2012 International Conference on Stochastic Processes in System Biology, Genetics, and Evolution

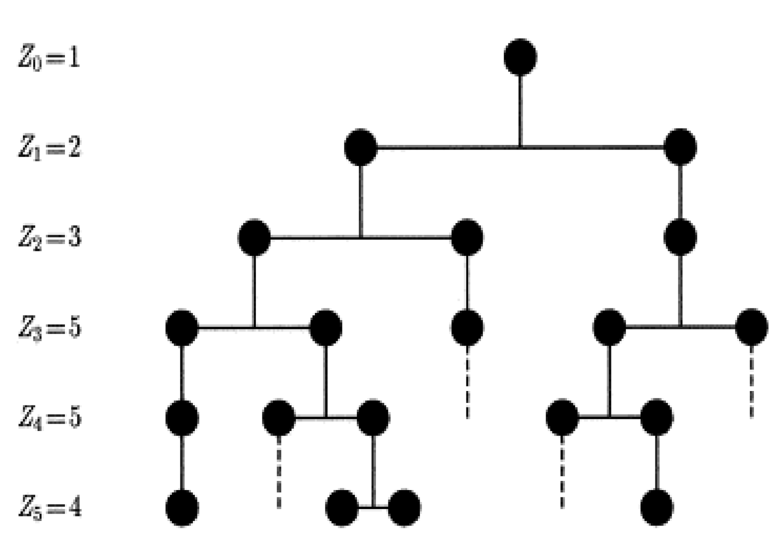
Offspring Variance Estimators in Branching Processes with Time Non-Homogeneous Immigration

George P. Yanev, Department of Mathematics, The University of Texas-Pan American,
Ibrahim Rahimov, Department of Mathematics and Statistics, Zayed University, Dubai

Introduction

Yakovlev and Yanev (2006) point out that *in vivo* cell kinetics requires stochastic modeling of renewing cell populations with non-homogeneous immigration. They analyze the population of terminally differentiated oligodendrocytes of the central nervous system and the population of leukemia cells. In both cases the cell population expands through both division of existing (progenitor) cells and differentiation of stem cells. This dynamics belongs to branching processes with non-homogeneous immigration. Furthermore, the population's viability is preserved by allowing the immigration distribution to vary in time, such that the number of immigrants increases to infinity on average .

Figure 1: Simple Branching Process



Summary

The asymptotic normality of conditional least squares estimators for the offspring variance in critical branching processes with non-homogeneous immigration is established, under moment assumptions on both reproduction and immigration. The proofs use martingale techniques and weak convergence results in Skorokhod spaces.

Model and Assumptions

Define

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_{n,i} + \xi_n, \quad n \geq 1;\ Z_0 = 0,$$

$X_{n,i}$ - number of offspring; ξ_n - immigrants.

CLSE for the offspring variance:

$$\widehat{b_n^2} := \frac{\sum_{k=1}^n ((Z_k - Z_{k-1} - \alpha_k)^2 - \beta_k^2) Z_{k-1}}{\sum_{k=1}^n Z_{k-1}^2}$$

where

$$\alpha_n = E\xi_n \ \text{and}\ \beta_n^2 = Var\xi_n.$$

Assume:

$$EX_{1,1} = 1 \ \text{and}\ EX_{1,1}^4 < \infty \quad (1)$$

$$\alpha_n = E\xi_n = n^\alpha L_\alpha(n),\ \beta_n^2 = Var\xi_n = n^\beta L_\beta(n),$$
$$\text{and}\ \gamma_n^4 = Var(\xi_n - \alpha_n)^2 = n^\gamma L_\gamma(n) \quad (2)$$

$$\lim_{n\to\infty} E\xi_n = \infty \quad (3)$$

$$(\xi_n - \alpha_n)^2 - \beta_n^2 \ \text{satisfies the Lindeberg condition}\ (4)$$

Main Result

Define

$$A_n := EZ_n, \quad \tau_n^2 := \sum_{k=1}^n \gamma_k^4, \quad \theta_n := nA_n^2(nA_n^2 + \tau_n^2)^{-1}$$

We have

$$\lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \frac{nA_n^2}{nA_n^2 + \tau_n^2} =: \theta \in [0,1]$$

Theorem

Assume Eq. (1)-(4) hold true. Then

$$\lim_{n\to\infty} (\theta_n n)^{1/2} (\widehat{b_n^2} - b^2) = N(0, \sigma^2),$$

where the limit is normally distributed with zero mean and variance

$$\sigma^2 = (2\alpha+3)^2 \left(\theta \frac{nA_n^2}{4\alpha+5} + (1-\theta) \frac{\gamma+1}{2\alpha+3+\gamma} \right).$$

Examples

Poisson Immigration

For Poisson distributed number of immigrants with mean $\alpha_n = o(n) \to \infty \ as\ n \to \infty$ under the assumptions of the Theorem, we have asymptotically normal CLSE with θ=1.

