Offspring Variance Estimators in Branching Processes with Time Non-Homogeneous Immigration

George P. Yanev, Department of Mathematics, The University of Texas-Pan American, Ibrahim Rahimov, Department of Mathematics and Statistics, Zayed University, Dubai

Introduction
Yakovlev and Yanev (2006) point out that in vivo cell kinetics requires stochastic modeling of renewing cell populations with non-homogeneous immigration. They analyze the population of terminally differentiated oligodendrocytes of the central nervous system and the population of leukemia cells. In both cases the cell population expands through both division of existing (progenitor) cells and differentiation of stem cells. This dynamics belongs to branching processes with non-homogeneous immigration. Furthermore, the population’s viability is preserved by allowing the immigration distribution to vary in time, such that the number of immigrants increases to infinity on average.

Main Result
Define
\[ A_l = E_{X_l} t_l^2 - \sum_{k=0}^{n_l} \theta_k (n_k^2 + t_k^2) \]
where \( A_l \) is the offspring variance for the offspring variance:
\[ \theta_k = n_k \sigma_k^2 (1 - \theta_k) \]
We have
\[ \lim_{n_l \to \infty} \theta_k - \lim_{n_l \to \infty} \frac{n_k \sigma_k^2 (1 - \theta_k)}{n_k} = \theta \in [0,1] \]

Theorem
Assume Eq. (1)-(4) hold true. Then
\[ \lim_{n_l \to \infty} \left( \frac{n_l \sigma_l^2}{4n + 5 + (1 - \theta) \frac{1 + \lambda}{2a + 1 + 3}} \right) = \mathcal{N}(0, \sigma^2) \]
where the limit is normally distributed with zero mean and variance
\[ \sigma^2 = (2a + 3) \left( \sigma_k^2 \left( 1 - \theta_k \right) \frac{1 + \lambda}{2a + 1 + 3} \right) \]

Examples
Poisson Immigration
For Poisson distributed number of immigrants with mean \( \theta_k = \xi_n \to 0 \) as \( n \to \infty \), the asymptotic normality of the proposed estimators is established with convergence rate \( n^{-1/2} \), where \( \xi_n \) is the number of immigrants.

Neumann Type A Immigration
Define the immigration by
\[ \lambda_k = E_{X_k} \phi_k \]
where \( \lambda_k = n_k \lambda_k (n_k) \to \infty \) and \( \phi_k = n_k \lambda_k (n_k) \).
If \( 0 < \lambda < 1/2 \), then the Theorem holds true with \( \theta = 0 \) and \( \lambda = 1/2 \).

Discussion
For the critical process with homogeneous immigration, Winnicki (1991) established the weak limit of the CLSE with rate of convergence \( n^{1/2} \). We extend this result allowing the immigration to vary with time such that it increases to infinity on average. The rate of convergence is \( (\theta_n)^{1/2} \) where \( \lim_{n \to \infty} \theta_n = \theta \in [0,1] \). The presented results complete those of Rahimov (2008) in that the limiting distribution of the CLSE for the offspring mean depends on the offspring variance.

Conclusion
Branching processes with time-dependent immigration are encountered in a variety of applications to population biology. We study conditional least-squares estimators for the offspring variance in the critical case, assuming that the immigration mean increases to infinity over time. The asymptotic normality of the proposed estimators is established with convergence rate \( n^{-1/2} \), where \( \xi_n \) is the number of immigrants.

A next question concerns the conditional consistency of the estimators with different weights.

References