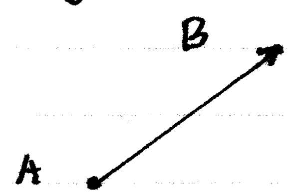
VECTORS

Definitions

- A vector is a line segment with an established direction. If A, B are two points, then $\vec{AB}$ represents the vector defined by the line segment AB with direction from A to B.

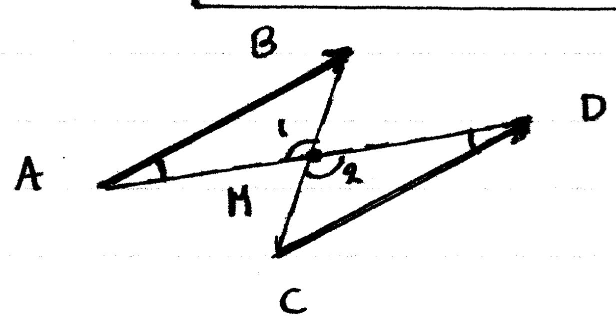


A = initial point
B = terminal point

Vector Equality

Def : Let $\vec{AB}$, $\vec{CD}$ be two vectors. Let $M = AD \cap BC$. We then define vector equality as follows:

$$\boxed{\vec{AB} = \vec{CD} \iff \begin{cases} AM = MD \\ BM = MC \end{cases}}$$



- interpretation: If $\vec{AB} = \vec{CD}$ then $AB = CD$ and $AB \parallel CD$ and $\vec{AB}$ and $\vec{CD}$ have "the same direction".

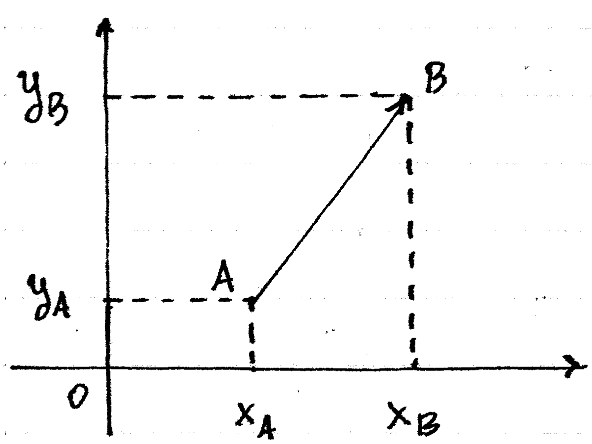
Prop : $\boxed{\vec{AB} = \vec{CD} \Rightarrow AB = CD \wedge AB \parallel CD}$

Proof

Let $\hat{M}_1 = A\hat{M}B$ and $\hat{M}_2 = C\hat{M}D$.
Let $\hat{C} = B\hat{C}D$ and $\hat{D} = A\hat{D}C$.
By definition : $\vec{AB} = \vec{CD} \Rightarrow AM = MD \wedge BM = MC$ (1)
We also note that : $\hat{M}_1 = \hat{M}_2$ (vertical angles) (2)
From (1) and (2) : $A\overset{\Delta}{M}B = C\overset{\Delta}{M}D$ (3)
From (3) : $\underline{AB = CD}$.
From (3) : $\hat{A} = \hat{D} \Rightarrow \underline{AB \parallel CD}$ (equal interior alternating angles) $\square$

- **Vector representation**

Consider a cartesian coordinate system with axis $x'Ox$ and $y'Oy$. Let $\vec{AB}$ be a vector with $A(x_A, y_A)$ and $B(x_B, y_B)$.



