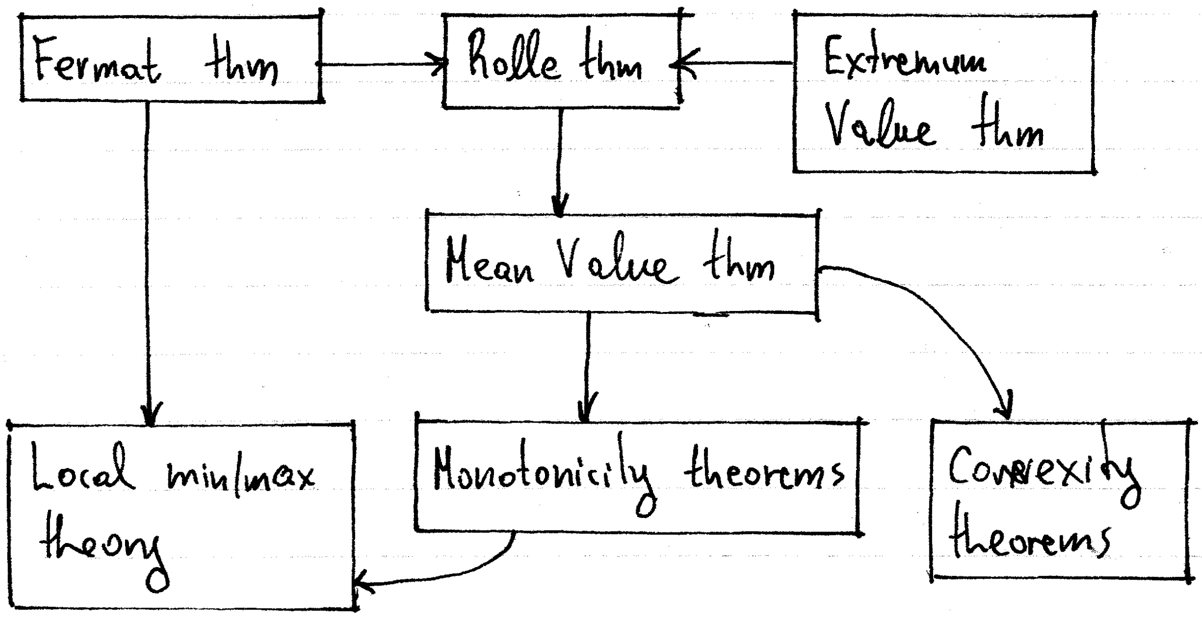
DIFFERENTIAL CALCULUS

Foundation of Differential Calculus

The applications of derivatives are based on a collection of theorems that have the following interdependence amongst themselves



(1) → Fermat theorem

Def: (Interior points)
Let A be a set $A \subseteq \mathbb{R}$. We say that
x_0 interior point of $A \iff \exists\, \delta \in (0, +\infty) : (x_0 - \delta, x_0 + \delta) \subseteq A$

notation: The set of all interior points of a set A is denoted as

$$\mathrm{int}(A) = \{x_0 \in A \mid x_0 \text{ interior to } A\}$$
$$= \{x_0 \in A \mid \exists \delta \in (0,+\infty) : (x_0-\delta, x_0+\delta) \subseteq A\}$$

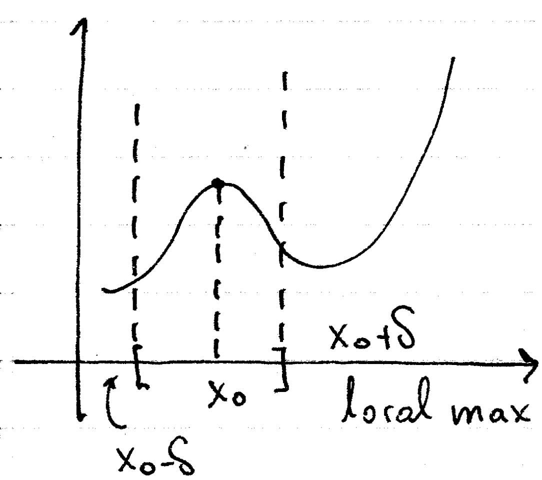
▸ In general, given a set defined as a union of intervals, $\mathrm{int}(A)$ can be obtained by changing all closed intervals to open intervals

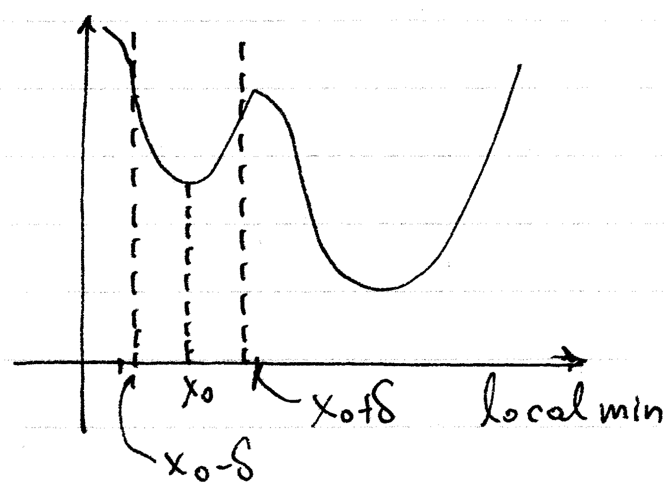
example: For $A = (1,3] \cup [5,+\infty)$, we have
$$\mathrm{int}(A) = (1,3) \cup (5,+\infty).$$
Consequently, 2 is interior to A but for $x_0 \in \{1,3,5\}$, x_0 is not interior to A.

