LIMITS AND CONTINUITY

▼ Weierstrass limit definition

Let $f: A \to \mathbb{R}$ be a function with domain $\mathrm{dom}(f) = A \subseteq \mathbb{R}$. In order to define $\lim_{x \to \sigma} f(x) = L$, we begin with the following notation:

Notation

a) The neighborhood $N(\sigma, \delta)$ is defined as

$$N(\sigma,\delta) = \begin{cases} (x_0-\delta, x_0) \cup (x_0, x_0+\delta) &, \text{ if } \sigma = x_0 \\ (x_0-\delta, x_0) &, \text{ if } \sigma = x_0^- \\ (x_0, x_0+\delta) &, \text{ if } \sigma = x_0^+ \\ (1/\delta, +\infty) &, \text{ if } \sigma = +\infty \\ (-\infty, -1/\delta) &, \text{ if } \sigma = -\infty \end{cases}$$

b) The interval $I(L, \varepsilon)$ is defined as

$$I(L,\varepsilon) = \begin{cases} (\ell-\varepsilon, \ell+\varepsilon) &, \text{ if } L = \ell \in \mathbb{R} \\ (1/\varepsilon, +\infty) &, \text{ if } L = +\infty \\ (-\infty, -1/\varepsilon) &, \text{ if } L = -\infty \end{cases}$$

Note that the corresponding belonging conditions are:

$x \in N(x_0, \delta) \iff 0 < \lvert x - x_0 \rvert < \delta$	$y \in I(\ell, \varepsilon) \iff \lvert y - \ell \rvert < \varepsilon$
$x \in N(x_0^-, \delta) \iff x_0 - \delta < x < x_0$	$y \in I(+\infty, \varepsilon) \iff y > 1/\varepsilon$
$x \in N(x_0^+, \delta) \iff x_0 < x < x_0 + \delta$	$y \in I(-\infty, \varepsilon) \iff y < -1/\varepsilon$
$x \in N(+\infty, \delta) \iff x > 1/\delta$	
$x \in N(-\infty, \delta) \iff x < -1/\delta$	

Remarks

a) It is easy to show that

$$\begin{cases} \forall \delta_1, \delta_2 \in (0,+\infty) : (\delta_1 < \delta_2 \Rightarrow N(\sigma,\delta_1) \subseteq N(\sigma,\delta_2)) \\ \forall \varepsilon_1, \varepsilon_2 \in (0,+\infty) : (\varepsilon_1 < \varepsilon_2 \Rightarrow I(L,\varepsilon_1) \subseteq I(L,\varepsilon_2)) \end{cases}$$

thus, decreasing δ or ε tends to make the neighborhood $N(\sigma,\delta)$ or interval $I(L,\varepsilon)$ tighter.

b) We can also show that

$$\begin{cases} \forall \delta_1, \delta_2 \in (0,+\infty) : N(\sigma,\delta_1) \cap N(\sigma,\delta_2) = N(\sigma, \min\{\delta_1,\delta_2\}) \\ \forall \varepsilon_1, \varepsilon_2 \in (0,+\infty) : I(L,\varepsilon_1) \cap I(L,\varepsilon_2) = I(L, \min\{\varepsilon_1,\varepsilon_2\}) \end{cases}$$

c) Relationship between neighborhoods and intervals:

$$\begin{cases} N(x_0,\delta) = I(x_0,\delta) - \{x_0\} \\ N(+\infty,\delta) = I(+\infty,\delta) \\ N(-\infty,\delta) = I(-\infty,\delta) \end{cases}$$

▸ We now give the following definitions:

> **Def**: Let $A \subseteq \mathbb{R}$ be a set. We say that
>
> σ limit point of $A \Leftrightarrow \forall \delta \in (0,+\infty) : N(\sigma,\delta) \cap A = \emptyset$

▸ __interpretation:__ σ is a limit point of A if and only if regardless of how much we "squeeze" $N(\sigma,\delta)$ by decreasing δ, it always overlaps with A.

> **Def**: Let $f : A \to \mathbb{R}$ be a function, σ a limit point of A, and $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Then
>
> $$\lim_{x \to \sigma} f(x) = L \Leftrightarrow \forall \varepsilon \in (0,+\infty) : \exists \delta \in (0,+\infty) : \forall x \in A : (x \in N(\sigma,\delta) \Rightarrow f(x) \in I(L,\varepsilon))$$

- interpretation: In the above definition:
  ε = how close we want $f(x)$ to be to L
  δ = how close x must be brought to σ so that $f(x)$ will be as close to L as required by our choice of ε.

  Thus, as we choose smaller ε, it should always be possible to find a smaller δ that works.
- For each choice of σ and L we get a corresponding Weierstrass limit definition (5 choices for σ, 3 choices for L, thus 15 possible definitions). For example, for $\sigma = x_0 \in \mathbb{R}$ and $L = \ell \in \mathbb{R}$, we have:

$$\lim_{x \to x_0} f(x) = \ell \Leftrightarrow \forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (0 < |x - x_0| < \delta \Rightarrow \Rightarrow |f(x) - \ell| < \varepsilon)$$

- Note the following immediate consequences of the limit definition

$$\boxed{\begin{aligned} &\lim_{x \to \sigma} (f(x) - \ell) = 0 \Leftrightarrow \lim_{x \to \sigma} f(x) = \ell \\ &\lim_{x \to \sigma} f(x) = \ell \Leftrightarrow \lim_{x \to \sigma} [-f(x)] = -\ell \\ &\lim_{x \to \sigma} f(x) = \pm\infty \Leftrightarrow \lim_{x \to \sigma} [-f(x)] = \mp\infty \end{aligned}}$$

Def: Let $f: A \to \mathbb{R}$ be a function, σ a limit point of A. We say that:

$$\lim_{x \to \sigma} f(x) \text{ does not exist} \Leftrightarrow \begin{cases} \forall \ell \in \mathbb{R} : \lim_{x \to \sigma} f(x) \neq \ell \\ \lim_{x \to \sigma} f(x) \neq +\infty \wedge \lim_{x \to \sigma} f(x) \neq -\infty \end{cases}$$

EXAMPLES

a) Use the limit definition to show that

$$\lim_{x \to -\infty} \frac{3x^2+2x-1}{x^2+2x+9} = 3$$

Solution

Define $\forall x \in \mathbb{R}: f(x) = \dfrac{3x^2+2x-1}{x^2+2x+9}$, and note that:

$$f(x)-3 = \frac{3x^2+2x-1}{x^2+2x+9} - 3 = \frac{(3x^2+2x-1)-3(x^2+2x+9)}{x^2+2x+9}$$

$$= \frac{3x^2+2x-1-3x^2-6x-27}{x^2+2x+9} =$$

$$= \frac{(3-3)x^2+(2-6)x+(-1-27)}{x^2+2x+9} =$$

$$= \frac{-4x-28}{x^2+2x+9} = \frac{-2(2x+9)}{x^2+2x+9}$$

with $\begin{cases} \forall x \in \mathbb{R}: x^2+2x+9 = (x+1)^2+8 > 0 \\ \forall x \in (-\infty,-5): 2x+9 < 0 \end{cases}$

Restrict the domain of f to $A = (-\infty,-5)$.

Let $\varepsilon \in (0,+\infty)$ be given. We have; for all $x \in (-\infty,-5)$:

$$|f(x)-3| = \left| \frac{-2(2x+9)}{x^2+2x+9} \right| = \frac{-2(2x+9)}{x^2+2x+9} \leqslant \frac{-4x}{x^2+2x+9}$$

$$\leqslant \frac{-4x}{x^2+2x} = \frac{-4x}{x(x+2)} = \frac{-4}{x+2} < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow -4 > \varepsilon(x+2) \qquad [\text{via } x+2<0]$$

$$\Leftrightarrow \varepsilon x + 2\varepsilon < -4 \Leftrightarrow \varepsilon x < -4 - 2\varepsilon \Leftrightarrow x < \frac{-4-2\varepsilon}{\varepsilon}$$

Choose $\underline{\delta = \varepsilon/(2\varepsilon+4) > 0}$. Let $\underline{x \in A}$ be given and assume that $\underline{x \in N(-\infty, \delta)}$. Then, we have:

$$x \in N(-\infty, \delta) \Rightarrow x \in (-\infty, -1/\delta) \Rightarrow x < -1/\delta \Rightarrow$$
$$\Rightarrow x < \frac{-4-2\varepsilon}{\varepsilon} \Rightarrow \underline{|f(x) - 3| < \varepsilon}$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (x \in N(-\infty, \delta) \Rightarrow |f(x) - 3| < \varepsilon)$$
$$\Rightarrow \lim_{x \to -\infty} f(x) = 3$$