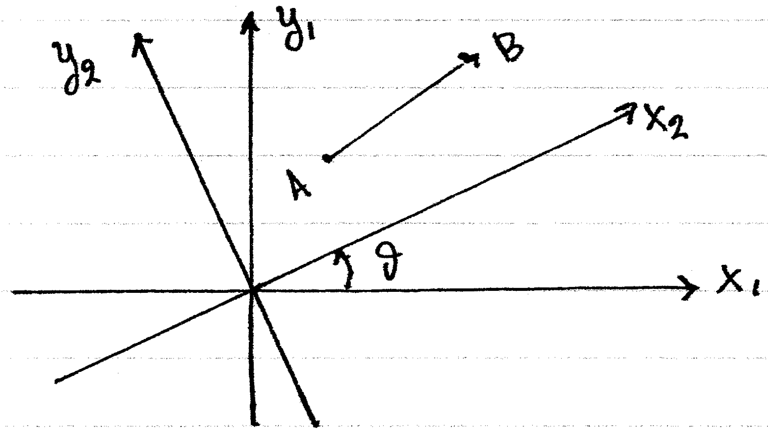
We represent :
$\vec{AB} = (x_B - x_A, y_B - y_A)$
Note that the same vector may have different representations in different coordinate systems.

- **Zero vector**

We define the zero vector as $\vec{0} = (0,0)$ for any coordinate system. Note that for any point A:

$$\vec{AA} = (x_A - x_A, y_A - y_A) = (0,0) = \vec{0}.$$

- **Rotation of coordinate system**



Consider a coordinate system consisting of an x_1-axis and y_1-axis. We define a new coordinate system, by rotating counterclockwise by angle ϑ, consisting of an x_2-axis and y_2-axis.
Let $\vec{AB}$ be a vector. If

$\vec{AB} = (x_1, y_1)$ in the x_1y_1 coordinate system

$\vec{AB} = (x_2, y_2)$ in the x_2y_2 coordinate system

then

$$\boxed{\begin{aligned} x_2 &= x_1 \cos\vartheta + y_1 \sin\vartheta \\ y_2 &= -x_1 \sin\vartheta + y_1 \cos\vartheta \end{aligned}}$$

We write equivalently: $\boxed{(x_2, y_2) = R(\vartheta)(x_1, y_1)}$

It can be shown that

$$\boxed{R(\vartheta_1)R(\vartheta_2)(x_1, y_1) = R(\vartheta_1+\vartheta_2)(x_1, y_1)}$$

- **<u>Magnitude of vector</u>**

- Let $\vec{\alpha} = (a_1, a_2)$ be a vector. We define:

$$\boxed{|\vec{a}| = \sqrt{a_1^2 + a_2^2}}$$

- $|\vec{a}|$ represents the <u>length</u> of the vector $\vec{\alpha}$.
  It follows that for two points A, B:
  $$|\vec{AB}| = |\vec{BA}| = AB.$$
- We note that $|\vec{a}|$ is invariant under rotation:

<u>Thm</u>: $\boxed{|R(\vartheta)\vec{\alpha}| = |\vec{a}|}$

<u>Proof</u>

Let $\vec{a} = (a_1, a_2)$ and $R(\vartheta)\vec{a} = (b_1, b_2)$. It follows that

$b_1 = a_1\cos\vartheta + a_2\sin\vartheta$
$b_2 = -a_1\sin\vartheta + a_2\cos\vartheta$

and therefore:

$$b_1^2+b_2^2=(a_1\cos\vartheta+a_2\sin\vartheta)^2+(-a_1\sin\vartheta+a_2\cos\vartheta)^2=$$
$$=\underline{a_1^2\cos^2\vartheta}+2a_1a_2\cos\vartheta\sin\vartheta+\underline{\underline{a_2^2\sin^2\vartheta}}+\underline{a_1^2\sin^2\vartheta}$$
$$-2a_1a_2\cos\vartheta\sin\vartheta+\underline{\underline{a_2^2\cos^2\vartheta}}=$$
$$=a_1^2(\cos^2\vartheta+\sin^2\vartheta)+a_2^2(\cos^2\vartheta+\sin^2\vartheta)=$$
$$=a_1^2+a_2^2\Rightarrow$$

$$\Rightarrow |R(\vartheta)\vec{a}|=\sqrt{b_1^2+b_2^2}=\sqrt{a_1^2+a_2^2}=|\vec{a}|. \quad \square$$

EXAMPLE

a) For $\vec{a}=(\sqrt{2}-1,\sqrt{2}+1)$, evaluate $|\vec{a}|$.