Def: (Local min/max)
Let $f: A \to \mathbb{R}$ be a function and let $x_0 \in A$.
We say that

a) x_0 local max of $f \iff$
$$\iff \exists \delta \in (0,+\infty) : \forall x \in (x_0-\delta, x_0+\delta) \cap A : f(x) \leq f(x_0)$$

b) x_0 local min of $f \iff$
$$\iff \exists \delta \in (0,+\infty) : \forall x \in (x_0-\delta, x_0+\delta) \cap A : f(x) \geq f(x_0)$$



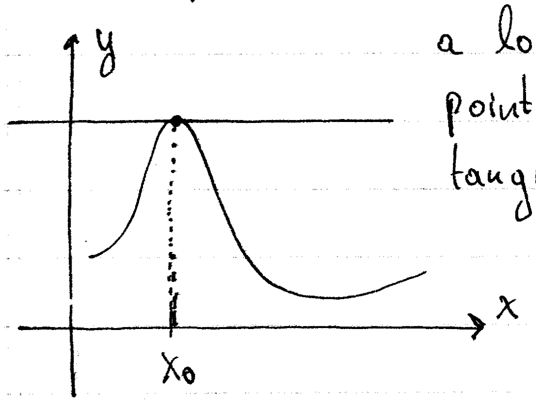


<u>interpretation</u>: A point $x_0 \in A$ is local min of $f: A \to \mathbb{R}$ if and only if $f(x_0)$ is the minimum value of f in a small enough interval around the point x_0. Likewise, a point $x_0 \in A$ is local max of $f: A \to \mathbb{R}$ if and only if $f(x_0)$ is the maximum value of f in a small enough interval around the point x_0.

<u>Thm</u>: (<u>Fermat theorem</u>)
Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ be a function and let $x_0 \in A$.
We have:

$$\begin{cases} x_0 \in \mathrm{int}(A) \\ x_0 \text{ local min or max of } f \\ f \text{ differentiable on } x_0 \end{cases} \Rightarrow f'(x_0) = 0$$

▸ <u>interpretation</u>: If a function is differentiable and has a local max or min at an interior point x_0 of its domain, then the tangent line (ℓ) to the graph of f at the point x_0 is horizontal.



<u>Proof</u>
With no loss of generality, assume that

$$\begin{cases} x_0 \in \mathrm{int}(A) \wedge x_0 \text{ local max of } f \\ f \text{ differentiable on } x_0 \end{cases}$$

It follows that

$x_0 \in \mathrm{int}(A) \Rightarrow \exists \delta_1 \in (0, +\infty) : (x_0 - \delta_1, x_0 + \delta_1) \subseteq A$

x_0 local max of $f \Rightarrow$

$\Rightarrow \exists \delta_2 \in (0, +\infty) : \forall x \in (x_0 - \delta_2, x_0 + \delta_2) \cap A : f(x) \leq f(x_0)$

Choose $\delta_1, \delta_2 \in (0, +\infty)$ such that

$$\begin{cases} (x_0 - \delta_1, x_0 + \delta_1) \subseteq A \\ \forall x \in (x_0 - \delta_2, x_0 + \delta_2) \cap A : f(x) \leq f(x_0) \end{cases}$$

Define $\delta = \min\{\delta_1, \delta_2\}$ and define

$$\forall x, x_0 \in A : \lambda(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

Since

$(x_0 - \delta, x_0 + \delta) \subseteq (x_0 - \delta_1, x_0 + \delta_1) \subseteq A \Rightarrow$

$\Rightarrow (x_0 - \delta, x_0 + \delta) \subseteq A \Rightarrow (x_0 - \delta, x_0 + \delta) \cap A = (x_0 - \delta, x_0 + \delta)$

$\Rightarrow \forall x \in (x_0 - \delta, x_0 + \delta) : f(x) \leq f(x_0)$

$\Rightarrow \forall x \in (x_0 - \delta, x_0 + \delta) : f(x) - f(x_0) \leq 0$

$$\Rightarrow \begin{cases} \forall x \in (x_0 - \delta, x_0) : \lambda(x, x_0) = \dfrac{f(x) - f(x_0)}{x - x_0} \geq 0 & (1) \\ \forall x \in (x_0, x_0 + \delta) : \lambda(x, x_0) = \dfrac{f(x) - f(x_0)}{x - x_0} \leq 0 & (2) \end{cases}$$

Since f differentiable at x_0

$$f'(x_0) = \lim_{x \to x_0^-} \lambda(x, x_0) \geq 0 \quad , \text{ from Eq.(1)}$$

$$f'(x_0) = \lim_{x \to x_0^+} \lambda(x, x_0) \leq 0 \quad , \text{ from Eq.(2)}$$

