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Characterizations of Logistic Distribution Through Order Statistics with Independent Exponential Shifts

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Abstract. Distributional properties of logistic order statistics subject to independent exponential one-sided and two-sided shifts are established. Utilizing these properties, we extend several known results and obtain new characterizations of the logistic distribution.

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1 Discussion of Main Results

The standard logistic cumulative distribution function (c.d.f.) is given by

$$F_L(x) = \frac{e^x}{1+e^x}, \quad -\infty < x < \infty. \tag{1}$$

The density curve resembles that of normal distribution with heavier tails (higher kurtosis). The mean and variance are 0 and $\frac{\pi^2}{3}$, respectively. The logistic distribution has an important place in both probability theory and statistics as a model for population growth. It has been successfully applied to modeling in such diverse areas as demography, biology, epidemiology, environmental studies, psychology, marketing, etc. Recent applications in economics include portfolio modeling (Bowden [3]), approximation of the fill rate of inventory systems (Zhang and Zhang [12]), and Hubbert models of production trends of various resources (Modis [10]).

Characterizations of distributions is an active area of contemporary probability theory. They reveal intrinsic properties of distributions as well as connections, sometimes unexpected, between them. Within the abandon literature, there are surprisingly few results about the logistic distribution (see the discussion in Galambos [5] and Lin and Hu [9]). George and Mudolkar [6,7] were first to obtain characterizations involving order statistics. Recently, Lin and Hu [9] generalized their results and derived new interesting ones. We extend some of George and Mudolkar's

[6] and Lin and Hu's [9] findings (Theorem 2 and the corollary) and obtain new characterizations through order statistics with independent exponential shifts.

Let $X_1, X_2, \ldots, X_n$ be independent copies of a random variable X with absolutely continuous (with respect to the Lebesgue measure) cumulative distribution function (c.d.f.) F. Denote the corresponding order statistics by $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$. Let ϕ_k and F_k (suppressing the dependence on n) be the characteristic function (ch.f.) and the c.d.f. of $X_{k,n}$ for $1 \le k \le n$, respectively. Equality in distribution is denoted by $\stackrel{d}{=}$. Let E'_j and E''_j, $1 \le j \le n$, be mutually independent standard exponential random variables, which are also independent from X_i for $1 \le i \le n$.

We characterize the logistic distribution by distribution equalities between order statistics with random exponential shifts. Three cases are considered depending on the relative positions of the order statistics: adjacent, two, and three spacings away.

Theorem 1. *Suppose F is an absolutely continuous (with respect to the Lebesgue measure) c.d.f. with $F(0) = \frac{1}{2}$. Choose r to be 1, 2, or 3 and let k_i for $1 \le i \le r$ be distinct integers in $[1, n-r]$. Then F is standard logistic if and only if*

(i) *$t^r \phi_{k_i}(t)$ and $t^r \phi_{k_i+r}(t)$ are absolutely integrable for any t; and*

(ii) *the following r equations hold true for $k_i \in [1, n-r]$ where $1 \le i \le r$:*

$$X_{k_i+r,n} - \sum_{j=k_i}^{k_i+r-1} \frac{E'_j}{j} \stackrel{d}{=} X_{k_i,n} + \sum_{j=k_i}^{k_i+r-1} \frac{E''_j}{n-j}. \tag{2}$$

Next we extend the characterization $X \stackrel{d}{=} X_{1,n} + \sum_{j=1}^{n-1} \frac{E'_j}{j}$ given in Lin and Hu [9].

Theorem 2. *Let F be absolutely continuous (with respect to the Lebesgue measure) and $F(0) = \frac{1}{2}$. Assume*

(i) *the c.d.f. $G_X(x) = P(e^X \le x)$ is analytic and strictly increasing in $[0, \infty)$;*

(ii) *the derivative $G_X^{(k)}(x)$, $k \ge 1$, is strictly monotone in some interval $[0, t_k)$.*

Choose an arbitrary but fixed $k \in \{1, 2, \ldots, n-1\}$. Then F is standard logistic if and only if

$$X \stackrel{d}{=} X_{k,n} + \sum_{j=1}^{n-k} \frac{E'_j}{j} - \sum_{j=1}^{k-1} \frac{E''_j}{j}, \tag{3}$$

where as usual $\sum_{j=1}^{0}(\cdot) = 0$.