8) Use the limit definition to show that

$$\lim_{x \to 1} (x^2+2x+3) = 6$$

Solution

Define $f(x) = x^2+2x+3$, $\forall x \in \mathbb{R}$. Then, we have:

$$f(x)-6 = (x^2+2x+3)-6 = x^2+2x+3-6 = x^2+2x-3$$
$$= (x+3)(x-1), \ \forall x \in \mathbb{R}.$$

Restrict the domain of f to $A = (0,1) \cup (1,2)$. Let $\varepsilon \in (0,+\infty)$ be given. Then, for all $x \in A$, we have:

$$|f(x)-6| = |(x+3)(x-1)| = |x+3||x-1| = |(x-1)+4||x-1|$$
$$\leq [|x-1|+4]|x-1| < [1+4]|x-1| = 5$$
$$= 5|x-1| < \varepsilon \iff |x-1| < \varepsilon/5.$$

Choose $\delta = \varepsilon/5$. Let $x \in A$ be given and assume that $x \in N(1,\delta)$. Then, we have:

$$x \in N(1,\delta) \Rightarrow 0 < |x-1| < \delta \Rightarrow 0 < |x-1| < \varepsilon/5 \Rightarrow$$
$$\Rightarrow |f(x)-6| < \varepsilon.$$

We have thus shown that

$$\forall \varepsilon \in (0,+\infty) : \exists \delta \in (0,+\infty) : \forall x \in A : (x \in N(1,\delta) \Rightarrow |f(x)-6| < \varepsilon)$$
$$\Rightarrow \lim_{x \to 1} (x^2+2x+3) = 6$$

THEORY QUESTIONS

① State the definition of the neighborhood $N(\sigma, \delta)$ for ~~an~~ arbitrary σ and the definition of the statement:

σ limit point of A

with $A \subseteq \mathbb{R}$, using quantifier notation.

② State the definition of the neighborhood $N(\sigma, \delta)$ and interval $I(L, \varepsilon)$ for an arbitrary choice of σ and L, and then state the general definition of the statement $\lim_{x \to \sigma} f(x) = L$, using quantifier notation.

③ State the definition of the statement

$\lim_{x \to \sigma} f(x)$ does not exist

for an arbitrary choice of σ.

④ Use quantifiers to write the specific defintions for the following statements, without using the neighborhood / interval notation

a) $\lim_{x \to x_0} f(x) = l \in \mathbb{R}$

b) $\lim_{x \to x_0^-} f(x) = -\infty$

c) $\lim_{x \to x_0^+} f(x) = +\infty$

d) $\lim_{x \to -\infty} f(x) = l \in \mathbb{R}$

e) $\lim_{x \to +\infty} f(x) = -\infty$

f) $\lim_{x \to -\infty} f(x) = +\infty$

g) $\lim_{x \to x_0^+} f(x) = \ell \in \mathbb{R}$

h) $\lim_{x \to 0} f(x) = 0$

EXERCISES

⑤ Use the neighborhood definition and proof by cases to show that

a) $\forall \delta_1, \delta_2 \in (0, +\infty) : (\delta_1 < \delta_2 \Rightarrow N(\sigma, \delta_1) \subseteq N(\sigma, \delta_2))$

b) $\forall \delta_1, \delta_2 \in (0, +\infty) : N(\sigma, \delta_1) \cap N(\sigma, \delta_2) = N(\sigma, \min\{\delta_1, \delta_2\})$

↳ Use (a) to prove (b).

⑥ Use the limit definition to show that:

a) $\lim_{x \to -\infty} \frac{2x-3}{x-5} = 2$

b) $\lim_{x \to +\infty} \frac{x^2}{x-2} = +\infty$

c) $\lim_{x \to +\infty} \frac{\sin(3x)}{x+1} = 0$

d) $\lim_{x \to +\infty} \frac{2x^2+3x+1}{3x^2+x+5} = \frac{2}{3}$

e) $\lim_{x \to 1} \frac{(x+1)(x+3)}{x^2+2x+5} = 1$

f) $\lim_{x \to 3^-} \frac{x^3}{2x+1} = \frac{27}{7}$

g) $\lim_{x \to 2^+} \frac{3x+1}{x-2} = +\infty$

h) $\lim_{x \to 3^-} \frac{5x+2}{x-3} = -\infty$

i) $\lim_{x \to -1^+} \frac{x^3}{2x+2} = -\infty$

j) $\lim_{x \to -\infty} \frac{2x \cos x}{x^2-2} = 0$

k) $\lim_{x \to +\infty} \frac{3x^2 \cos x}{x^3+2} = 0$

l) $\lim_{x \to 0} x^3 \sin(1/x) = 0$

m) $\lim_{x \to +\infty} \frac{\cos x}{x^3} = 0$

(7) Use the limit definition to show that

a) $\lim_{x \to -\infty} \frac{2x\cos x}{x^2-2} = 0$

b) $\lim_{x \to +\infty} \frac{3x^2 \cos x}{x^3+2} = 0$

c) $\lim_{x \to 0} x^3 \sin(1/x) = 0$

d) $\lim_{x \to +\infty} \frac{\cos x}{x^3} = 0$

Relation between side limits $x \to x_0^{\pm}$ and $x \to x_0$

Thm: Let $f: A \to \mathbb{R}$ be a function, let x_0 be a limit point of A, and let $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Then, we have

$$\lim_{x \to x_0} f(x) = L \iff \lim_{x \to x_0^+} f(x) = L \wedge \lim_{x \to x_0^-} f(x) = L$$

Proof

($\Rightarrow$): Assume that $\lim_{x \to x_0} f(x) = L$. Then, we have:

$$\lim_{x \to x_0} f(x) = L \Rightarrow$$
$$\Rightarrow \forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$$
$$\Rightarrow f(x) \in I(L, \varepsilon))$$

Let $\varepsilon \in (0, +\infty)$ be given. Choose $\delta \in (0, +\infty)$ such that $\forall x \in A : (x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta) \Rightarrow f(x) \in I(L, \varepsilon))$

Let $x \in A$ be given. Then, we have:

$$x \in (x_0, x_0 + \delta) \Rightarrow x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta) \Rightarrow$$
$$\Rightarrow f(x) \in I(L, \varepsilon)$$

and

$$x \in (x_0 - \delta, x_0) \Rightarrow x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta) \Rightarrow$$
$$\Rightarrow f(x) \in I(L, \varepsilon)$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : \begin{cases} x \in (x_0 - \delta, x_0) \Rightarrow f(x) \in I(L, \varepsilon) \\ x \in (x_0, x_0 + \delta) \Rightarrow f(x) \in I(L, \varepsilon) \end{cases}$$

$$\Rightarrow \lim_{x \to x_0^+} f(x) = L \wedge \lim_{x \to x_0^-} f(x) = L.$$

$(\Rightarrow)$: Assume that $\lim\limits_{x\to x_0^+} f(x)=L \wedge \lim\limits_{x\to x_0^-} f(x)=L$.

Let $\underline{\varepsilon\in(0,+\infty)}$ be given. Then, we have:

$$\begin{cases} \lim\limits_{x\to x_0^+} f(x)=L \\ \lim\limits_{x\to x_0^-} f(x)=L \end{cases} \Rightarrow \begin{cases} \exists\delta\in(0,+\infty):\forall x\in A:(x\in(x_0-\delta,x_0)\Rightarrow f(x)\in I(L,\varepsilon)) \\ \exists\delta\in(0,+\infty):\forall x\in A:(x\in(x_0,x_0+\delta)\Rightarrow f(x)\in I(L,\varepsilon)) \end{cases}$$

Choose $\delta_1,\delta_2\in(0,+\infty)$ such that

$$\begin{cases} \forall x\in A:(x\in(x_0-\delta_1,x_0)\Rightarrow f(x)\in I(L,\varepsilon)) \\ \forall x\in A:(x\in(x_0,x_0+\delta_2)\Rightarrow f(x)\in I(L,\varepsilon)) \end{cases}$$

Choose $\underline{\delta=\min\{\delta_1,\delta_2\}}$. Let $\underline{x\in A}$ be given and assume that $\underline{x\in(x_0-\delta,x_0)\cup(x_0,x_0+\delta)}$. Then, we have:

$$\begin{aligned} x\in(x_0-\delta,x_0)\cup(x_0,x_0+\delta) &\Rightarrow x\in(x_0-\delta,x_0)\vee x\in(x_0,x_0+\delta) \\ &\Rightarrow x\in(x_0-\delta_1,x_0)\vee x\in(x_0,x_0+\delta_2) \\ &\Rightarrow f(x)\in I(L,\varepsilon)\vee f(x)\in I(L,\varepsilon) \\ &\Rightarrow \underline{f(x)\in I(L,\varepsilon)} \end{aligned}$$