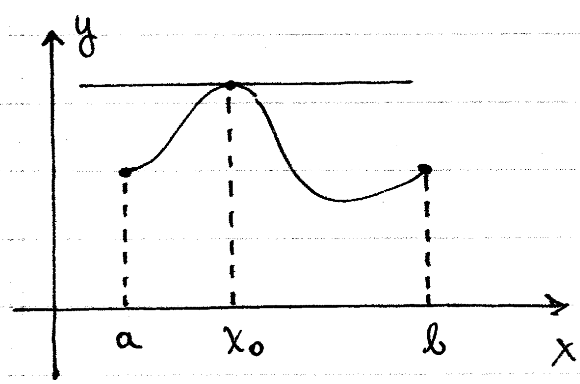
and it follows that $f'(x_0) = 0$. $\square$

② → Rolle theorem

Thm: Let $f: A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$ and let $a, b \in A$ with $[a,b] \subseteq A$. Then,

$$\left.\begin{array}{l} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \\ f(a) = f(b) \end{array}\right\} \Rightarrow \exists x_0 \in (a,b) : f'(x_0) = 0$$

interpretation:



If a function f is continuous on $[a,b]$ and differentiable on (a,b) and if $f(a) = f(b)$, then there is a point $x_0 \in (a,b)$ where the tangent line to the graph of the function becomes horizontal.

Proof

Assume that

$$\begin{cases} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \\ f(a) = f(b) \end{cases}$$

Using the Extremum Value Theorem,

f continuous on $[a,b] \Rightarrow$

$\Rightarrow \exists x_1, x_2 \in [a,b] : \forall x \in [a,b] : f(x_1) \leqslant f(x) \leqslant f(x_2)$.

Choose $x_1, x_2 \in [a,b]$ such that

$\forall x \in [a,b] : f(x_1) \leqslant f(x) \leqslant f(x_2)$

We distinguish between the following cases.

Case 1: Assume that $x_1 \in (a,b)$. Then

$(\forall x \in [a,b] : f(x) \geqslant f(x_1)) \Rightarrow x_1$ local min of f (1)

We also know that

$\begin{cases} x_1 \text{ interior to } (a,b) & (2) \\ f \text{ differentiable on } (a,b) \end{cases}$

From Eq. (1) and Eq. (2), via the Fermat theorem:

$f'(x_1) = 0 \Rightarrow \exists x_0 \in (a,b) : f'(x_0) = 0$ (for $x_0 = x_1$)

Case 2: Assume that $x_2 \in (a,b)$. Then

$(\forall x \in [a,b] : f(x) \leqslant f(x_2)) \Rightarrow x_2$ local max of f (3)

We also know that

$\begin{cases} x_2 \text{ interior to } (a,b) & (4) \\ f \text{ differentiable on } (a,b) \end{cases}$

From Eq. (3) and Eq. (4), via the Fermat theorem:

$f'(x_2) = 0 \Rightarrow \exists x_0 \in (a,b) : f'(x_0) = 0$ (for $x_0 = x_2$).

Case 3: Assume that $x_1 = a \wedge x_2 = b$.

We define $c = f(a) = f(b)$. Then:

$\forall x \in [a,b] : f(x_1) \leqslant f(x) \leqslant f(x_2)$

$\Rightarrow \forall x \in [a,b] : f(a) \leqslant f(x) \leqslant f(b)$

$\Rightarrow \forall x \in [a,b] : c \leqslant f(x) \leqslant c$

$\Rightarrow \forall x \in [a,b] : f(x) = c$

$\Rightarrow \forall x \in [a,b] : f'(x) = c$

$\Rightarrow \exists x_0 \in [a,b] : f'(x_0) = c.$

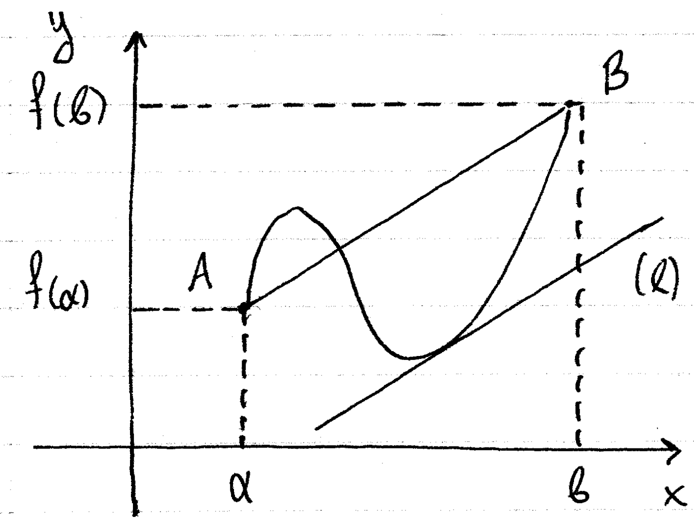
In all cases we conclude that $\exists x_0 \in [a,b] : f'(x_0) = c$.

③ → Mean Value Theorem

Thm: (Lagrange's Mean Value Theorem)
Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ be a function and let $a, b \in A$ such that $[a,b] \subseteq A$. Then

$$\left\{\begin{array}{l} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \end{array}\right. \Rightarrow \exists\, x_0 \in (a,b) : f'(x_0) = \frac{f(b)-f(a)}{b-a}$$

Interpretation:



If the function f is continuous on $[a,b]$ and differentiable on (a,b), then given the points $A(a, f(a))$ and $B(b, f(b))$ on the graph of f, there is at least one $x_0 \in (a,b)$ such that the tangent line (l) at $x = x_0$ to the graph of f satisfies $(l) \parallel (AB)$.