Solution

$$|\vec{a}|=\sqrt{(\sqrt{2}-1)^2+(\sqrt{2}+1)^2}=$$
$$=\sqrt{2-2\sqrt{2}+1+2+2\sqrt{2}+1}=\sqrt{6} \quad \square$$

b) Rotate the vector $\vec{a}=(2,1)$ by -30°

Solution

Rotate the axis in the opposite direction: $+30^\circ$!!

Let $(x,y)=R(30^\circ)\vec{a}=R(30^\circ)(2,1)$. Then

$$x=2\cos 30^\circ+1\sin 30^\circ=2(\sqrt{3}/2)+1\cdot(1/2)=$$
$$=\sqrt{3}+1/2=\frac{2\sqrt{3}+1}{2}$$

$$y=-2\sin 30^\circ+1\cos 30^\circ=-2\cdot(1/2)+1\cdot(\sqrt{3}/2)=$$
$$=-1+\sqrt{3}/2=\frac{\sqrt{3}-2}{2}$$

. Thus $R(30^\circ)\vec{a}=\left(\frac{2\sqrt{3}+1}{2},\frac{\sqrt{3}-2}{2}\right)$.

EXERCISES

① Let A, B, C be three points with $A(2,1)$, $B(3,3)$, $C(1,5)$.

a) Evaluate $|\vec{AB}|$.

b) Rotate $\vec{BC}$ by 45°.

c) Rotate $\vec{AC}$ by 15°.

② Evaluate $|\vec{a}|$ with

a) $\vec{a} = \left(\sqrt{2+\sqrt{2}}, \sqrt{2-\sqrt{2}}\right)$

b) $\vec{a} = (3+\sqrt{2}, 3-\sqrt{2})$

c) $\vec{a} = (2, 1-\sqrt{2})$

d) $\vec{a} = (2+3\sqrt{2}, 1-\sqrt{2})$

③ Let $A(1+\sqrt{2}, 1-\sqrt{2})$ and $B(1-\sqrt{2}, 1+\sqrt{2})$
Rotate $\vec{AB}$ by 30°.

④ Let $A(2,-1)$ and $B(-1,-1)$.
Rotate $\vec{AB}$ by 15°.

→ To rotate a vector by angle ϑ we must rotate the axises by angle $-\vartheta$. Thus to rotate $\vec{a}$ by angle ϑ we calculate
$\vec{b} = R(-\vartheta)\,\vec{a}$.

Vector operations

We define 3 vector operations:

a) Vector sum
b) Scalar product
c) Inner product (dot product).

Vector sum

Let $\vec{a}, \vec{b}$ be vectors with $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$. Then we define

$$\boxed{\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2)}$$

Properties

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$	commutative
$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$	associative
$\vec{a} + \vec{0} = \vec{a}$	neutral element

We also define $\underline{-\vec{a} = (-a_1, -a_2)}$ and therefore

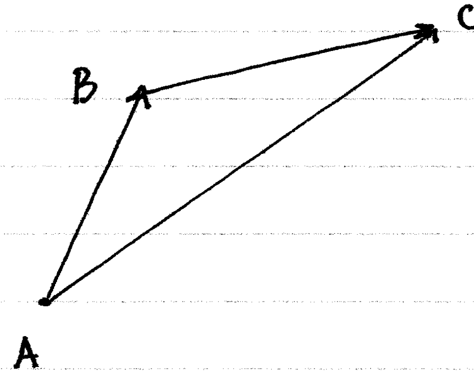
$\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$	inverse element

▸ Geometric interpretation

Thm : For any three points A, B, C:

$$\boxed{\vec{AB} + \vec{BC} = \vec{AC}}$$

Proof



Let $A(x_A, y_A)$,
$B(x_B, y_B)$,
$C(x_C, y_C)$.