We have thus shown that

$$\forall\varepsilon\in(0,+\infty):\exists\delta\in(0,+\infty):\forall x\in A:(x\in(x_0-\delta,x_0)\cup(x_0,x_0+\delta)\Rightarrow$$
$$\Rightarrow f(x)\in I(L,\varepsilon))$$

$$\Rightarrow \lim_{x\to x_0} f(x)=L$$ □

▸ An immediate consequence of this result is the following statement:

$$\boxed{\lim_{x\to x_0^+} f(x)=L_1\wedge\lim_{x\to x_0^-} f(x)=L_2\wedge L_1\neq L_2\Rightarrow \lim_{x\to x_0} f(x) \text{ does not exist}}$$

→ <u>Methodology</u>: To show that $\lim_{x \to \sigma} f(x) = L$ by definition

- 1 Investigate $f(x) \in I(L, \varepsilon)$ and, if needed, restrict the domain A of f to $A = A_0 \cap N(\sigma, \delta)$ for an appropriate δ with A_0 the widest possible domain.
- 2 Let $\underline{\varepsilon \in (0, +\infty)}$ be given. Derive an equivalence
  $f(x) \in \$ \subseteq I(L, \varepsilon) \Leftrightarrow x \in N(\sigma, g(\varepsilon))$
- 3 Choose $\underline{\delta = g(\varepsilon)}$. Let $\underline{x \in A}$ be given and assume that $\underline{x \in N(\sigma, \delta)}$. Then, we have
  $x \in N(\sigma, \delta) \Rightarrow x \in N(\sigma, g(\varepsilon)) \Rightarrow f(x) \in \$ \Rightarrow$
  $\Rightarrow \underline{f(x) \in I(L, \varepsilon)}$.
- 4 We have thus shown that

$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (x \in N(\sigma, \delta) \Rightarrow f(x) \in I(L, \varepsilon))$

$\Rightarrow \lim_{x \to \sigma} f(x) = L$.

EXAMPLES

a) Use the limit definition to show that for

$$f(x)=\begin{cases} 3x & , \text{ if } x\in[1,+\infty) \\ -x+4 & , \text{ if } x\in(-\infty,1) \end{cases}$$

we have $\lim\limits_{x\to 1} f(x)=3$

<u>Solution</u>

• <u>Limit $x\to 1^+$</u>: Restrict the domain of f to $A=[1,+\infty)$.
Let <u>$\varepsilon\in(0,+\infty)$</u> be given. Then for all $x\in A$, we have
$|f(x)-3|=|3x-3|=|3(x-1)|=3|x-1|<\varepsilon \Leftrightarrow |x-1|<\varepsilon/3$.
Choose <u>$\delta=\varepsilon/3$.</u> Let <u>$x\in A$</u> be given and assume that <u>$x\in N(1^+,\delta)$.</u> Then, we have:

$$x\in N(1^+,\delta)\Rightarrow x\in N(1,\delta)\Rightarrow 0<|x-1|<\delta \Rightarrow$$
$$\Rightarrow |x-1|<\varepsilon/3 \Rightarrow \underline{|f(x)-3|<\varepsilon.}$$

We have thus shown that

$$\forall \varepsilon\in(0,+\infty):\exists\delta\in(0,+\infty):\forall x\in A:(x\in N(1^+,\delta)\Rightarrow|f(x)-3|<\varepsilon)$$
$$\Rightarrow \lim_{x\to 1^+} f(x)=3.$$

• <u>Limit $x\to 1^-$</u>: Restrict the domain of f to $A=(-\infty,1)$.
Let <u>$\varepsilon\in(0,+\infty)$</u> be given. Then, for all $x\in A$, we have
$|f(x)-3|=|(-x+4)-3|=|-x+1|=|x-1|$.
Choose <u>$\delta=\varepsilon$.</u> Let <u>$x\in A$</u> be given and assume that <u>$x\in N(1^-,\delta)$.</u> Then, we have:

$$x\in N(1^-,\delta)\Rightarrow x\in N(1,\delta)\Rightarrow 0<|x-1|<\delta\Rightarrow$$
$$\Rightarrow |x-1|<\varepsilon \Rightarrow \underline{|f(x)-3|<\varepsilon}$$

We have thus shown that

$\forall \varepsilon \in (0,+\infty) : \exists \delta \in (0,+\infty) : \forall x \in A : (x \in N(1^-,\delta) \Rightarrow |f(x)-3|<\varepsilon)$

$\Rightarrow \lim_{x \to 1^-} f(x) = 3.$

- From the above, we conclude that $\lim_{x\to 1^+} f(x) = 3 \wedge \lim_{x \to 1^-} f(x) = 3 \Rightarrow \lim_{x \to 1} f(x) = 3.$

THEORY QUESTIONS

(8) Let $f: A \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$ be a limit point of A and let $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Show that

a) $\lim_{x \to x_0} f(x) = L \Rightarrow \left(\lim_{x \to x_0^+} f(x) = L \wedge \lim_{x \to x_0^-} f(x) = L \right)$

b) $\left(\lim_{x \to x_0^+} f(x) = L \wedge \lim_{x \to x_0^-} f(x) = L \right) \Rightarrow \lim_{x \to x_0} f(x) = L$

EXERCISES

(9) Let $f: A \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ limit point of A and let $L_1, L_2 \in \mathbb{R} \cup \{+\infty, -\infty\}$. Show that

$$\begin{cases} \lim_{x \to x_0^+} f(x) = L_1 \\ \lim_{x \to x_0^-} f(x) = L_2 \\ L_1 \neq L_2 \end{cases} \Rightarrow \lim_{x \to x_0} f(x) \text{ does not exist.}$$

(10) Use the limit definition, in conjunction with side limits, to show that

a) $\lim_{x \to 2} f(x) = 7$ with $f(x) = \begin{cases} 3x+1, & \text{if } x \in [2, +\infty) \\ 4x-1, & \text{if } x \in (-\infty, 2) \end{cases}$

b) $\lim_{x \to 1} f(x) = 2$ with $f(x) = \begin{cases} x^2+x, & \text{if } x \in (-\infty, 1) \\ 2x^3, & \text{if } x \in (1, +\infty) \end{cases}$

c) $\lim\limits_{x \to -1} f(x)$ does not exist, with

$$f(x) = \begin{cases} 3x^2, & \text{if } x \in (-\infty, -1) \\ 3x, & \text{if } x \in (-1, +\infty) \end{cases}$$

d) $\lim\limits_{x \to 3} f(x)$ does not exist, with

$$f(x) = \begin{cases} x(x+2), & \text{if } x \in (3, +\infty) \\ x^2-2, & \text{if } x \in (-\infty, 3). \end{cases}$$

▼ Function limits as net limits

Function limits are a special case of a net limit, and as such they inherit all the properties that we have previously established on convergent nets. The connection between the two concepts is established by the following theorem:

Thm: Let $f: A \to \mathbb{R}$ be a function, let σ be a limit point of A, let $\delta_0 \in (0, +\infty)$, and let $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Then, define $(D, <_\sigma)$ such that

$$\begin{cases} D = N(\sigma, \delta_0) \cap A \\ \|x\|_\sigma = \inf \{\delta \in (0, +\infty) \mid x \in N(\sigma, \delta) \cap A\} \\ \forall x_1, x_2 \in D : (x_1 <_\sigma x_2 \Leftrightarrow \|x_1\|_\sigma \geq \|x_2\|_\sigma) \end{cases}$$

Then, we have:

$$\begin{cases} (D, <_\sigma) \text{ is a directed set} \\ \lim\limits_{x \to \sigma} f(x) = L \Leftrightarrow \lim\limits_{x \in D} f(x) = L \end{cases}$$

↳ Note that $\|x\|_\sigma$ represents how close x is to the limit point σ. Also, $x_1 <_\sigma x_2$ is the statement that x_2 is closer to the limit point σ than x_1.

Proof

Since σ limit point of A, it follows that $D = N(\sigma, \delta_0) \cap A \neq \emptyset$.
Define $\$(x) = \{\delta \in (0, +\infty) \mid x \in N(\sigma, \delta) \cap A\}$.

▸ We claim that $\|x\|_\sigma = \inf \$(x)$ is well-defined. for all $x \in D$.

Let $x \in D$ be given. Then, we have:

$$S(x) = \{\delta \in (0, +\infty) \mid x \in N(\sigma, \delta)\} \subseteq (0, +\infty) \subseteq \mathbb{R} \Rightarrow S(x) \subseteq \mathbb{R} \quad (1)$$

and

$$x \in D \Rightarrow x \in N(\sigma, \delta_0) \cap A \Rightarrow \delta_0 \in S(x) \Rightarrow S(x) \neq \emptyset \quad (2)$$

and

$$\begin{aligned} S(x) \subseteq (0, +\infty) &\Rightarrow \forall \delta \in S(x) : \delta \in (0, +\infty) \\ &\Rightarrow \forall \delta \in S(x) : \delta \geq 0 \\ &\Rightarrow S(x) \text{ lower bounded} \quad (3) \end{aligned}$$

From Eq. (1), Eq. (2), Eq. (3) via the axiom of completeness it follows that $\|x\|_\sigma = \inf S(x)$ is well-defined.