Proof

Assume that

$$\left\{\begin{array}{l} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \end{array}\right.$$

Define

$\forall x \in [a,b]: F(x) = (a-b)f(x) + [f(b)-f(a)]x + [bf(a) - af(b)]$

and note that

f continuous on $[a,b] \Rightarrow F$ continuous on $[a,b]$ (1)

and

f differentiable on $(a,b) \Rightarrow F$ differentiable on (a,b) (2)

with $\forall x \in (a,b): F'(x) = (a-b)f'(x) - [f(a)-f(b)]$ (3)

We also have

$$
\begin{aligned}
F(a) &= (a-b)f(a) + [f(b)-f(a)]a + [bf(a) - af(b)] = \\
&= (a-b)f(a) + af(b) - af(a) + bf(a) - af(b) = \\
&= (a-b-a+b)f(a) + (a-a)f(b) = \\
&= 0f(a) + 0f(b) = 0 \qquad (4)
\end{aligned}
$$

and

$$
\begin{aligned}
F(b) &= (a-b)f(b) + [f(b)-f(a)]b + [bf(a) - af(b)] = \\
&= (a-b)f(b) + bf(b) - bf(a) + bf(a) - af(b) = \\
&= (-b+b)f(a) + (a-b+b-a)f(b) = \\
&= 0f(a) + 0f(b) = 0 \qquad (5)
\end{aligned}
$$

From Eq.(4) and Eq.(5): $F(a) = F(b) = 0$ (6).

From Eq.(1) and Eq.(2) and Eq.(6), via the Rolle theorem:

$$
\left\{
\begin{array}{l}
F \text{ continuous on } [a,b] \\
F \text{ differentiable on } (a,b) \\
F(a) = F(b)
\end{array}
\right.
\Rightarrow \exists x_0 \in (a,b): F'(x_0) = 0
$$

$$
\begin{aligned}
&\Rightarrow \exists x_0 \in (a,b): (a-b)f'(x_0) - [f(a)-f(b)] = 0 \\
&\Rightarrow \exists x_0 \in (a,b): (b-a)f'(x_0) = f(b)-f(a) \\
&\Rightarrow \exists x_0 \in (a,b): f'(x_0) = \frac{f(b)-f(a)}{b-a} \qquad \square
\end{aligned}
$$

Remark: During the early development of Calculus, many arguments were based on the concept of the linear approximation

$$f(x+\Delta x) \approx f(x) + \Delta x f'(x)$$

where Δx is very small relative to x (i.e. $\Delta x \ll x$). The linear approximation assumes that the graph of the function f in the interval $[x, x+\Delta x]$ is approximately a straight line as long as Δx is small enough, and can be therefore represented by a linear function with respect to Δx. The linear approximation can be used to argue, e.g. that if a function has $f'(x) > 0$, then it is increasing from x to $x+\Delta x$. The problem is that such arguments are not rigorous because they are based on a statement that is true only approximately.

According to the Mean Value Theorem, if f satisfies

$$\begin{cases} f \text{ continuous on } [a,b] & \text{with } a = x \text{ and } b = x+\Delta x \\ f \text{ differentiable on } (a,b) \end{cases}$$

then we conclude that

$$\exists x_0 \in (x, x+\Delta x) : f(x+\Delta x) = f(x) + \Delta x f'(x_0)$$

It follows that the linear approximation statement becomes exact if we replace $f'(x)$ with $f'(x_0)$ for some choice of $x_0 \in (x, x+\Delta x)$. This in turn makes it possible to formulate rigorous arguments based on the overall linear approximation concept.

Immediate corollaries of the Mean Value Theorem

The following theorems are immediate consequences of the Mean Value Theorem. We use the assumption that a set $I \subseteq \mathbb{R}$ is an interval, as opposed to a union of disjoint intervals (e.g. $I = [a,b]$ or $I = (a,b]$ or $I = [a,b)$ etc....). A practical definition that encompasses all possibilities is the following:

Def: Let $I \subseteq \mathbb{R}$. We say that

$$I \text{ interval} \Leftrightarrow \forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow [x_1, x_2] \subseteq I)$$

We also define the concept of a constant function:

Def: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $I \subseteq A$. We say that

$$f \text{ constant on } I \Leftrightarrow \forall x_1, x_2 \in I : f(x_1) = f(x_2)$$

We will now show that

Thm: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $I \subseteq A$. Then:

$$\left.\begin{cases} I \text{ interval} \\ f \text{ differentiable on } I \\ \forall x \in I : f'(x) = 0 \end{cases}\right\} \Rightarrow f \text{ constant on } I.$$

Proof

Assume that

$$\begin{cases} I \text{ interval} \\ f \text{ differentiable on } I \\ \forall x \in I : f'(x) = 0 \end{cases}$$

▸ We will show that $\forall x_1, x_2 \in I : f(x_1) = f(x_2)$.