Corollary 1. *Under the assumptions of Theorem 2 for $k \geq 2$, F is standard logistic if and only if*

$$X \stackrel{d}{=} X_{k,2k-1} + \sum_{j=1}^{k-1} La_j, \tag{4}$$

where La_j for $1 \leq j \leq k-1$ are mutually independent Laplace random variables with density function $f_j(x) = j\frac{e^{-j|x|}}{2}$ for $|x| < \infty$ and independent from $X_{k,2k-1}$.

This extends George and Mudholkar's [6] characterization $X \stackrel{d}{=} X_{2,3} + E_1' - E_2'$.

In Section 2 we derive two relations among logistic order statistics with independent exponential shifts. In the following two sections we proof Theorem 1. The proof of Theorem 2 is given in Section 5, followed by some concluding remarks.

2 Preliminaries

If F is absolutely continuous, then (Ahsanullah and Nevzorov [1, (1.1.2)])

$$F_k'(x) = k\binom{n}{k} F^{k-1}(x)(1-F(x))^{n-k} F'(x), \quad 1 \leq k \leq n. \tag{5}$$

Integrating (5) and iterating, one obtains the recurrence for $1 \leq r \leq n-k$

$$F_k(x) = \sum_{j=k}^{k+r} \binom{n}{j} F^j(x)(1-F(x))^{n-j} + F_{k+r+1}(x). \tag{6}$$

Furthermore, the following inversion formula holds true for $1 \leq j \leq n$:

$$F_j'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi_j(t)\,dt, \quad F_j'(\infty) = F_j'(-\infty) = 0. \tag{7}$$

If $\int_{-\infty}^{\infty} |t^{m-1}\phi_j(t)|\,dt < \infty$ for $m \geq 2$, then the Dominated Convergence Theorem implies that $F^{(m-1)}(x)$ is differentiable and thus for $m \geq 2$

$$F_j^{(m)}(x) = \frac{(-i)^{m-1}}{2\pi}\int_{-\infty}^{\infty} e^{-itx} t^{m-1}\phi_k(t)dt, \quad F_j^{(m)}(\infty) = F_j^{(m)}(-\infty) = 0. \tag{8}$$

The following characterization property of the logistic distribution is well known (Lin and Hu [9]). If F is absolutely continuous and

$$F'(x) = F(x)(1-F(x)), \quad x \in (l_F, r_F), \tag{9}$$

where $l_F = \inf\{x : F(x) > 0\}$ and $r_F = \sup\{x : F(x) < 1\}$, then for some μ

$$F(x) = \frac{1}{1 + e^{-(x-\mu)}}. \tag{10}$$

We end this section with relations for logistic order statistics with exponential shifts.

Lemma 1. *Under the assumptions for X, E'_j, and E''_j in Section 1, we have:*

(i) *For $1 \le k < m \le n$,*

$$X_{m,n} - \sum_{j=k}^{m-1} \frac{E'_j}{j} \overset{d}{=} X_{k,n} + \sum_{j=k}^{m-1} \frac{E''_j}{n-j}. \tag{11}$$

(ii) *For $1 \le k \le n$,*

$$X \overset{d}{=} X_{k,n} + \sum_{j=1}^{n-k} \frac{E'_j}{j} - \sum_{j=1}^{k-1} \frac{E''_j}{j}, \tag{12}$$

where as usual $\sum_{j=1}^{0}(\cdot) = 0$.