Neyman Type A Immigration

Define the immigration by

$$Ez^{\xi_n} = exp\left(\lambda_n\left(e^{\varphi_n(z-1)} - 1\right)\right)$$

where

$$\lambda_n = n^\lambda L_\lambda(n) \to \infty \ and\ \varphi_n = n^\varphi L_\varphi(n),$$

If $0<\lambda+\varphi\leq 1/2$, then the Theorem holds true with $\theta=1$ and $\alpha=\lambda+\varphi$. If $1<\lambda+\varphi<3/2$, then the Theorems holds true with $\theta=0$ and $\alpha=\lambda+\varphi$.

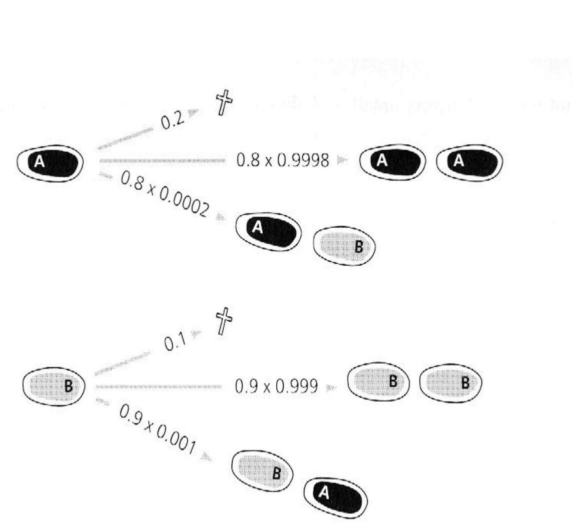


Figure 2: Reproduction of bacteria with two types of alleles, A or B. The allele affects mortality rate. (Source: Haccou, Jagers, Vatutin , 2005, Branching Processes: Variation, Growth, and Extinction of Populations, Fig. 2.5, p.24.)

Discussion

For the critical process with homogeneous immigration, Winnicki (1991) established the weak limit of the CLSE with rate of convergence $n^{1/2}$ We extend this result allowing the immigration to vary with time such that it increases to infinity on average. The rate of convergence is $(\theta_n n)^{1/2}$ where $\lim_{n\to\infty} \theta_n = \theta \in [0,1]$. The presented results complete those of Rahimov (2008) in that the limiting distribution of the CLSE for the offspring mean depends on the offspring variance.

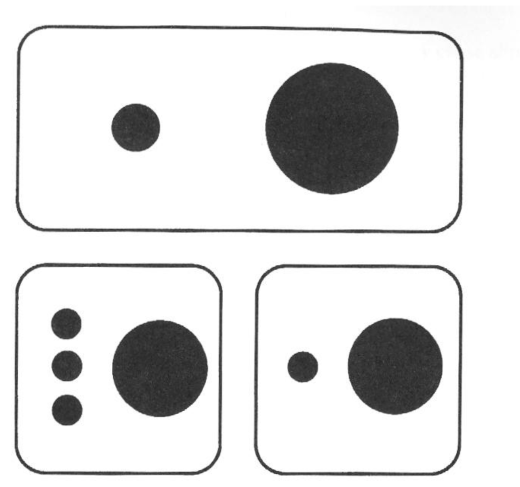


Figure 3: Cell division and partition of contents. The amount of DNA in the nucleus (large circles) doubles into two daughter cells. However, other cell constituents may not exactly double, resulting in different numbers of these constituents. (Source: Kimmel and Axelrod, 2002, Branching Processes in Biology, Fig. 2.5 on p. 26.)

Conclusion

Branching processes with time-dependent immigration are encountered in a variety of applications to population biology. We study conditional least-squares estimators for the offspring variance in the critical case, assuming that the immigration mean increases to infinity over time. The asymptotic normality of the proposed estimators is established with convergence rate , where . A next question concerns the conditional consistency of the estimators with different weights.

References

[1] Winnicki, J. (1991). Estimation of the variances in the branching process with immigration. Probab. Theory Related Fields, 88:77-106.

[2] Rahimov, I. (2008). Asymptotic distribution of the CLSE in critical process with immigration. Stoch. Proc. Appl., 118:1892-1908.

[3] Rahimov and Yanev , G.P. Variance estimators in critical branching processes with non-homogeneous immigration. Math. Pop. Studies, in press.

[4] Yakovlev, A. and Yanev, N.M. (2006). Branching stochastic processes with immigration in analysis of renewing cell populations. Math. Biosci. 203:37-63 .