Then:

$$\begin{aligned}\vec{AB} + \vec{BC} &= (x_B - x_A, y_B - y_A) + (x_C - x_B, y_C - y_B) = \\ &= (x_B - x_A + x_C - x_B, y_B - y_A + y_C - y_B) = \\ &= (x_C - x_A, y_C - y_A) = \vec{AC}. \quad \square\end{aligned}$$

● Scalar product

Let $\vec{a} = (a_1, a_2)$. Then we define:

$$\boxed{\lambda \vec{a} = (\lambda a_1, \lambda a_2), \ \forall \lambda \in \mathbb{R}}$$

- **Properties**

$\lambda(\vec{a}+\vec{b}) = \lambda\vec{a} + \lambda\vec{b}$	distributive
$(\lambda+\mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$	distributive
$(\lambda\mu)\vec{a} = \lambda(\mu\vec{a}) = \mu(\lambda\vec{a})$	associative
$1\vec{a} = \vec{a}$	neutral element
$0\vec{a} = \mathbb{0}$	
$\lambda\vec{\mathbb{0}} = \mathbb{0}$	

- We also define:
  $\vec{a} - \vec{b} = \vec{a} + (-1)\vec{b} = (a_1 - b_1, a_2 - b_2)$.

- Define the unit vectors: $\vec{i} = (1,0)$ and $\vec{j} = (0,1)$.
  Then:
  $$\vec{a} = (a_1, a_2) = a_1\vec{i} + a_2\vec{j}.$$

EXAMPLES

a) If $\vec{a} = (2,1)$ and $\vec{b} = (3,2)$, evaluate $\vec{c} = 2\vec{a} + 3\vec{b}$.

Solution

$$\vec{c} = 2\vec{a} + 3\vec{b} = 2(2,1) + 3(3,2) =$$
$$= (4,2) + (9,6) = (4+9, 2+6) = (13,8).$$

b) If $\vec{a} = (x+1, y)$ and $\vec{b} = (x-1, x+y)$, find all x, y such that $\vec{a} - 2\vec{b} = \vec{0}$.

Solution

$$\vec{a} - 2\vec{b} = (x+1, y) - 2(x-1, x+y) =$$
$$= (x+1, y) + (-2x+2, -2x-2y)$$
$$= (x+1-2x+2, y-2x-2y) =$$
$$= (-x+3, -2x-y)$$

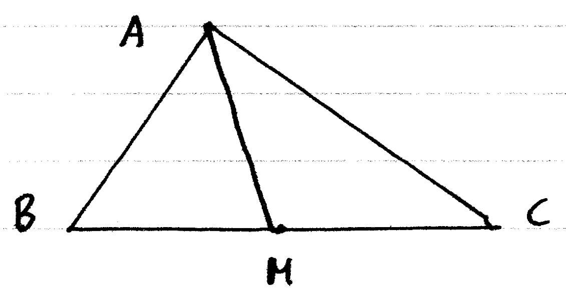
It follows that

$$\vec{a} - 2\vec{b} = \vec{0} \Leftrightarrow \begin{cases} -x+3=0 \\ -2x-y=0 \end{cases} \Leftrightarrow \begin{cases} x=3 \\ -2\cdot 3 - y = 0 \end{cases} \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} x=3 \\ -6-y=0 \end{cases} \Leftrightarrow \begin{cases} x=3 \\ y=-6 \end{cases}$$
$$\Leftrightarrow (x,y) = (3,-6).$$

c) Let $\hat{ABC}$ be a triangle and let M be the midpoint of BC. Show that

$$\vec{AM} = (1/2)(\vec{AB} + \vec{AC}).$$

Solution



M midpoint of $BC \Rightarrow BM = (1/2)\,BC \Rightarrow$

$$\Rightarrow \vec{BM} = (1/2)\,\vec{BC}.$$

It follows that

$$\vec{AM} = \vec{AB} + \vec{BM} = \vec{AB} + (1/2)\vec{BC} = \vec{AB} + (1/2)(\vec{BA} + \vec{AC})$$
$$= \vec{AB} + (1/2)(-\vec{AB} + \vec{AC}) =$$
$$= (1 - 1/2)\vec{AB} + (1/2)\vec{AC} = (1/2)\vec{AB} + (1/2)\vec{AC} =$$
$$= (1/2)(\vec{AB} + \vec{AC}).$$