Proof. For the logistic c.d.f. $F_L(x)$ in (1), equation (5) yields

$$\begin{aligned}\phi_k(t) &= k\binom{n}{k}\int_{-\infty}^{\infty} e^{itx}\Big(\frac{e^x}{1+e^x}\Big)^{k-1}\Big(\frac{1}{1+e^x}\Big)^{n-k}\frac{e^{-x}}{(1+e^{-x})^2}\,dx \\ &= k\binom{n}{k}\int_0^1 u^{k-1+it}(1-u)^{n-k-it}\,du \\ &= \frac{\Gamma(k+it)}{\Gamma(k)}\,\frac{\Gamma(n-k+1-it)}{\Gamma(n-k+1)}. \end{aligned} \tag{13}$$

Denote the logistic ch.f. by $\phi(t) := \Gamma(1+it)\Gamma(1-it)$. Thus, the ch.f. of the logistic kth order statistic for $2 \le k \le n-1$ can be factored as

$$\begin{aligned}\phi_k(t) &= \frac{(k-1+it)\dots(1+it)\Gamma(1+it)}{(k-1)!}\,\frac{(n-k-it)\dots(1-it)\Gamma(1-it)}{(n-k)!} \\ &= \Big(1+\frac{it}{k-1}\Big)\dots(1+it)\Big(1-\frac{it}{n-k}\Big)\dots(1-it)\phi(t). \end{aligned} \tag{14}$$

Similarly, (5) with $k = 1$ and $k = n$ yields

$$\phi_1(t) = \prod_{j=1}^{n-1}\Big(1-\frac{it}{j}\Big)\phi(t) \quad \text{and} \quad \phi_n(t) = \prod_{j=1}^{n-1}\Big(1+\frac{it}{j}\Big)\phi(t). \tag{15}$$

(i) The relations (14) and (15) imply for $1 \le k < m \le n$

$$\frac{\phi_m(t)}{\phi_k(t)} = \frac{(1-\frac{it}{n-k})^{-1}\dots(1-\frac{it}{n-m+1})^{-1}}{(1+\frac{it}{m-1})^{-1}\dots(1+\frac{it}{k})^{-1}}$$
$$= \frac{\phi_E(\frac{t}{n-k})\dots\phi_E(\frac{t}{n-m+1})}{\phi_E(-\frac{t}{m-1})\dots\phi_E(-\frac{t}{k})}, \tag{16}$$

where $\phi_E(t) = \frac{1}{1-it}$ is the standard exponential ch.f. This is equivalent to

$$\phi_m(t)\prod_{j=k}^{m-1}\phi_E\Big(-\frac{t}{j}\Big) = \phi_k(t)\prod_{j=k}^{m-1}\phi_E\Big(\frac{t}{n-j}\Big). \tag{17}$$

Since for independent random variables the product of their ch.f.'s equals the ch.f. of their sum, (17) yields (11). The assertion in (ii) follows from (14) and (15). □

3 Proof of Theorem 1 for $r = 1$

To prove the sufficiency assume (2) with $r = 1$ and set $k_1 = k$. Relation (2) yields

$$\phi_{k+1}(t)\Big(1-\frac{it}{n-k}\Big) = \phi_k(t)\Big(1+\frac{it}{k}\Big). \tag{18}$$

Equation (18) and the inversion formula (8) imply

$$F'_{k+1}(x) + \frac{1}{n-k}F''_{k+1}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\Big(1-\frac{it}{n-k}\Big)\phi_{k+1}(t)\,dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\Big(1+\frac{it}{k}\Big)\phi_k(t)\,dt$$
$$= F'_k(x) - \frac{1}{k}F''_k(x). \tag{19}$$

Integrating (19) and taking into account the boundary conditions, we obtain

$$F_k(x) - F_{k+1}(x) = \frac{F'_k(x)}{k} + \frac{F'_{k+1}(x)}{n-k}. \tag{20}$$

Using (5) and (6), equation (20) can be simplified to

$$F'(x) - F(x)(1-F(x)) = 0, \tag{21}$$

i.e., (9) holds true, which, along with the condition $F(0) = \frac{1}{2}$, completes the proof of the sufficiency. The necessity follows from Lemma 1 (i).

4 Proof of Theorem 1 for $r = 2$ and $r = 3$

Let us introduce the notation

$$w(x) := \frac{F'(x)}{F(x)(1-F(x))}, \quad x \in (l_F, r_F). \tag{22}$$

We will suppress the dependence on x in expressions for w, F, and F_k.