Let $x_1, x_2 \in I$ be given and assume with no loss of generality that $x_1 < x_2$. Then

$$\begin{cases} I \text{ interval} \\ x_1, x_2 \in I \wedge x_1 < x_2 \end{cases} \Rightarrow [x_1, x_2] \subseteq I$$

and therefore:

f differentiable on $I \Rightarrow f$ differentiable on $[x_1, x_2] \Rightarrow$

$$\Rightarrow \begin{cases} f \text{ continuous on } [x_1, x_2] \\ f \text{ differentiable on } (x_1, x_2) \end{cases}$$

$$\Rightarrow \exists x_0 \in (x_1, x_2) : f'(x_0) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Choose $x_0 \in (x_1, x_2)$ such that $f'(x_0) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

It follows that

$$\begin{aligned} f(x_2) - f(x_1) &= f'(x_0)(x_2 - x_1) \\ &= 0(x_2 - x_1) \qquad [\text{via } \forall x \in I : f'(x) = 0] \\ &= 0 \Rightarrow f(x_1) = f(x_2) \end{aligned}$$

and therefore:

$(\forall x_1, x_2 \in I : f(x_1) = f(x_2)) \Rightarrow$

$\Rightarrow f$ constant on I.

$\underline{\text{Thm}}$: Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $I \subseteq A$. Then:

$$\left\{\begin{array}{l} I \text{ interval} \\ f, g \text{ differentiable on } I \\ \forall x \in I: f'(x) = g'(x) \end{array}\right. \Rightarrow \exists\, c \in \mathbb{R}: \forall x \in I: f(x) = g(x) + c$$

<u>Proof</u>

Assume that

$$\left\{\begin{array}{ll} I \text{ interval} & (1) \\ f, g \text{ differentiable on } I \\ \forall x \in I: f'(x) = g'(x) \end{array}\right.$$

Define $\forall x \in I: h(x) = f(x) - g(x)$. Then

f, g differentiable on $I \Rightarrow h$ differentiable on I (2)

with

$$\forall x \in I: h'(x) = [f(x) - g(x)]' = f'(x) - g'(x)$$
$$= f'(x) - f'(x) = 0 \qquad (3)$$

From Eq.(1), Eq.(2), Eq.(3):

h constant on $I \Rightarrow \exists\, c \in \mathbb{R}: \forall x \in I: h(x) = c$

$\Rightarrow \exists\, c \in \mathbb{R}: \forall x \in I: f(x) - g(x) = c$

$\Rightarrow \exists\, c \in \mathbb{R}: \forall x \in I: f(x) = g(x) + c$

Method - Examples

(1) To show that an equation has a unique solution (i.e. $f(x)=0$) in (a,b).

- 1 Use the Bolzano theorem to establish EXISTENCE of a solution $x_0 \in (a,b)$.
- 2 Show that $f'(x) \neq 0$, $\forall x \in (a,b)$
- 3 Assume there are two solutions $x_0, x_1 \in (a,b)$ with $x_0 \neq x_1$ and use the Rolle theorem to reach a contradiction.

EXAMPLES

a) Show that $x^3-3x+1=0$ has a unique solution at $(-1,1)$

Solution

- Existence: Let $f(x) = x^3-3x+1$. Then

$$\left.\begin{array}{l} f(-1) = (-1)^3-3(-1)+1 = -1+3+1=3 \\ f(1) = 1^3-3\cdot 1+1 = -1 \end{array}\right\} \Rightarrow$$

$\Rightarrow f(-1)f(1) = 3\cdot(-1) < 0$ (1)

f continuous at $[-1,1]$ (2)

From (1) and (2):

$\exists\, x_0 \in (-1,1) : f(x_0)=0$

• <u>Uniqueness</u> : Assume that the equation is satisfied by $x_0, x_1 \in (-1,1)$ with $x_0 < x_1$

We note that

$f'(x) = (x^3-3x+1)' = 3x^2-3 = 3(x^2-1) < 0, \forall x \in (-1,1) \Rightarrow$

$\Rightarrow f'(x) \neq 0, \forall x \in (-1,1).$ (3)

$$\text{Since } \left.\begin{array}{l} f(x_0) = f(x_1) = 0 \\ f \text{ continuous at } [x_0, x_1] \\ f \text{ differentiable at } (x_0, x_1) \end{array}\right\} \Rightarrow$$

$\Rightarrow \exists x_2 \in (x_0, x_1) : \underline{f'(x_2) = 0.}$

From (3): $f'(x_2) \neq 0$, thus we have a contradiction.

It follows that the solution x_0 is unique.

B) Show that $x^5 + 2x^3 + 7x + 12 = 0$ has a unique solution in $\mathbb{R}$.

<u>Solution</u>

• <u>Existence</u> : Let $f(x) = x^5 + 2x^3 + 7x + 12$.

We note that:

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (x^5 + 2x^3 + 7x + 12) = \lim_{x \to +\infty} x^5 = +\infty \Rightarrow$$

$\Rightarrow \exists b \in (0, +\infty) : f(b) > 0$ (1)

and

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (x^5 + 2x^3 + 7x + 12) = \lim_{x \to -\infty} x^5 = -\infty \Rightarrow$$

$\Rightarrow \exists a \in (-\infty, 0) : f(a) < 0$ (2)

From (1) and (2):

$$\left.\begin{array}{l} f(a)f(b)<0 \\ f \text{ continuous at } [a,b] \end{array}\right\} \Rightarrow \exists x_0 \in (a,b) : f(x_0)=0 \Rightarrow$$

$\Rightarrow$ x_0 solves the equation.