This proves the claim

▶ We will show that $(D, <_\sigma)$ is a directed set.

• <u>$<_\sigma$ reflective property.</u>

Let <u>$x \in D$</u> be given. Then $\|x\|_\sigma \geq \|x\|_\sigma \Rightarrow$ <u>$x <_\sigma x$</u>. It follows that:

$$\forall x \in D : x <_\sigma x$$

• <u>$<_\sigma$ transitive property.</u>

Let <u>$x, y, z \in D$</u> be given and assume that <u>$x <_\sigma y$</u> and <u>$y <_\sigma z$</u>. Then, we have:

$$\begin{cases} x <_\sigma y \\ y <_\sigma z \end{cases} \Rightarrow \begin{cases} \|x\|_\sigma \geq \|y\|_\sigma \\ \|y\|_\sigma \geq \|z\|_\sigma \end{cases} \Rightarrow \|x\|_\sigma \geq \|z\|_\sigma \Rightarrow \underline{x <_\sigma z}$$

We have thus shown that

$$\forall x, y, z \in D : ((x <_\sigma y \land y <_\sigma z) \Rightarrow x <_\sigma z)$$

• <u>$<_\sigma$ refinement property</u>

Let <u>$x, y \in D$</u> be given. Since σ is a limit point of A, choose $\underline{z \in N(\sigma, \min\{\|x\|_\sigma, \|y\|_\sigma\}) \cap A}$. Then, we have:

$$\|z\|_\sigma \leq \min\{\|x\|_\sigma, \|y\|_\sigma\} \Rightarrow \begin{cases} \|z\|_\sigma \leq \|x\|_\sigma \\ \|z\|_\sigma \leq \|y\|_\sigma \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \underline{x <_\sigma z} \\ \underline{y <_\sigma z} \end{cases}$$

We have thus shown that

$\forall x, y \in D : \exists z \in D : (x <_\sigma z \wedge y <_\sigma z)$

From the above, we conclude that $(D, <_\sigma)$ is a directed set.

▸ We will show that $\lim_{x \to \sigma} f(x) = L \Leftrightarrow \lim_{x \in D} f(x) = L$.

$(\Rightarrow)$: Assume that $\lim_{x \to \sigma} f(x) = L$. It follows that

$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (x \in N(\sigma, \delta) \Rightarrow f(x) \in I(L, \varepsilon))$

Let $\underline{\varepsilon \in (0, +\infty)}$ be given. Choose $\delta \in (0, +\infty)$ such that

$\forall x \in A : (x \in N(\sigma, \delta) \Rightarrow f(x) \in I(L, \varepsilon))$

Since σ limit point of A, we can choose

$n_0 \in N(\sigma, \min\{\delta_0, \delta\}) \cap A \subseteq D. \Rightarrow \underline{n_0 \in D}.$

and note that $\|n_0\|_\sigma \leq \min\{\delta_0, \delta\}$. Let $\underline{n \in D}$ be given and assume that $\underline{n >_\sigma n_0}$. Then, we have:

$$n >_\sigma n_0 \Rightarrow \|n\|_\sigma \leq \|n_0\|_\sigma \leq \min\{\delta_0, \delta\} \leq \delta \Rightarrow$$

$$\Rightarrow \|n\|_\sigma \leq \delta \Rightarrow n \in N(\sigma, \delta) \cap A \Rightarrow \underline{f(n) \in I(L, \varepsilon)}.$$

We have thus shown that

$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n >_\sigma n_0 \Rightarrow f(n) \in I(L, \varepsilon))$

$\Rightarrow \lim_{n \in D} f(n) = L.$

$(\Leftarrow)$: Assume that $\lim_{n \in D} f(n) = L$. Then, we have:

$\forall \varepsilon \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n >_\sigma n_0 \Rightarrow f(n) \in I(L,\varepsilon))$

Let $\varepsilon \in (0,+\infty)$ be given. Chose $n_0 \in D$ such that $\forall n \in D : (n >_\sigma n_0 \Rightarrow f(n) \in I(L,\varepsilon))$

Choose $\delta = \|n_0\|_\sigma \in (0,+\infty)$. Let $x \in A$ be given and assume that $x \in N(\sigma,\delta)$. Then, we have:

$$\begin{cases} x \in A \\ x \in N(\sigma,\delta) \end{cases} \Rightarrow x \in N(\sigma,\delta) \cap A \Rightarrow \|x\|_\sigma \leqslant \delta = \|n_0\|_\sigma$$

$$\Rightarrow \|x\|_\sigma \leqslant \|n_0\|_\sigma \Rightarrow x >_\sigma n_0 \Rightarrow f(x) \in I(L,\varepsilon).$$

We have thus shown that:

$\forall \varepsilon \in (0,+\infty) : \exists \delta \in (0,+\infty) : \forall x \in A : (x \in N(\sigma,\delta) \Rightarrow f(x) \in I(L,\varepsilon))$

$\Rightarrow \lim_{x \to \sigma} f(x) = L.$

This concludes the proof $\square$

Properties of limits of functions

Since limits of functions are special cases of net limits, the following properties of function limits are immediately obtained:

① → Uniqueness

Let $f: A \to \mathbb{R}$ with σ limit point of A and let $L_1, L_2 \in \mathbb{R} \cup \{+\infty, -\infty\}$. Then, we have:

$$\left(\lim_{x\to\sigma} f(x) = L_1 \wedge \lim_{x\to\sigma} f(x) = L_2\right) \Rightarrow L_1 = L_2.$$

② → Functions with finite limits

Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and let σ be a limit point of both A and B and assume that:

$$\lim_{x\to\sigma} f(x) = l_1 \in \mathbb{R} \wedge \lim_{x\to\sigma} g(x) = l_2 \in \mathbb{R}$$

Then, we have:

a) $\lim_{x\to\sigma} [f(x) + g(x)] = l_1 + l_2$

b) $\lim_{x\to\sigma} [f(x)\, g(x)] = l_1 l_2$

c) $\forall a \in \mathbb{R}: \lim_{x\to\sigma} [a f(x)] = a l_1$

d) $l_2 \neq 0 \Rightarrow \lim_{x\to\sigma} \left(\frac{f(x)}{g(x)}\right) = \frac{l_1}{l_2}$

e) $\lim_{x\to\sigma} |f(x)| = |l_1|$

f) $l_1 > 0 \Rightarrow \forall k \in \mathbb{N}^*: \lim_{x\to\sigma} \sqrt[k]{f(x)} = \sqrt[k]{l_1}$

(3) → Functions with limits going to infinity.

Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and let σ be a limit point of A and B. Let $\delta \in (0, +\infty)$ and $a \in \mathbb{R}$. Then, we have:

a) $\begin{cases} \forall x \in N(\sigma, \delta) \cap B : g(x) > a \\ \lim\limits_{x \to \sigma} f(x) = +\infty \end{cases} \Rightarrow \lim\limits_{x \to \sigma} [f(x) + g(x)] = +\infty$

b) $\begin{cases} \forall x \in N(\sigma, \delta) \cap B : g(x) < a \\ \lim\limits_{x \to \sigma} f(x) = -\infty \end{cases} \Rightarrow \lim\limits_{x \to \sigma} [f(x) + g(x)] = -\infty$

c) $\begin{cases} \forall x \in N(\sigma, \delta) \cap B : g(x) > a > 0 \\ \lim\limits_{x \to \sigma} f(x) = \pm\infty, \end{cases} \Rightarrow \lim\limits_{x \to \sigma} [f(x) g(x)] = \pm\infty$

d) $\begin{cases} \forall x \in N(\sigma, \delta) \cap B : g(x) < a < 0 \\ \lim\limits_{x \to \sigma} f(x) = \pm\infty \end{cases} \Rightarrow \lim\limits_{x \to \sigma} [f(x) g(x)] = \mp\infty$

When the limit of $g(x)$ is also known, then this result can be combined with the following statements:

$$\lim_{x \to \sigma} g(x) > 0 \lor \lim_{x \to \sigma} g(x) = +\infty \Rightarrow \exists \delta \in (0, +\infty) : \exists a \in \mathbb{R} : \forall x \in N(\sigma, \delta) \cap B : g(x) > a > 0$$

$$\lim_{x \to \sigma} g(x) < 0 \lor \lim_{x \to \sigma} g(x) = -\infty \Rightarrow \exists \delta \in (0, +\infty) : \exists a \in \mathbb{R} : \forall x \in N(\sigma, \delta) \cap B : g(x) < a < 0$$

giving several deductions that are summarized in the tables given below:

$f(x) \downarrow \setminus g(x) \rightarrow$	a	$+\infty$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	?
$-\infty$	$-\infty$	?	$-\infty$

$$\lim_{x \to \sigma} [f(x)+g(x)]$$

$f(x) \downarrow g(x) \rightarrow$	0	$p>0$	$n<0$	$+\infty$	$-\infty$
$+\infty$	?	$+\infty$	$-\infty$	$+\infty$	$-\infty$
$-\infty$	?	$-\infty$	$+\infty$	$-\infty$	$+\infty$

$$\lim_{x \to \sigma} [f(x)g(x)]$$

↳ The "?" correspond to indeterminate forms. It means that the limit cannot be determined without more information, and the limit may or may not exist.