EXERCISES

⑤ Given the vectors

$$\vec{a} = (\sqrt{3}-2, \sqrt{3}+2)$$
$$\vec{b} = (\sqrt{3}+1, \sqrt{3}-1)$$

evaluate:

$$\vec{c} = (\sqrt{3}-1)(\vec{a}+\vec{b}).$$

⑥ Given the vectors

$$\vec{a} = (x+y\sqrt{2}, x-y\sqrt{2})$$
$$\vec{b} = (x-y\sqrt{3}, x+y\sqrt{3})$$
$$\vec{c} = (1,2)$$

find all values of $x, y \in \mathbb{R}$ such that

$$2\vec{a} - \vec{b} = \vec{c}.$$

⑦ Let $\vec{a}, \vec{b}$ be two vectors. Let O, A, B, C be points such that

$$\vec{OA} = \vec{a}+\vec{b}, \quad \vec{OB} = 2\vec{a}+3\vec{b}, \quad \vec{OC} = 5\vec{a}+9\vec{b}.$$

Show that $\vec{AB} = 4\vec{AC}$.

⑧ Let $\hat{ABC}$ be a triangle with $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$. Find the coordinates of the point G that satisfies

$$\vec{GA} + \vec{GB} + \vec{GC} = 0.$$

(9) Let $\overset{\Delta}{ABC}$ be a triangle. If D is the midpoint of AB and E the midpoint of AC, show that

$$\vec{DE} = (1/2)\,\vec{AB}$$

(10) Let $\overset{\Delta}{ABC}$ be a triangle. If D is the midpoint of BC, E the midpoint of CA, F the midpoint of AB, then show that

$$\vec{AD} + \vec{BE} + \vec{CF} = \vec{0}$$

(Hint: First show that $\vec{AD} = (1/2)(\vec{AB} + \vec{AC})$, etc.)

Inner Product

Let $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ be two vectors.
We define

$$\boxed{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2}$$

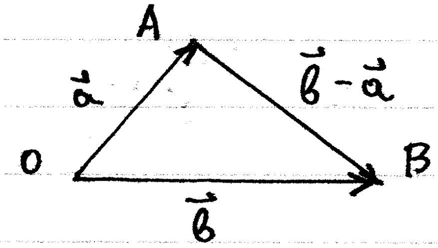
Properties

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$	commutative
$(\lambda \vec{a}) \cdot \vec{b} = \vec{a} \cdot (\lambda \vec{b}) = \lambda (\vec{a} \cdot \vec{b})$	associative
$(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$	distributive
$\lvert a \rvert^2 = \vec{a} \cdot \vec{a}$	norm

inner product theorem

- Let $\vec{a}, \vec{b}$ be two vectors and let θ be the angle between $\vec{a}$ and $\vec{b}$. Then:

$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos\theta}$$



Proof

Let $\vec{a} = \vec{OA}$ and $\vec{b} = \vec{OB}$. From the law of cosines on $\overset{\triangle}{OAB}$:

$$|\vec{b}-\vec{a}|^2 = AB^2 = OA^2+OB^2-2OA\cdot OB\cdot\cos\vartheta =$$
$$= |\vec{a}|^2+|\vec{b}|^2-2|\vec{a}||\vec{b}|\cos\vartheta \qquad (1)$$

Also note that:

$$|\vec{b}-\vec{a}|^2 = (\vec{b}-\vec{a})\cdot(\vec{b}-\vec{a}) =$$
$$= \vec{b}\cdot\vec{b}-\vec{b}\cdot\vec{a}-\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{a} =$$
$$= |b|^2+|a|^2-2(\vec{a}\cdot\vec{b}) \qquad (2)$$