- Uniqueness : Assume that $x_0, x_1 \in \mathbb{R}$ solve the equation with $x_0 < x_1$. We note that

$$f'(x) = (x^5+2x^3+7x+12)' = 5x^4+6x^2+7 >$$
$$> 5x^4+6x^2 \geq 0 \,,\ \forall x \in \mathbb{R} \Rightarrow$$
$$\Rightarrow \forall x \in \mathbb{R} : f'(x) > 0 \qquad (3)$$

Furthermore:

$$\left.\begin{array}{l} f(x_0)=f(x_1)=0 \\ f \text{ continuous at } [x_0,x_1] \\ f \text{ differentiable at } (x_0,x_1) \end{array}\right\} \Rightarrow \exists x_2 \in (x_0,x_1) : \underline{f'(x_2)=0}.$$

From (3): $\underline{f'(x_2)>0}$, so we have a contradiction. It follows that the equation cannot have more than one solution in $\mathbb{R}$.

$\hookrightarrow$ In the above solution we have used the statements:

$$\lim_{x\to+\infty} f(x) = +\infty \Rightarrow \exists a \in (0,+\infty) : f(a) > 0$$

$$\lim_{x\to-\infty} f(x) = -\infty \Rightarrow \exists a \in (-\infty,0) : f(a) < 0$$

which are immediate consequences of the limit definition. More generally:

$$\lim_{x\to\sigma} f(x) = L \Rightarrow \exists a \in N(\sigma,\delta) : f(a) \in I(L,\varepsilon)$$

② **Inequalities**: In general, using the Mean Value Theorem, an inequality satisfied by $f'(x)$ implies an inequality satisfied by $f(x)$.

EXAMPLES

a) Let f be a function differentiable in $\mathbb{R}$. Show that if $\forall x \in \mathbb{R}: 3 \leq f'(x) \leq 5$, then $18 \leq f(8) - f(2) \leq 30$.

Solution

f differentiable in $\mathbb{R} \Rightarrow$ MVT applies on $[2,8] \Rightarrow$

$$\Rightarrow \exists x_0 \in (2,8): f(8) - f(2) = f'(x_0)(8-2) = 6f'(x_0) \quad (1)$$

It follows that

$$3 \leq f'(x) \leq 5, \forall x \in \mathbb{R} \Rightarrow 3 \leq f'(x_0) \leq 5 \Rightarrow$$

$$\Rightarrow 18 \leq 6f'(x_0) \leq 30 \Rightarrow 18 \leq f(8) - f(2) \leq 30.$$

↳ Inequalities involving $\boxed{\text{two}}$ variables can be proved via the Mean Value Theorem if it is possible, with or without, some manipulation, to produce an expression of the form $f(b) - f(a)$. Then we can use:

$$f(b) - f(a) = f'(x_0)(b-a)$$

for some $x_0 \in (a,b)$.

b) Show that:

$$0 < a < b < \pi/2 \Rightarrow \frac{a}{b} < \frac{\sin a}{\sin b}$$

Solution

Since $0 < a < b < \pi/2 \Rightarrow b\sin b > 0$ and $ab > 0$.

It follows that

$$\frac{a}{b} < \frac{\sin a}{\sin b} \overset{*}{\Leftrightarrow} \frac{a}{b}(b\sin b) < \frac{\sin a}{\sin b}(b\sin b) \Leftrightarrow$$

$$\Leftrightarrow a\sin b < b\sin a \Leftrightarrow a\sin b - b\sin a < 0 \overset{*}{\Leftrightarrow}$$

$$\Leftrightarrow \frac{a\sin b - b\sin a}{ab} < 0 \Leftrightarrow \frac{\sin b}{b} - \frac{\sin a}{a} < 0 \quad (1).$$

Define $f(x) = \frac{\sin x}{x}$. It follows that:

$$f'(x) = \left(\frac{\sin x}{x}\right)' = \frac{(\sin x)' x - \sin x\,(x)'}{x^2} =$$

$$= \frac{x\cos x - \sin x}{x^2}$$

Since:

f continuous on $[a,b]$

f differentiable on $[a,b]$ $\Big\} \Rightarrow$ The Mean-Value-Theorem applies on $[a,b] \Rightarrow$

$\Rightarrow \exists x_0 \in (a,b) : f(b) - f(a) = f'(x_0)(b-a) \Rightarrow$

$$\Rightarrow \frac{\sin b}{b} - \frac{\sin a}{a} = f(b) - f(a) = f'(x_0)(b-a) =$$

$$= \frac{x_0\cos x_0 - \sin x_0}{x_0^2} \cdot (b-a) =$$

$$= \frac{(x_0 \cos x_0 - \sin x_0)(b-a)}{x_0^2} \qquad (2)$$

Note that

$a < b \Rightarrow b - a > 0 \qquad (3)$

and $\quad x_0^2 > 0 \qquad (4)$

and

$$\left.\begin{array}{l} |\tan x_0| > |x_0| \\ x_0 \in (0, \pi/2) \end{array}\right\} \Rightarrow \tan x_0 > x_0 \Rightarrow \frac{\sin x_0}{\cos x_0} > x_0 \Rightarrow$$

$$\Rightarrow \sin x_0 > x_0 \cos x_0 \Rightarrow x_0 \cos x_0 - \sin x_0 < 0 \qquad (5)$$