The case $r = 2$. Assume (2) with $r = 2$ and set $k_1 = k$. Hence,

$$\phi_{k+2}(t)\Big(1 - \frac{it}{n-k-1}\Big)\Big(1 - \frac{it}{n-k}\Big) = \phi_k(t)\Big(1 + \frac{it}{k}\Big)\Big(1 + \frac{it}{k+1}\Big). \tag{23}$$

Similarly to (19), using the inversion formula (8), one can see that (23) implies

$$F_k' - F_{k+2}' = \sum_{j=k}^{k+1}\Big(\frac{F_k''}{j} + \frac{F_{k+2}''}{n-j}\Big) - \frac{F_k'''}{k(k+1)} + \frac{F_{k+2}'''}{(n-k)(n-k-1)}. \tag{24}$$

Integrating and taking into account the boundary conditions, we obtain

$$F_k - F_{k+2} = \sum_{j=k}^{k+1}\Big(\frac{F_k'}{j} + \frac{F_{k+2}'}{n-j}\Big) - \frac{F_k''}{k(k+1)} + \frac{F_{k+2}''}{(n-k)(n-k-1)}. \tag{25}$$

Applying (6) to the left-hand side of (25) yields

$$\sum_{j=k}^{k+1}\binom{n}{j}F^j(1-F)^{n-j} = \sum_{j=k}^{k+1}\Big(\frac{F_k'}{j} + \frac{F_{k+2}'}{n-j}\Big) - \frac{F_k''}{k(k+1)} + \frac{F_{k+2}''}{(n-k)(n-k-1)}. \tag{26}$$

Using (5), we write the right-hand side of (26) in terms of F, F', and F'' only, and after some algebra, obtain

$$(2F-1)w' = P(n, F, w)(w-1) - k(2F-1)(w-1)^2, \tag{27}$$

where $w = w(x)$ is defined by (22) and $P(n, F, w) = (2nF - n - 1)Fw - (n-1)F - 1$. Clearly $w \equiv 1$ is a solution of (27). This solution corresponds to the standard logistic c.d.f. F_L. To prove the sufficiency part of the theorem, it remains to show that this solution is unique. According to the assumptions of the theorem, there are two distinct integers k_1 and k_2 such that (27) holds true for both $k = k_1$

and $k = k_2$. Writing (27) for these two values of k, subtracting the two equations from each other, and dividing by $k_1 - k_2 \neq 0$, we obtain

$$(2F - 1)(w - 1)^2 = 0. \tag{28}$$

Since $F(x) \neq \frac{1}{2}$ for $x \neq 0$, the only solution of (28) is $w(x) \equiv 1$, which implies the characteristic property (9). This, along with $F(0) = \frac{1}{2}$, completes the proof of the sufficiency. The necessity follows from Lemma 1 (i).

The case $r = 3$. Assuming (2) with $r = 3$ and setting $k_1 = k$, we have

$$\phi_{k+3}(t) \prod_{j=0}^{2} \Big(1 - \frac{it}{n-k-j}\Big) = \phi_k(t) \prod_{j=0}^{2} \Big(1 + \frac{it}{k+j}\Big). \tag{29}$$

Similarly to (25), using the inversion formula (8) and integrating the resulting expression, one can see that (29) implies

$$\begin{aligned} F_k - F_{k+3} = {} & \sum_{j=k}^{k+2} \Big(\frac{F_k'}{j} + \frac{F_{k+3}'}{n-j}\Big) - \sum{}' \Big(\frac{F_k''}{ij} - \frac{F_{k+3}''}{(n-i)(n-j)}\Big) \\ & + \frac{F_k'''}{k(k+1)(k+2)} + \frac{F_{k+3}'''}{(n-k)(n-k-1)(n-k-2)}, \end{aligned} \tag{30}$$

where the summation in $\sum'$ is over all $k \leq i < j \leq k+2$. Applying (6) to the left-hand side of (30) we obtain

$$\begin{aligned} \sum_{j=k}^{k+2} \binom{n}{j} F^j (1-F)^{n-j} = {} & \sum_{j=k}^{k+2} \Big(\frac{F_k'}{j} + \frac{F_{k+3}'}{n-j}\Big) - \sum{}' \Big(\frac{F_k''}{ij} - \frac{F_{k+3}''}{(n-i)(n-j)}\Big) \\ & + \frac{F_k'''}{k(k+1)(k+2)} + \frac{F_{k+3}'''}{(n-k)(n-k-1)(n-k-2)}, \end{aligned} \tag{31}$$