From (1) and (2):

$$|a|^2+|b|^2-2|\vec{a}||\vec{b}|\cos\vartheta = |\vec{a}|^2+|\vec{b}|^2-2(\vec{a}\cdot\vec{b}) \Rightarrow$$
$$\Rightarrow -2(\vec{a}\cdot\vec{b}) = -2|\vec{a}||\vec{b}|\cos\vartheta \Rightarrow$$
$$\Rightarrow \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\vartheta \qquad \square$$

It follows that the angle ϑ between two vectors $\vec{a}=(a_1,a_2)$ and $\vec{b}=(b_1,b_2)$ satisfies:

$$\boxed{\cos\vartheta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_1b_1+a_2b_2}{\sqrt{a_1^2+a_2^2}\,\sqrt{b_1^2+b_2^2}}}$$

- Orthogonal vectors

Thm : $\boxed{\vec{a}\perp\vec{b} \Leftrightarrow \vec{a}\cdot\vec{b}=0}$

Proof

$$\vec{a}\perp\vec{b} \Leftrightarrow \vartheta=\pi/2 \vee \vartheta=3\pi/2 \Leftrightarrow \cos\vartheta=0 \Leftrightarrow$$
$$\Leftrightarrow \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=0 \Leftrightarrow \vec{a}\cdot\vec{b}=0 \qquad \square$$

EXAMPLES

a) If $\vec{a}=(3,1)$ and $\vec{b}=(2,4)$, then evaluate $\lambda=(\vec{a}-\vec{b})\cdot\vec{b}$.

Solution

$$\lambda=(\vec{a}-\vec{b})\cdot\vec{b}=[(3,1)-(2,4)]\cdot(2,4)=$$
$$=(3-2,1-4)\cdot(2,4)=(1,-3)\cdot(2,4)=$$
$$=1\cdot 2+(-3)\cdot 4=2-12=-10.$$

b) If $\vec{a}=(1,2)$ and $\vec{b}=(2,3)$, then find $\cos\vartheta$ of the angle ϑ between $\vec{a}$ and $\vec{b}$.

Solution

$$\cos\vartheta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{(1,2)\cdot(2,3)}{|(1,2)|\cdot|(2,3)|}=$$
$$=\frac{1\cdot 2+2\cdot 3}{\sqrt{1^2+2^2}\sqrt{2^2+3^2}}=\frac{2+6}{\sqrt{1+4}\sqrt{4+9}}=$$
$$=\frac{8}{\sqrt{5}\sqrt{13}}=\frac{8}{\sqrt{65}}=\frac{8\sqrt{65}}{65}$$

c) Find all x such that $\vec{a}=(x,x+1)$ and $\vec{b}=(x+1,3)$ are orthogonal.

Solution

We note that:

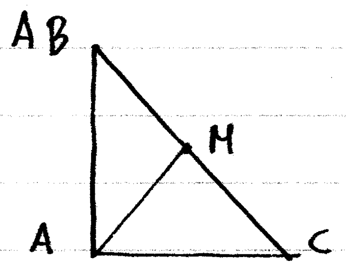
$$\vec{a}\cdot\vec{b} = (x, x+1)\cdot(x+1, 3) =$$
$$= x(x+1) + (x+1)3 = (x+1)(x+3).$$

It follows that

$$\vec{a}\perp\vec{b} \Leftrightarrow \vec{a}\cdot\vec{b}=0 \Leftrightarrow (x+1)(x+3)=0 \Leftrightarrow$$
$$\Leftrightarrow x+1=0 \vee x+3=0 \Leftrightarrow$$
$$\Leftrightarrow x=-1 \vee x=-3.$$

d) Consider a triangle $A\hat{B}C$ with $\hat{A}=90^\circ$. Let M be the midpoint of BC. Show that $AM = BC/2$.

