their limiting distributions.

As mentioned in the Introduction, randomly indexed BGW branching processes were successfully applied as stock price models. The new extensions introduced here open possibilities for more diverse applications. Although the randomly indexed branching processes appeared in financial mathematics, it seems that they could also be applied in cell biology studies, especially in the analysis of clonal data, PCR processes, and cell proliferation models.

The investigation of the non-critical processes as well as the critical processes under different conditions is an open problem. It would also be interesting to obtain more detailed results for some particular CBP classes. Another open problem is to define a CBP with CT along the lines of the construction in [22] for two-sex branching processes.

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