Using (5), we write $F_k^{(i)}$ and $F_{k+3}^{(i)}$ for $1 \leq i \leq 3$ in terms of $F^{(i)}$ for $0 \leq i \leq 3$. After some tedious algebra, one can see that (31) is equivalent to

$$\begin{aligned} & Q(n, F, w) \\ & = kP_1(F, w)(w-1) + kP_2(n, F, w)(w-1)^2 + k^2 P_3(F)(w-1)^3, \end{aligned} \tag{32}$$

where $w = w(x)$ is defined by (22) and the polynomials $Q(n, F, w)$,

$$P_1(F, w) = 3(3F^2 - 3F + 1)w', \tag{33}$$

$$P_2(n, F, w) = 3(n-2)F^2 - (n-8)F - 3 \\ - 2F[3nF^2 - 3(n+1)F + n + 1]w, \tag{34}$$

$$P_3(F) = (3F^2 - 3F + 1) \tag{35}$$

do not depend on k. One solution of (32) is $w \equiv 1$, which yields $F_L(x)$. It remains to prove that this is the only solution. By the assumptions, (32) holds true for three distinct values of k. Writing (32) for $k = k_1$ and $k = k_2$ and subtracting the two equations from each other, we obtain

$$P_1(F, w)(w-1) + P_2(n, F, w)(w-1)^2 + (k_1 + k_2)P_3(F)(w-1)^3 = 0. \tag{36}$$

Next, writing (32) for $k = k_2$ and $k = k_3$ and subtracting the two equations from each other, we have

$$P_1(F, w)(w-1) + P_2(n, F, w)(w-1)^2 + (k_2 + k_3)P_3(F)(w-1)^3 = 0. \tag{37}$$

Finally, subtracting (36) from (37) and dividing by $k_3 - k_1 > 0$, we have

$$(3F^2 - 3F + 1)(w-1)^3 = 0. \tag{38}$$

Since, $3F^2 - 3F + 1 \neq 0$, the only solution of (38) is $w \equiv 1$, i.e., (9) holds true. Now, taking into account $F(0) = \frac{1}{2}$ we complete the proof of the sufficiency. The necessity is a straightforward corollary of Lemma 1 (i).

5 Proof of Theorem 2

We follow the scheme of the proof of Theorem 6 in Lin and Hu [9]. Recall the notion of intensively monotone operators and $\mathcal{E}$-positive families (Kakosyan et al. [8]).

Definition 1. Let $\mathcal{C} = \mathcal{C}\,[0, \infty)$ be the space of all real-valued functions defined and continuous in the interval $[0, \infty)$. The notation $f \geq g$ for $f, g \in \mathcal{C}$ means that $f(t) \geq g(t)$ for all $t \in [0, \infty)$. Let A be an operator mapping some set $\mathcal{E} \subset \mathcal{C}$ into $\mathcal{C}$. We say that the operator A is intensively monotone, if for any f_1 and f_2 belonging to $\mathcal{E}$, the condition $f_1(\tau) \geq f_2(\tau)$ for all $\tau \in (0, t)$ implies that $(Af_1)(\tau) \geq (Af_2)(\tau)$ for $\tau \in (0, t)$ and, in addition, the condition $f_1(\tau) > f_2(\tau)$ for all $\tau \in (0, t)$ implies that $(Af_1)(t) > (Af_2)(t)$.