<u>Solution</u>



Since $\hat{A}=90^\circ \Rightarrow AB \perp AC \Rightarrow$
$$\Rightarrow \underline{\vec{AB}\cdot\vec{AC}=0} \quad (1)$$

We also recall that

$$\vec{AM} = \frac{1}{2}(\vec{AB}+\vec{AC})$$

Note that

$$|\vec{AB}+\vec{AC}|^2 = (\vec{AB}+\vec{AC})\cdot(\vec{AB}+\vec{AC}) =$$
$$= \vec{AB}\cdot\vec{AB} + 2(\vec{AB}\cdot\vec{AC}) + \vec{AC}\cdot\vec{AC} =$$
$$= |\vec{AB}|^2 + 2\cdot 0 + |\vec{AC}|^2 = AB^2 + AC^2 =$$
$$= BC^2 \Rightarrow |\vec{AB}+\vec{AC}| = BC \Rightarrow$$
$$\Rightarrow AM = |\vec{AM}| = \left|\frac{1}{2}(\vec{AB}+\vec{AC})\right| = \frac{1}{2}|\vec{AB}+\vec{AC}| =$$
$$= \frac{BC}{2}.$$

EXERCISES

(11) Evaluate $\vec{a}\cdot\vec{b}$ given

a) $\vec{a} = (1+\sqrt{2}, 1-\sqrt{3})$
   $\vec{b} = (1-\sqrt{2}, 1+\sqrt{3})$

b) $\vec{a} = (x+y, 2x)$, $\vec{b} = (x+y, -y)$

c) $\vec{a} = (x+y, 3xy)$, $\vec{b} = (x+y)(x+y, -1)$

(12) Show that $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2$.

(13) Let $\vec{a}, \vec{b}$ such that $|\vec{a}| = 2$ and $|\vec{b}| = 3$ and let $\theta = \pi/3$ be the angle from $\vec{a}$ to $\vec{b}$. Show that:

$$(\vec{a}-2\vec{b})\cdot(3\vec{a}+2\vec{b}) = -21.$$

(14) If $|\vec{a}| = 1$ and $|\vec{b}| = \sqrt{2}$ and the angle from $\vec{a}$ to $\vec{b}$ is $\theta = 3\pi/4$, then evaluate $|\vec{c}|$ with $\vec{c} = 3\vec{a} - 2\vec{b}$.

(15) Let $\vec{a} = (2,1)$ and $\vec{b} = (2+\sqrt{3}, 1-2\sqrt{3})$. Show that the angle θ between $\vec{a}$ and $\vec{b}$ satisfies $\cos\theta = 1/2$.

(16) If $|\vec{a}| = |\vec{b}| = 1$ and θ is the angle from $\vec{a}$ to $\vec{b}$ is $\theta = 2\pi/3$, show that the angle from $\vec{c} = 2\vec{a}+\vec{b}$ to $\vec{d} = \vec{a}-2\vec{b}$ satisfies $\cos\varphi = \sqrt{21}/14$.

(17) Let $\vec{a} = ((x-1)\sqrt{3}, 2x)$ and $\vec{b} = (-\sqrt{3}, 1)$.
If ϑ is the angle from $\vec{a}$ to $\vec{b}$ show that
$\cos\vartheta = 1/2 \Leftrightarrow x = \pm 1$.

(18) Given the points $A(-2,2)$ and $B(1,1)$, find a point C on the y-axis such that $AC \perp BC$.

(19) Let (c) be a circle with center O. Let AB be a diameter and let C be another point on the circle. Show that $AC \perp CB$.

(20) Let $\vec{a}, \vec{b}, \vec{c}$ be vectors. Show that:

a) $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}| \Rightarrow \vec{a} \perp \vec{b}$.

b) $\vec{a} \perp [(\vec{a}\cdot\vec{b})\vec{c} - (\vec{a}\cdot\vec{c})\vec{b}]$

c) $\vec{a} \perp (\vec{b}-\vec{c})$ and $\vec{b} \perp (\vec{c}-\vec{a}) \Rightarrow \vec{c} \perp (\vec{a}-\vec{b})$