(9) → <u>Limit forms $k/0$, $1/(\pm\infty)$.</u>

Let $f: A \to \mathbb{R}$ and $\delta \in (0, +\infty)$ and let σ be a limit point of A.

a) $\begin{cases} \forall x \in N(\sigma, \delta) \cap A : f(x) > 0 \\ \lim_{x \to \sigma} f(x) = 0 \end{cases} \Rightarrow \lim_{x \to \sigma} \dfrac{1}{f(x)} = +\infty$

b) $\begin{cases} \forall x \in N(\sigma, \delta) \cap A : f(x) < 0 \\ \lim_{x \to \sigma} f(x) = 0 \end{cases} \Rightarrow \lim_{x \to \sigma} \dfrac{1}{f(x)} = -\infty$

c) $\lim_{x \to \sigma} f(x) \in \{+\infty, -\infty\} \Rightarrow \lim_{x \to \sigma} \dfrac{1}{f(x)} = 0$

Immediate consequences of limit properties

The following results are immediate consequences of the limit properties

(1) Monomial function

$$\forall x_0 \in \mathbb{R} : \forall k \in \mathbb{N}^* : \lim_{x \to x_0} x^k = x_0^k$$

$$\forall k \in \mathbb{N}^* : \lim_{x \to +\infty} x^k = +\infty \qquad \forall k \in \mathbb{N}^* : \lim_{x \to -\infty} x^{2k+1} = -\infty$$

$$\forall k \in \mathbb{N}^* : \lim_{x \to -\infty} x^{2k} = +\infty \qquad \forall k \in \mathbb{N}^* : \lim_{x \to \pm\infty} x^{-k} = 0$$

(2) Polynomial function

Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$.
Then, we have:

a) $\forall x_0 \in \mathbb{R} : \lim_{x \to x_0} f(x) = f(x_0)$

b) $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} a_n x^n$

3 → Rational function

Let $P: \mathbb{R} \to \mathbb{R}$ and $Q: \mathbb{R} \to \mathbb{R}$ with

$$\begin{cases} P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \\ Q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0 \end{cases}$$

Then, we have:

a) $\forall x_0 \in \mathbb{R}: \left(Q(x_0) \neq 0 \Rightarrow \lim\limits_{x \to x_0} \dfrac{P(x)}{Q(x)} = \dfrac{P(x_0)}{Q(x_0)} \right)$

b) $\lim\limits_{x \to \pm\infty} \dfrac{P(x)}{Q(x)} = \lim\limits_{x \to \pm\infty} \dfrac{a_n x^n}{b_m x^m}$

4 → Rational k/0 limits

$\forall a \in \mathbb{R}: \lim\limits_{x \to a^+} \dfrac{1}{x-a} = +\infty$

$\forall a \in \mathbb{R}: \lim\limits_{x \to a^-} \dfrac{1}{x-a} = -\infty$

$\forall a \in \mathbb{R}: \lim\limits_{x \to a} \dfrac{1}{x-a} = +\infty$

EXAMPLES

a) Let $f(x) = \dfrac{ax^3+x-2}{(a-1)x^2+x+1}$. Use the properties of limits to calculate $\lim\limits_{x\to+\infty} f(x)$

Solution

We distinguish between the following cases:

Case 1: Assume that $a \in \mathbb{R} - \{0,1\}$. Then, we have:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{ax^3+x-2}{(a-1)x^2+x+1} = \lim_{x\to+\infty} \frac{ax^3}{(a-1)x^2} =$$

$$= \frac{a}{a-1} \lim_{x\to+\infty} x = \frac{a}{a-1}(+\infty)$$

We use a sign table for the sign of $a/(a-1)$:

a		0		1	
a	$-$	0	$+$		$+$
$a-1$	$-$		$-$	0	$+$
	$+$	0	$-$	$\Vert$	$+$

to conclude that:

$$\lim_{x\to+\infty} f(x) = \begin{cases} +\infty, & \text{if } a \in (-\infty,0)\cup(1,+\infty) \\ -\infty, & \text{if } a \in (0,1) \end{cases}$$

Case 2: Assume that $a=0$. Then, we have:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x-2}{-x^2+x+1} = \lim_{x\to+\infty} \frac{x}{-x^2} =$$

$$= \lim_{x\to+\infty} \frac{-1}{x} = 0$$

Case 3: Assume that $a=1$. Then, we have:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^3+x-2}{x+1} = \lim_{x\to+\infty} \frac{x^3}{x} = \lim_{x\to+\infty} x^2$$

$$= +\infty .$$

We conclude that:

$$\lim_{x\to+\infty} f(x) = \begin{cases} +\infty, & \text{if } a\in(-\infty,0)\cup[1,+\infty) \\ -\infty, & \text{if } a\in(0,1) \\ 0, & \text{if } a=0 \end{cases}$$

6) Use the limit properties to calculate $\lim_{x\to-\infty} f(x)$ for $f(x) = \sqrt{x^2-5x+6} + ax$, for all $a \in \mathbb{R}$.

Solution

Since

$$\lim_{x\to-\infty} (x^2-5x+6) = \lim_{x\to-\infty} x^2 = +\infty \Rightarrow$$

$$\Rightarrow \exists \mu \in (0,+\infty) : \forall x \in (-\infty,-\mu) : x^2-5x+6 > 0$$

it follows that $-\infty$ is a limit point of the domain of f for all $a \in \mathbb{R}$. Choose $\mu \in (0,+\infty)$ such that $\forall x \in (-\infty,-\mu) : x^2-5x+6 > 0$.

Then, we have:

$$\forall x \in (-\infty,-\mu) : f(x) = \sqrt{x^2-5x+6} + ax =$$
$$= \sqrt{x^2(1-5x^{-1}+6x^{-2})} + ax =$$
$$= |x|\sqrt{1-5x^{-1}+6x^{-2}} + ax =$$
$$= -x\sqrt{1-5x^{-1}+6x^{-2}} + ax = \quad [\text{via } x<-\mu<0]$$
$$= x(a-\sqrt{1-5x^{-1}+6x^{-2}}) \equiv xg(x)$$

where we define $g(x) = a-\sqrt{1-5x^{-1}+6x^{-2}}$

and

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} [a-\sqrt{1-5x^{-1}+6x^{-2}}] = a-\sqrt{1-0+0}$$
$$= a-1$$

We distinguish between the following cases:

Case 1: Assume that $a \in (1,+\infty)$. Then, we have:

$$\lim_{x\to-\infty} x = -\infty \wedge \lim_{x\to-\infty} g(x) = a-1 > 0 \Rightarrow \lim_{x\to-\infty} f(x) = -\infty$$

Case 2: Assume that $a \in (-\infty, 1)$. Then, we have:

$$\lim_{x \to -\infty} x = -\infty \wedge \lim_{x \to -\infty} g(x) = a-1 < 0 \Rightarrow \lim_{x \to +\infty} f(x) = +\infty$$

Case 3: Assume that $a = 1$. Then, we have:

$$f(x) = \sqrt{x^2-5x+6} + x = \frac{(\sqrt{x^2-5x+6})^2 - x^2}{\sqrt{x^2-5x+6} - x} =$$

$$= \frac{x^2-5x+6-x^2}{|x|\sqrt{1-5x^{-1}+6x^{-2}} - x} = \frac{-5x+6}{-x\sqrt{1-5x^{-1}+6x^{-2}} - x}$$

$$= \frac{-5+6x^{-1}}{-\sqrt{1-5x^{-1}+6x^{-2}} - 1}, \quad \forall x \in (-\infty, -p)$$

$$\rightarrow \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{-5+6x^{-1}}{-\sqrt{1-5x^{-1}+6x^{-2}} - 1}$$

$$= \frac{-5+0}{-\sqrt{1-0+0} - 1} = \frac{-5}{-1-1} = \frac{5}{2}$$

From all of the above, we conclude that

$$\lim_{x \to -\infty} f(x) = \begin{cases} -\infty, & \text{if } a \in (1, +\infty) \\ +\infty, & \text{if } a \in (-\infty, 1) \\ 5/2, & \text{if } a = 1 \end{cases}$$

THEORY QUESTION

(11) Let $f: A \to \mathbb{R}$ and let σ be a limit point of A and let $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. State the construction of the limit statement $\lim_{x \to \sigma} f(x) = L$ in terms of a directed set $(D, <)$ and the corresponding directed set limits.