Definition 2. Let $\mathcal{E} \subset \mathcal{C}$ and $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of elements of $\mathcal{E}$. We say that the family $\{f_\lambda\}_{\lambda\in\Lambda}$ is strongly $\mathcal{E}$-positive if the following conditions hold true:

(i) for any $f \in \mathcal{E}$ there are $t_0 > 0$ and $\lambda_0 \in \Lambda$ such that $f(t_0) = f_{\lambda_0}(t_0)$;

(ii) for any $f \in \mathcal{E}$ and any $\lambda \in \Lambda$ either $f(t) = f_\lambda(t)$ for all $t \in [0,\infty)$, or there is $\delta > 0$ such that the difference $f(t) - f_\lambda(t)$ does not vanish (preserves its sign) in the interval $(0,\delta]$.

The following example and lemma play an essential role in the proof.

Example 1 (Lin and Hu [9]). Define $\mathcal{E}$ to be the set of all survivor functions $\overline{G}$, which are real, analytic, and strictly decreasing in $[0,\infty)$. For each k, assume that the kth derivative $\overline{G}^{(k)}$ is strictly monotone in some interval $[0,\delta_k)$. If $\overline{G}_\lambda(x) = \frac{1}{1+\lambda x}$, $x > 0$, where $\lambda \in \Lambda = (0,\infty)$, then the family $\{\overline{G}_\lambda\}_{\lambda\in\Lambda}$ is strongly $\mathcal{E}$-positive.

Lemma 2 (Kakosyan et al. [8, Theorem 1.1.1]). *Let A be an intensively monotone operator on $\mathcal{E} \subset \mathcal{C}$ and let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a strongly $\mathcal{E}$-positive family. Assume that $Af_\lambda = f_\lambda$ for all $\lambda \in \Lambda$. Then the condition $Af = f$, where $f \in \mathcal{E}$, implies that there is $\lambda \in \Lambda$ such that $f = f_\lambda$. In other words, all solutions of the equation $Af = f$, belonging to $\mathcal{E}$, coincide with elements of the family $\{f_\lambda\}_{\lambda\in\Lambda}$.*

Proof of Theorem 2. The case $k = 1$ is Theorem 6 in Lin and Hu [9]. Assume $2 \le k \le n$. The necessity statement follows from Lemma 1 (ii). We shall prove the sufficiency. For E'_j and E''_j, $1 \le j \le n$ i.i.d. standard exponential, recall (David and Nagaraja [4, pp. 17f.]) the following formula for the maximum $E'_{n,n} := \max\{E'_1,\dots,E'_n\}$:

$$E'_{n,n} \overset{d}{=} \sum_{j=1}^{n} \frac{E'_j}{j}, \quad n \ge 1. \tag{39}$$

Since $U'_j := e^{-E'_j}$ for $1 \le j \le n$ are uniform on $[0,1]$, we have that

$$U'_{1,n} := \min\{U'_1,\dots,U'_n\} \overset{d}{=} e^{-E'_{n,n}} \tag{40}$$

is distributed as the minimum of n i.i.d. uniform $[0,1]$ variables. (3) and (39) yield

$$e^X \overset{d}{=} \frac{e^{X_{k,n}} e^{-\sum_{j=1}^{k-1} \frac{E''_j}{j}}}{e^{-\sum_{j=1}^{n-k} \frac{E'_j}{j}}} \overset{d}{=} \frac{e^{X_{k,n}} e^{-E''_{k-1,k-1}}}{e^{-E'_{n-k,n-k}}} \overset{d}{=} \frac{U''_{1,k-1} e^{X_{k,n}}}{U'_{1,n-k}}. \tag{41}$$

Let us make the change of variables

$$\xi = \frac{U''_{1,k-1} e^{X_{k,n}}}{U'_{1,n-k}}, \quad \eta = U''_{1,k-1}, \quad \zeta = U'_{1,n-k}. \tag{42}$$

Denote $G_X(x) := P(e^X \leq x)$ and $\overline{G}_X(x) := 1 - G_X(x)$. For the p.d.f.'s $f_{e^{X_{k,n}}}$ and $f_{U_{1,j}}$ of $e^{X_{k,n}}$ and $U_{1,j}$, respectively, we have

$$f_{e^{X_{k,n}}}(x) = k \binom{n}{k} G_X^{k-1}(x) \overline{G}_X^{n-k}(x) G'_X(x), \quad f_{U_{1,j}}(y) = j(1-y)^{j-1}. \tag{43}$$