From $(2), (3), (4), (5)$:

$$\frac{\sin b}{b} - \frac{\sin a}{a} < 0 \underset{(1)}{\Rightarrow} \frac{a}{b} < \frac{\sin a}{\sin b} \qquad \square$$

↳ Note the 3-step process:

- 1 Reduce the inequality to be shown to an equivalent simpler inequality that exposes the $f(b) - f(a)$ expression
- 2 Define $f(x)$ and calculate $f'(x)$.
- 3 Apply the MVT and establish a relation between f and f'.
- 4 Determine if $f'(x_0)$ is positive or negative and backtrack your way back to the original inequality.

↳ Also recall the inequalities:

$|\tan x| > |x|, \ \forall x \in (-\pi/2, 0) \cup (0, \pi/2)$

$|\sin x| < |x|, \ \forall x \in \mathbb{R} - \{0\}.$

EXERCISES

Problems on the Rolle theorem

① Use the Bolzano and Rolle theorems to show that the following equations have a unique solution in the corresponding sets

a) $\dfrac{\cos x}{2} + \dfrac{1}{(1+x)^2} = 0$ on $A = (2\pi, 3\pi)$

b) $x^5 + x^3 + x = a^2(b-x) + b^2(c-x) + c^2(a-x)$
on $\mathbb{R}$ with $a, b, c \in \mathbb{R}$.

c) $\cos x = x$ on $A = (0, \pi)$

② Show that the equation $x^2 = x\sin x + \cos x$ has only 2 distinct solutions on $A = (-\pi, \pi)$

③ Let $f : \mathbb{R} \to \mathbb{R}$ such that
$$\begin{cases} f \text{ twice differentiable on } \mathbb{R} \\ \forall x \in \mathbb{R} : f''(x) \neq 0 \end{cases}$$
Show that the equation $f(x) = 0$ cannot have more than two distinct solutions on $\mathbb{R}$.

④ Show that the equation $x^n + ax + b = 0$ with $n \in \mathbb{N}^*$ has
a) at most 2 real solutions when n even and $n \geq 2$.
b) no more than 3 real solutions when n odd with $n \geq 3$.

⑤ Show that the equation $x^n + nx + 1 = 0$ with $n \in \mathbb{N}^*$ has

a) only one real solution when n odd

b) at most 2 real solutions when n even

⑥ Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ and let $a, b \in \mathbb{R}$ with $[a,b] \subseteq A$ such that

$$\begin{cases} f, g \text{ differentiable on } (a,b) \\ f, g \text{ continuous on } [a,b] \\ \dfrac{f(a)}{g(a)} = \dfrac{f(b)}{g(b)} \\ \forall x \in [a,b]: g(x) \neq 0 \\ \forall x \in (a,b): g'(x) \neq 0 \end{cases}$$

Show that:

$$\exists x_0 \in (a,b): \frac{f'(x_0)}{g'(x_0)} = \frac{f(x_0)}{g(x_0)}$$

⑦ Let $f: A \to \mathbb{R}$ and $a \in (0, +\infty)$ with $[-a, a] \subseteq A$ such that

$$\begin{cases} f \text{ continuous on } [-a,a] \\ f \text{ twice-differentiable on } (-a,a) \\ f(-a) = a \wedge f(a) = -a \wedge f(0) = 0 \end{cases}$$

Show that

$$\exists x_0 \in (-a,a): f''(x_0) = 0$$

⑧ Let $f: A \to \mathbb{R}$ and let $a, b \in \mathbb{R}$ with $[a,b] \subseteq A$ such that

$$\begin{cases} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \\ f(a) = f(b) \end{cases}$$

Let $c \in \mathbb{R} - [a,b]$ and define $g : [a,b] \to \mathbb{R}$ such that

$$\forall x \in [a,b] : g(x) = \frac{f(x)}{x - c}$$

Show that : $\exists x_0 \in (a,b) : g'(x_0) = 0$

⑨ Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ and $a, b \in \mathbb{R}$ with $[a,b] \subseteq A$ and $0 \notin [a,b]$ such that

$$\begin{cases} f, g \text{ differentiable on } [a,b] \\ f, g \text{ continuous on } [a,b] \\ f(a) = g(b) = 0 \\ \forall x \in [a,b] : f(x) g(x) \neq 0 \end{cases}$$

Show that:

$$\exists x_0 \in (a,b) : \frac{f'(x_0)}{f(x_0)} + \frac{g'(x_0)}{g(x_0)} = \frac{1}{x_0}$$

↳ Hint: Apply the Rolle theorem on the function $h(x) = f(x) g(x) / x$

⑩ Let $f: A \to \mathbb{R}$ and $a, b \in (0, +\infty)$ with $[a,b] \subseteq A$ such that

$$\begin{cases} f \text{ twice-differentiable on } [a,b] \\ f(a) = f(b) = 0 \\ \forall x \in (a,b) : f'(x) \neq 0 \end{cases}$$

Show that the equation $x f'(x) - f(x) = 0$ has a unique solution on the interval (a,b).