EXERCISES

(12) Let $f: \mathbb{R} \to \mathbb{R}$ such that

$\forall x \in \mathbb{R}: f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

Show that: $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} a_n x^n$

(13) Let $p: \mathbb{R} \to \mathbb{R}$ and $q: \mathbb{R} \to \mathbb{R}$ such that

$$\forall x \in \mathbb{R}: \begin{cases} p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \\ q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0 \end{cases}$$

Show that: $\lim_{x \to \pm\infty} \frac{p(x)}{q(x)} = \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m}$

(14) Use the limit properties to evaluate the following limits for all $a \in \mathbb{R}$.

a) $\lim_{x \to +\infty} f(x)$ with $f(x) = \frac{(a^2-1)x^2 + ax}{(a-1)x^2 + (a+1)x + 3}$

b) $\lim_{x \to -\infty} f(x)$ with $f(x) = \frac{(a-1)x^3 + (a+1)x}{(a+1)x^3 + (a-1)x}$

c) $\lim_{x\to+\infty} f(x)$ with $f(x) = \dfrac{ax^3+(a+1)x^2-ax+3}{(a+1)x^2+2ax+1}$

d) $\lim_{x\to-\infty} f(x)$ with $f(x) = \dfrac{x^2}{x+a} - \dfrac{x^2}{x-a}$

(15) Consider the function $f: A\to\mathbb{R}$ with $f(x)=\sqrt{a-x^2-2x}-x$ with $a\in\mathbb{R}$. Show that:
$\forall a\in\mathbb{R}: \lim_{x\to+\infty} f(x)$ not well-defined.

→ We say that for $f: A\to\mathbb{R}$, the limit $\lim_{x\to\sigma} f(x)$ is not well-defined $\Leftrightarrow$ σ not a limit point of A.

(16) Find the set $S\subseteq\mathbb{R}$ of all $a\in\mathbb{R}$ for which the following limits are well-defined, and then evaluate the limit in terms of the parameter $a\in S$.

a) $\lim_{x\to+\infty} f(x)$ with $f(x) = x\left(\sqrt{ax^2+6x+3}-x\right)$

b) $\lim_{x\to-\infty} f(x)$ with $f(x) = \sqrt{ax^2+2x-1}-\sqrt{x^2+1}$

c) $\lim_{x\to+\infty} f(x)$ with $f(x) = \sqrt{x^2-4x+a}-x$

d) $\lim_{x\to-\infty} f(x)$ with $f(x) = |x|\left[\sqrt{ax^2+2x+1}-x\right]$

(17) Use the limit properties to show that

a) $\lim_{x\to+\infty}\left[\sqrt{4x^2-3x+1}-ax+b\right] = 1/4 \Leftrightarrow (a,b)=(2,1)$

b) $\lim_{x\to-\infty}\left[\sqrt[3]{x^3+1}-ax-b\right]=0 \Leftrightarrow (a,b)=(1,0)$

Limit composition theorem

Def: Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$. We define the composition $f \circ g: G \to \mathbb{R}$ such that

$$\begin{cases} G = \{x \in B \mid g(x) \in A\} \\ \forall x \in G: (f \circ g)(x) = f(g(x)) \end{cases}$$

Note that the belonging condition for the domain G of $f \circ g$ is given by

$$\boxed{x \in G \Leftrightarrow \begin{cases} x \in B \\ g(x) \in A \end{cases}}$$

The following theorems make it possible to calculate the limit of the composition $f \circ g$:

Thm: Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and $f \circ g: G \to \mathbb{R}$. Let σ be a limit point of G. Then, we have:

$$\begin{cases} \lim_{x \to \sigma} g(x) = a \in \mathbb{R} \\ \lim_{x \to a} f(x) = f(a) \end{cases} \Rightarrow \lim_{x \to \sigma} f(g(x)) = f(a)$$

Proof

Let $\varepsilon \in (0, +\infty)$ be given. Since

$$\lim_{x \to a} f(x) = f(a) \Rightarrow \exists \delta \in (0, +\infty): \forall x \in A: (0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

We choose $\delta_1 \in (0, +\infty)$ such that
$\forall x \in A : (0 < |x-a| < \delta_1 \Rightarrow |f(x) - f(a)| < \varepsilon)$
Since:
$\lim_{x \to \sigma} g(x) = a \Rightarrow \exists \delta \in (0, +\infty) : \forall x \in B : (x \in N(\sigma, \delta) \Rightarrow |g(x) - a| < \delta_1)$

choose $\underline{\delta \in (0, +\infty)}$ such that
$\forall x \in B : (x \in N(\sigma, \delta) \Rightarrow |g(x) - a| < \delta_1)$
Let $\underline{x \in G}$ be given and assume that $\underline{x \in N(\sigma, \delta)}$. We will show that $|f(g(x)) - f(a)| < \varepsilon$. We have:

$$\begin{cases} x \in G \\ x \in N(\sigma, \delta) \end{cases} \Rightarrow \begin{cases} x \in B \\ x \in N(\sigma, \delta) \end{cases} \quad [\text{via } G \subseteq B]$$
$$\Rightarrow |g(x) - a| < \delta_1$$

We need the stronger condition $0 < |g(x) - a| < \delta_1$, so we distinguish between the following cases.
Case 1: Assume that $g(x) = a$. Then, we have
$|f(g(x)) - f(a)| = |f(a) - f(a)| = 0 < \varepsilon \Rightarrow \underline{|f(g(x)) - f(a)| < \varepsilon}$.
Case 2: Assume that $g(x) \neq a$. Then, we have:
$$\begin{cases} 0 < |g(x) - a| < \delta_1 \\ g(x) \in A \end{cases} \Rightarrow \underline{|f(g(x)) - f(a)| < \varepsilon}$$
We have thus shown that
$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in G : (x \in N(\sigma, \delta) \Rightarrow |f(g(x)) - f(a)| < \varepsilon)$
$\Rightarrow \lim_{x \to \sigma} f(g(x)) = f(a)$. $\square$

If we replace the condition $\lim_{x \to a} f(x) = f(a)$ (compare with definition of continuity, given later) with the more general statement $\lim_{x \to a} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\}$, then we need an additional assumption to ensure the composition theorem still works:

Prop: Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and $f \circ g: C \to \mathbb{R}$, and let σ be a limit point of C. Then, we have:

$$\begin{cases} \lim_{x \to \sigma} g(x) = a \in \mathbb{R} \\ \lim_{x \to a} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\} \\ \exists \delta \in (0, +\infty) : \forall x \in B \cap N(\sigma, \delta) : g(x) \neq a \end{cases} \Rightarrow \lim_{x \to \sigma} f(g(x)) = L$$

Proof

Let $\varepsilon \in (0, +\infty)$ be given. By hypothesis, choose $\delta_0 \in (0, +\infty)$ such that

$\forall x \in B \cap N(\sigma, \delta_0) : g(x) \neq a$.

Since:

$$\lim_{x \to a} f(x) = L \Rightarrow \exists \delta \in (0, +\infty) : \forall x \in A : (0 < |x - a| < \delta \Rightarrow f(x) \in I(L, \varepsilon))$$

choose $\delta_1 \in (0, +\infty)$ such that

$\forall x \in A : (0 < |x - a| < \delta_1 \Rightarrow f(x) \in I(L, \varepsilon))$

Since:

$$\lim_{x \to \sigma} g(x) = a \Rightarrow \exists \delta \in (0, +\infty) : \forall x \in B : (x \in N(\sigma, \delta) \Rightarrow |g(x) - a| < \delta_1)$$

choose $\delta_2 \in (0, +\infty)$ such that

$\forall x \in B : (x \in N(\sigma, \delta_2) \Rightarrow |g(x) - a| < \delta_1)$

Choose $\underline{\delta \in (0,+\infty)}$ such that $\delta = \min\{\delta_0, \delta_2\}$. Let $\underline{x \in G}$ be given and assume that $\underline{x \in N(\sigma, \delta)}$. Then we have

$$x \in N(\sigma,\delta) \cap G \Rightarrow \begin{cases} x \in N(\sigma, \delta_0) \cap B \\ x \in N(\sigma, \delta_2) \cap G \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} g(x) \neq a \\ |g(x) - a| < \delta_1 \\ x \in G \end{cases} \Rightarrow \begin{cases} 0 < |g(x) - a| < \delta_1 \\ g(x) \in A \end{cases} \Rightarrow$$

$$\Rightarrow \underline{f(g(x)) \in I(L, \varepsilon)}$$

We have thus shown that

$$\forall \varepsilon \in (0,+\infty) : \exists \delta \in (0,+\infty) : \forall x \in G : (x \in N(\sigma,\delta) \Rightarrow f(g(x)) \in I(L,\varepsilon))$$

$$\Rightarrow \lim_{x \to \sigma} f(g(x)) = L. \qquad \square$$

For the case $\lim_{x \to \sigma} g(x) \in \{+\infty, -\infty\}$, the previous proof can be modified to show the following statement:

<u>Prop</u>: Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ and $f \circ g : G \to \mathbb{R}$, and let σ be a limit point of G. Then, we have:

a) $$\begin{cases} \lim_{x \to \sigma} g(x) = +\infty \\ \lim_{x \to +\infty} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\} \end{cases} \Rightarrow \lim_{x \to \sigma} f(g(x)) = L$$

b) $$\begin{cases} \lim_{x \to \sigma} g(x) = -\infty \\ \lim_{x \to -\infty} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\} \end{cases} \Rightarrow \lim_{x \to \sigma} f(g(x)) = L$$

Proof : $\boxed{\text{Homework.}}$

→ The following statements are immediate consequences of the limit composition theorems:

1) $\lim\limits_{x \to +\infty} f(x) = L \Rightarrow \lim\limits_{n \in \mathbb{N}^*} f(n) = L$
2) $\lim\limits_{x \to x_0} f(x) = L \Leftrightarrow \lim\limits_{h \to 0} f(x_0 + h) = L$
3) $\lim\limits_{x \to x_0} f(x) = L \Leftrightarrow \lim\limits_{h \to 1} f(x_0 h) = L$

→ When the function g is a sequence with $g: \mathbb{N}^* \to \mathbb{R}$ and we choose $\sigma = +\infty$, then the composition theorems give the following statements:

1) $\begin{cases} \lim\limits_{n \in \mathbb{N}^*} a_n = x_0 \in \mathbb{R} \\ \lim\limits_{x \to x_0} f(x) = f(x_0) \end{cases} \Rightarrow \lim\limits_{n \in \mathbb{N}^*} f(a_n) = f(x_0)$
2) $\begin{cases} \lim\limits_{n \in \mathbb{N}^*} a_n = x_0 \in \mathbb{R} \\ \lim\limits_{x \to x_0} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\} \\ \forall n \in \mathbb{N}^* : a_n \neq x_0 \end{cases} \Rightarrow \lim\limits_{n \in \mathbb{N}^*} f(a_n) = L$

3) $\begin{cases} \lim\limits_{n\in\mathbb{N}^*} a_n = +\infty \\ \lim\limits_{x\to+\infty} f(x) = L \in \mathbb{R}\cup\{+\infty, -\infty\} \end{cases}$	$\Rightarrow \lim\limits_{n\in\mathbb{N}^*} f(a_n) = L$
4) $\begin{cases} \lim\limits_{n\in\mathbb{N}^*} a_n = -\infty \\ \lim\limits_{x\to-\infty} f(x) = L \in \mathbb{R}\cup\{+\infty, -\infty\} \end{cases}$	$\Rightarrow \lim\limits_{n\in\mathbb{N}^*} f(a_n) = L$

THEORY QUESTIONS

(18) Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$. State the domain and definition of the function composition $f \circ g$.

(19) Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and $f \circ g: C \to \mathbb{R}$ and let σ be a limit point of C. Prove that

a) $\lim\limits_{x \to \sigma} g(x) = a \in \mathbb{R} \wedge \lim\limits_{x \to a} f(x) = f(a) \Rightarrow \lim\limits_{x \to \sigma} f(g(x)) = f(a)$

b) $\begin{cases} \lim\limits_{x \to \sigma} g(x) = a \in \mathbb{R} \\ \lim\limits_{x \to a} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\} \\ \exists \delta \in (0, +\infty) : \forall x \in B \cap N(\sigma, \delta) : g(x) \neq a \end{cases} \Rightarrow \lim\limits_{x \to \sigma} f(g(x)) = L$

EXERCISES

(20) Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and $f \circ g: C \to \mathbb{R}$, let σ be a limit point of C and let $L \in \mathbb{R} \cup \{+\infty, -\infty\}$.
Prove that:

a) $\lim\limits_{x \to \sigma} g(x) = +\infty \wedge \lim\limits_{x \to +\infty} f(x) = L \Rightarrow \lim\limits_{x \to \sigma} f(g(x)) = L$

b) $\lim\limits_{x \to \sigma} g(x) = -\infty \wedge \lim\limits_{x \to -\infty} f(x) = L \Rightarrow \lim\limits_{x \to \sigma} f(g(x)) = L$

(21) Let $f: A \to \mathbb{R}$ with $A = (0, +\infty)$. Use the composition theorem to show that

a) $\lim_{x \to +\infty} f(x) = L \iff \lim_{x \to a^+} f\left(\frac{1}{x-a}\right) = L$

b) $\lim_{x \to -\infty} f(x) = L \iff \lim_{x \to a^-} f\left(\frac{1}{x-a}\right) = L$

↳ Note that you have to show both the "$\Rightarrow$" and "$\Leftarrow$" statements using separate arguments.

Trigonometric limits

Limits of trigonometric functions are established via the inequality:

$$\boxed{\forall x \in (-\pi/2, 0) \cup (0, \pi/2): |\sin x| < |x| < |\tan x|}$$

via the limit definition, as follows:

Basic trigonometric limits

(1) $$\boxed{\lim_{x \to x_0} \sin x = \sin x_0 \quad , \forall x_0 \in \mathbb{R}}$$

Proof

Let $x_0 \in \mathbb{R}$ be given. Let $\varepsilon \in (0, +\infty)$ be given. Choose $\delta = \min\{\varepsilon, \pi/2\}$. Let $x \in \mathbb{R}$ be given and assume that $x \in N(x_0, \delta)$. Then, we have:

$$|\sin x - \sin x_0| = \left| 2\sin\left(\frac{x - x_0}{2}\right)\cos\left(\frac{x + x_0}{2}\right)\right| =$$

$$= 2\left|\sin\left(\frac{x - x_0}{2}\right)\right| \cdot \left|\cos\left(\frac{x + x_0}{2}\right)\right|$$

$$\leq 2\left|\sin\left(\frac{x - x_0}{2}\right)\right| \leq 2\left|\frac{x - x_0}{2}\right| =$$

$$= 2 \cdot \frac{|x - x_0|}{2} = |x - x_0| \Rightarrow |\sin x - \sin x_0| \leq |x - x_0|.$$

and it follows that

$$x \in N(x_0,\delta) \Rightarrow 0 < |x-x_0| < \delta \leq \varepsilon \Rightarrow |x-x_0| < \varepsilon \Rightarrow$$
$$\Rightarrow \underline{|\sin x - \sin x_0| < \varepsilon}$$

We have thus shown that:

$\forall \varepsilon \in (0,+\infty) : \exists \delta \in (0,+\infty) : \forall x \in \mathbb{R} : (x \in N(x_0,\delta) \Rightarrow |\sin x - \sin x_0| < \varepsilon)$

$\Rightarrow \lim_{x \to x_0} \sin x = \sin x_0$ $\quad\quad\quad\quad$ O

(2) $\boxed{\forall x_0 \in \mathbb{R} : \lim_{x \to x_0} \cos x = \cos x_0}$

Proof

Let $\underline{x_0 \in \mathbb{R}}$ be given. Let $\underline{\varepsilon \in (0,+\infty)}$ be given. Choose $\underline{\delta = \min\{\varepsilon, \pi/2\}}$. Let $\underline{x \in \mathbb{R}}$ be given and assume that $\underline{x \in N(x_0,\delta)}$. Then, we have:

$$|\cos x - \cos x_0| = \left| 2 \sin\left(\frac{x+x_0}{2}\right) \sin\left(\frac{x_0-x}{2}\right)\right| =$$
$$= 2 \left|\sin\left(\frac{x+x_0}{2}\right)\right| \cdot \left|\sin\left(\frac{x_0-x}{2}\right)\right|$$
$$\leq 2 \left|\sin\left(\frac{x_0-x}{2}\right)\right| \leq 2 \left|\frac{x_0-x}{2}\right| =$$
$$= 2 \frac{|x-x_0|}{2} = |x-x_0|$$

and it follows that

$$x \in N(x_0,\delta) \Rightarrow 0 < |x-x_0| < \delta \leq \varepsilon \Rightarrow |x-x_0| < \varepsilon \Rightarrow$$
$$\Rightarrow \underline{|\cos x - \cos x_0| < \varepsilon}$$

We have thus shown that
$\forall \varepsilon \in (0,+\infty) : \exists \delta \in (0,+\infty) : \forall x \in \mathbb{R} : (x \in N(\sigma,\delta) \Rightarrow |\cos x - \cos x_0| < \varepsilon)$
$\Rightarrow \lim_{x \to x_0} \cos x = \cos x_0$ $\square$

- Using the limit properties, from the previous two result we immediately obtain:

3	$\forall x_0 \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\} : \lim_{x \to x_0} \tan x = \tan x_0$
4	$\forall x_0 \in \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\} : \lim_{x \to x_0} \cot x = \cot x_0$