Therefore, for the density f_ξ of ξ, we obtain

$$\begin{aligned} f_\xi(u) &= \int_0^1 \int_0^1 f_{\xi,\eta,\zeta}(u, v, w)\, dv\, dw \\ &= \int_0^1 \int_0^1 C(v, w)\, G_X^{k-1}\Big(\frac{uw}{v}\Big) \overline{G}_X^{n-k}\Big(\frac{uw}{v}\Big) G'_X\Big(\frac{uw}{v}\Big) \frac{w}{v}\, dv\, dw, \end{aligned} \tag{44}$$

where $C(v, w) = n!(1-v)^{k-2}(1-w)^{n-k-1}/(k-2)!(n-k-1)!$. Now, referring to (41) and (44), one can see that $\overline{G}_X(x)$ satisfies for $n \geq 3$

$$\begin{aligned} \overline{G}_X(x) &= \int_x^\infty f_\xi(u)\, du \\ &= \int_0^1 \int_0^1 \frac{C(v, w)}{k} \Big[\int_{G_X(xw/v)} \overline{G}_X^{n-k}\Big(\frac{uw}{v}\Big) dG_X^k\Big(\frac{uw}{v}\Big) \Big] dv\, dw, \\ &= \int_0^1 \int_0^1 \frac{C(v, w)}{k} H_k\Big(\overline{G}_X\Big(\frac{xw}{v}\Big)\Big)\, dv\, dw, \end{aligned} \tag{45}$$

where, making the substitution $t = G_X(\frac{uw}{v})$, one can write $H_k(y)$ as

$$H_k(y) = \int_{1-y}^1 (1-t)^{n-k}\, dt^k, \quad 0 < y < 1. \tag{46}$$

Define $\mathcal{E}$ to be the set of all survival functions $\overline{G}$ as in Example 1 above. Define also an operator A on $\mathcal{E}$ by

$$A\overline{G}_X(x) := \int_0^1 \int_0^1 C(v, w)\, H_k\Big(\overline{G}_X\Big(\frac{xw}{v}\Big)\Big)\, dv\, dw. \tag{47}$$

Since $H'_k(y) = ky^{n-k}(1-y)^{k-1} > 0$ for $0 < y < 1$, we have that $H_k(y)$ is increasing for any $y \in (0, 1)$ and $1 \leq k \leq n$, which implies that A is an intensively monotone operator on $\mathcal{E}$. By Example 1, the family $\{\overline{G}_\lambda\}_{\lambda \in \Lambda}$ is strongly

$\mathcal{E}$-positive. We have that $A\overline{G}_\lambda = \overline{G}_\lambda$ by the necessity part of the theorem. Finally, by Lemma 2 and (45) (which means $A\overline{G}_X = \overline{G}_X$), we conclude that $\overline{G}_X = \overline{G}_\lambda$. The condition $F(0) = \frac{1}{2}$ implies $\lambda = 1$ and completes the proof. □

6 Concluding Remarks

Theorem 1 characterizes the standard logistic distribution through equalities in distribution between two order statistics plus or minus sums of independent exponential variables. In case of adjacent order statistics, the characterization involves a single equality. If the order statistics are two or three spacings away, two or three equalities are needed, respectively. Our second result characterizes the logistic distribution by expressing the parent variable X as sum of one order statistic and a linear combination of independent exponential variables. In the corollary we singled out the expression involving the sample median.

One open question is if the number of characterizing equalities in the non-adjacent cases of Theorem 1 can be reduced. Another area of future study are other identities involving order statistics subject to exponential shifts. Some results in this direction are given in Ahsanullah et al. [2]. One can also look for connections between random shifts and contractions (Wesołowski and Ahsanullah [11]).

From the view point of application, it would be desirable to give an easily understandable interpretation of the conditions on order statistics that characterize the logistic distribution. Such an interpretation could help to select and apply the logistic distribution to real-world problems.

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