↳ Use the Rolle theorem on the function $g(x) = f(x)/x$.

⑪ Let $f: A \to \mathbb{R}$ with $[0,1] \subseteq A$ such that

$$\begin{cases} f \text{ continuous on } [0,1] \\ f \text{ differentiable on } (0,1) \\ f(1) = f(0) + 1/2 \end{cases}$$

Show that the equation $f'(x) = x$ has at least one solution on the interval $(0,1)$

↳ Use Rolle theorem on the appropriate function $g(x)$ to establish the existence of at least one solution.

⑫ Let $f: A \to \mathbb{R}$ with $[a,b] \subseteq A$ such that

$$\begin{cases} f \text{ twice-differentiable on } [a,b] \\ \forall x \in [a,b]: f(x) f'(x) \neq 0 \\ \dfrac{f(a)}{f'(a)} = \dfrac{f'(b)}{f'(b)} \end{cases}$$

Show that

$$\exists c_1, c_2 \in (a,b): f'(c_1) f''(c_1) + f(c_2) f''(c_2) > 0$$

↳ Use the Rolle theorem on the functions $g(x) = \dfrac{f(x)}{f'(x)}$ and $h(x) = \dfrac{f'(x)}{f(x)}$

(13) Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ and $h: A \to \mathbb{R}$ and $a, b \in \mathbb{R}$ with $[a,b] \subseteq A$ such that

$$\begin{cases} f, g, h \text{ continuous on } [a,b] \\ f, g, h \text{ differentiable on } (a,b) \end{cases}$$

Show that the equation

$$\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

has at least one solution on (a,b).

(14) Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ and let $a, b \in \mathbb{R}$ with $[a,b] \subseteq A$ such that

$$\begin{cases} f, g \text{ continuous on } [a,b] \\ f, g \text{ differentiable on } (a,b) \end{cases}$$

Show that:

$$\exists x_0 \in (a,b): \frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Problems on the Mean Value Theorem

(15) Use the mean-value theorem to prove the following inequalities

a) $a,b \in (-\pi/2, \pi/2) \Rightarrow |\sin a - \sin b| \leq |a-b|$

b) $\forall n \in \mathbb{N}^*: (0<a<b \Rightarrow n(b-a)a^{n-1} \leq b^n - a^n \leq n(b-a)b^{n-1})$

c) $0<a\leq b<\pi/2 \Rightarrow \dfrac{a-b}{\cos^2 b} \leq \tan a - \tan b \leq \dfrac{a-b}{\cos^2 a}$

d) $0<a<a+b<\pi/2 \Rightarrow \sin(a+b) < \sin a + b\cos a$

e) $0<a<b<\pi/2 \Rightarrow \dfrac{\tan a}{\tan b} < \dfrac{b}{a}$

Use the mean-value theorem on $g(x) = x\tan x$

(16) Let $f: A \to \mathbb{R}$ and let $a,b \in \mathbb{R}$ with $[a,b] \subseteq A$ such that

$$\begin{cases} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \end{cases}$$

Show that: $\exists c_1, c_2 \in (a,b): f'(c_1) + f'(c_2) = 0$

(17) Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} f, g \text{ differentiable on } \mathbb{R} \\ f(0) = 0 \wedge g(0) = 1 \\ \forall x \in \mathbb{R}: (f'(x) - g(x) = 0 \wedge f(x) + g'(x) = 0) \end{cases}$$

Show that: $\forall x \in \mathbb{R}: f^2(x) + g^2(x) = 1$.

(18) Let $f:\mathbb{R}\to\mathbb{R}$ such that
$\forall x,y\in\mathbb{R}: |f(x)-f(y)|\leq |x-y|^2$
Show that f is constant on $\mathbb{R}$.

(19) Let $f:\mathbb{R}\to\mathbb{R}$ such that
$$\begin{cases} f \text{ twice-differentiable on } \mathbb{R} \\ \forall x\in\mathbb{R}: f''(x)+f(x)=0 \\ f(0)=f'(0)=0 \end{cases}$$
Show that
$\exists c\in\mathbb{R}: \forall x\in\mathbb{R}: [f(x)]^2+[f'(x)]^2=c$

(20) Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ such that
$$\begin{cases} f,g \text{ differentiable on } \mathbb{R} \\ \forall x\in\mathbb{R}: (f'(x)=g(x) \wedge g'(x)=f(x)) \\ f(0)=1 \wedge g(0)=1 \end{cases}$$
Show that:
$\forall x\in\mathbb{R}: [f(x)]^2=[g(x)]^2+1$

(21) Let $f:A\to\mathbb{R}$ and let $a,b\in\mathbb{R}$ with $[a,b]\subseteq A$ such that
$$\begin{cases} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \end{cases}$$
Show that:
$$\exists x_1,x_2,x_3\in(a,b): \begin{cases} x_1\neq x_2\neq x_3\neq x_1 \\ (b-a)(f'(x_1)+f'(x_2)+f'(x_3))=f(b)-f(a) \end{cases}$$