Upgrades via limit composition theorem

Using the composition theorem, these results can be upgraded to obtain:

1	$\lim_{x \to \sigma} g(x) = a \in \mathbb{R} \Rightarrow \lim_{x \to \sigma} \sin(g(x)) = \sin a$
2	$\lim_{x \to \sigma} g(x) = a \in \mathbb{R} \Rightarrow \lim_{x \to \sigma} \cos(g(x)) = \cos a$
3	$\lim_{x \to \sigma} g(x) = a \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\} \Rightarrow \lim_{x \to \sigma} \tan(g(x)) = \tan a$
4	$\lim_{x \to \sigma} g(x) = a \in \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\} \Rightarrow \lim_{x \to \sigma} \cot(g(x)) = \cot a$

0/0 trigonometric limits

The squeeze theorem for function limits follows from the squeeze theorem for convergent nets, and it reads:

Thm: Let $f: A \to \mathbb{R}$ and $g_1: A \to \mathbb{R}$ and $g_2: A \to \mathbb{R}$ and let σ be a limit point of $\bar{A}$. Then, we have:

$$\begin{cases} \forall x \in A \cap N(\sigma, \delta): g_1(x) \leq f(x) \leq g_2(x) \\ \lim_{x \to \sigma} g_1(x) = \lim_{x \to \sigma} g_2(x) = \ell \in \mathbb{R} \end{cases} \Rightarrow \lim_{x \to \sigma} f(x) = \ell.$$

We now use the squeeze theorem to calculate the following limits:

① $$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

Proof

Define $\forall x \in \mathbb{R}^*: f(x) = (\sin x)/x$. Let $x \in (-\pi/2, 0) \cup (0, \pi/2)$ be given. Then, we have:

$\sin x$, x equisigned $\Rightarrow x \sin x > 0 \Rightarrow \dfrac{\sin x}{x} > 0$

and therefore:

$$f(x) = \frac{\sin x}{x} = \left| \frac{\sin x}{x} \right| = \frac{|\sin x|}{|x|} \leq \frac{|x|}{|x|} = 1$$

and

$$f(x) = \frac{\sin x}{x} = \left| \frac{\sin x}{x} \right| = \frac{|\sin x|}{|x|} \geq \frac{|\sin x|}{|\tan x|} = \left| \frac{\sin x}{\tan x} \right|$$

$$= \left| \frac{\sin x}{(\sin x)/(\cos x)} \right| = \left| \frac{1}{1/(\cos x)} \right| = |\cos x|$$

We have thus shown that

$$\forall x \in (-\pi/2, 0) \cup (0, \pi/2) : |\cos x| \leq f(x) \leq 1 \qquad (1)$$

It follows that

$$\lim_{x\to 0} \cos x = \cos 0 = 1 \Rightarrow \lim_{x\to 0} |\cos x| = |1| = 1 \Rightarrow$$

$$\Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1 \quad [\text{via Eq. (1)}] \quad \square$$

An immediate corollary is:

② $\boxed{\lim_{x\to 0} \frac{\tan x}{x} = 1}$

Using the composition theorem, these results can be upgraded to give:

$$\boxed{\begin{cases} \forall x \in N(\sigma, \delta) \cap \mathrm{dom}(g) : g(x) \neq 0 \\ \lim_{x\to\sigma} g(x) = 0 \end{cases} \Rightarrow \lim_{x\to\sigma} \frac{\sin(g(x))}{g(x)} = \lim_{x\to\sigma} \frac{\tan(g(x))}{g(x)} = 1}$$

An immediate consequence of these generalizations is that:

$\forall a \in \mathbb{R}^*$:	$\lim_{x\to 0} \frac{\sin(ax)}{ax} = \lim_{x\to 0} \frac{\tan(ax)}{ax} = 1$
$\forall a \in \mathbb{R}$:	$\lim_{x\to a} \frac{\sin(x-a)}{x-a} = \lim_{x\to a} \frac{\tan(x-a)}{x-a} = 1$

THEORY QUESTIONS

(22) Prove the following statements

a) $\forall x_0 \in \mathbb{R}: \lim\limits_{x \to x_0} \sin x = \sin x_0$

b) $\forall x_0 \in \mathbb{R}: \lim\limits_{x \to x_0} \cos x = \cos x_0$

c) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

EXERCISES

(23) Use the limit properties to evaluate the following limits (WITHOUT use of the De L'Hospital theorem).

a) $\lim\limits_{x \to +\infty} \dfrac{\sin x}{x} = 0$

b) $\lim\limits_{x \to 0} \dfrac{\cos x - \cos(5x)}{x \sin x} = 12$

c) $\lim\limits_{x \to 0} \dfrac{\sqrt{x+4} - 2}{\sin(5x)} = \dfrac{1}{20}$

d) $\lim\limits_{x \to 0^+} \dfrac{2x - \sin x}{\sqrt{1 - \cos x}} = \sqrt{2}$

e) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x \sin x} = \dfrac{1}{2}$

f) $\lim\limits_{x \to \pi/2} \dfrac{1 - \sin x}{\sqrt{1 + \cos(2x)}} = 0$

g) $\lim\limits_{x \to 0} \dfrac{\sqrt{2} - \sqrt{1 + \cos(2x)}}{\sin^2 x} = \dfrac{\sqrt{2}}{2}$

h) $\lim\limits_{x \to 0} \dfrac{1 - \cos^3 x}{x \sin(2x)} = \dfrac{3}{4}$

i) $\lim\limits_{x \to 0} \left[\dfrac{2}{\sin^2 x} - \dfrac{1}{1 - \cos x} \right] = \dfrac{1}{2}$

j) $\lim\limits_{x \to 0} \dfrac{\sqrt{\cos x} - 1}{x^2} = \dfrac{-1}{4}$

k) $\forall n \in \mathbb{N}^*: \lim\limits_{x \to 0} \dfrac{1 - \cos^n x}{x^2} = \dfrac{n}{2}$

Trigonometric limits with $x \to \pm\infty$

These limits usually <u>do not exist</u> and we can show that using proof by contradiction as follows:

- 1 To show a contradiction, assume that $\lim_{x \to +\infty} f(x) = L$.
- 2 Define sequences $(a_n), (b_n)$ such that $\lim_{n \in \mathbb{N}^*} f(a_n) = L_1 \wedge \lim_{n \in \mathbb{N}^*} f(b_n) = L_2 \wedge L_1 \neq L_2$
- 3 Use the composition theorem to show that $L_1 = L_2 = L$ and thus derive a contradiction.

EXAMPLE

Show that $\lim_{x \to +\infty} \sin x$ does not exist.

<u>Solution</u>

To show a contradiction, assume that $\lim_{x \to +\infty} \sin x = L$ with $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Define $(a_n), (b_n)$ such that

$$\forall n \in \mathbb{N}^* : (a_n = 2n\pi \wedge b_n = 2n\pi + \pi/4)$$

Since

$$\lim_{n \in \mathbb{N}^*} a_n = +\infty \wedge \lim_{n \in \mathbb{N}^*} b_n = +\infty \Rightarrow \lim_{n \in \mathbb{N}^*} f(a_n) = \lim_{n \in \mathbb{N}^*} f(b_n) = L$$

We also have:

$$l_1 = \lim_{n \in \mathbb{N}^*} f(a_n) = \lim_{n \in \mathbb{N}^*} f(2n\pi) = \lim_{n \in \mathbb{N}^*} \sin(2n\pi) =$$

$$= \lim_{n \in \mathbb{N}^*} \sin 0 = \lim_{n \in \mathbb{N}^*} 0 = 0$$

$$l_2 = \lim_{n\in\mathbb{N}^*} f(b_n) = \lim_{n\in\mathbb{N}^*} \sin(2n\pi + \pi/4) = \lim_{n\in\mathbb{N}^*} \sin(\pi/4)$$

$$= \sin(\pi/4) = \sqrt{2}/2$$

It follows that $\lim_{n\in\mathbb{N}^*} a_n \neq \lim_{n\in\mathbb{N}^*} b_n$, which is a contradiction

We conclude that $\lim_{x\to+\infty} \sin x$ does not exist. $\square$

EXERCISES

(24) Show that the following limits do not exist

a) $\lim_{x \to 0} \sin(1/x)$

b) $\lim_{x \to 1} \cos\left(\frac{1}{x-1}\right)$

c) $\lim_{x \to \pi/2} \frac{x - \pi/2}{\sqrt{1-\sin x}}$

d) $\lim_{x \to 0} \left[\frac{1}{x} \sin\left(\frac{1}{x}\right)\right]$

e) $\lim_{x \to +\infty} [2\cos(3x) - 1]$

f) $\lim_{x \to +\infty} \sqrt{3 + \cos(x/2)}$

g) $\lim_{x \to +\infty} \tan x$

h) $\lim_{x \to \pi} [2x \tan(x/2) + 3]$.