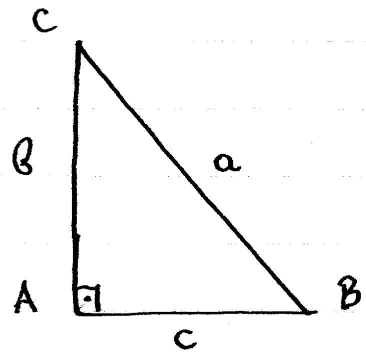
# APPLICATION TO TRIANGLES

## Right Triangles

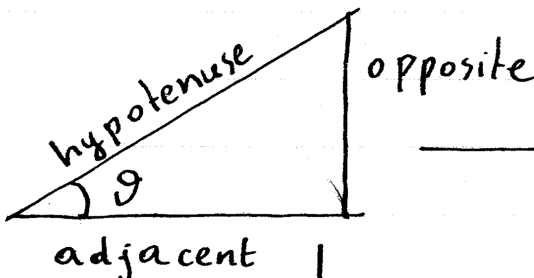



- $\boxed{\hat{A} = 90^\circ \Leftrightarrow \hat{B} + \hat{C} = 90^\circ}$
- $A = 90^\circ \Leftrightarrow a^2 = b^2 + c^2$

  OR

  $A = 90^\circ \Leftrightarrow BC^2 = AC^2 + BC^2$

### Mnemonic Rule for trig. relations




| | |
|---|---|
| $\sin\vartheta = \dfrac{opp}{hyp}$ | $\tan\vartheta = \dfrac{opp}{adj}$ |
| $\cos\vartheta = \dfrac{adj}{hyp}$ | $\cot\vartheta = \dfrac{adj}{opp}$ |

| |
|---|
| $\sin B = \dfrac{b}{a} = \cos C$ |
| $\cos B = \dfrac{c}{a} = \sin C$ |
| $\tan B = \dfrac{b}{c} = \cot C$ |
| $\cot B = \dfrac{c}{b} = \tan C$ |

$$b = a\sin B = a\cos C$$
$$c = a\cos B = a\sin C$$
$$b = c\tan B = c\cot C$$
$$c = b\cot B = b\tan C$$

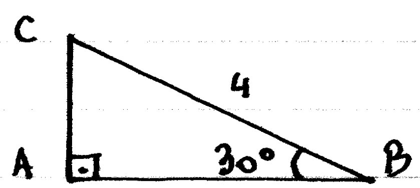
## Solving right triangles

Given $A = 90^\circ$ and two other élements, with one of them being a side, it is possible to calculate all other elements of the triangle.
By elements we mean:
a) the angles: $A, B, C$
b) The sides: $a, b, c$

## EXAMPLES

### 1) Hypotenuse + Angle:

Given: $B = 30^\circ$, $A = 90^\circ$, $a = 4$.



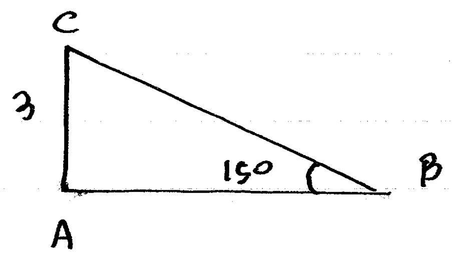

$b = AC = BC \sin B = 4 \sin 30^\circ = 4 \cdot (1/2) = 2$

$c^2 = a^2 - b^2 = 4^2 - 2^2 = 16 - 4 = 12 = 4 \cdot 3 \Rightarrow c = 2\sqrt{3}$.

$C = 90 - B = 90^\circ - 30^\circ = 60^\circ$

### 2) Side + Angle

Given: $B = 15^\circ$, $A = 90^\circ$, $b = 3$




$$C = 90 - B = 90^\circ - 15^\circ = 75^\circ$$

$$\sin B = \frac{AC}{BC} = \frac{3}{a} \Rightarrow a = \frac{3}{\sin 15^\circ} \qquad (1)$$

Note that

$$\sin^2 15^\circ = \frac{1 - \cos 30^\circ}{2} = \frac{1 - \sqrt{3}/2}{2} = \frac{2 - \sqrt{3}}{4} \Rightarrow$$

$$\Rightarrow \sin 15^\circ = \frac{\sqrt{2 - \sqrt{3}}}{2} \qquad (2)$$

From (1) and (2):

$$a = \frac{3}{\frac{\sqrt{2-\sqrt{3}}}{2}} = \frac{6}{\sqrt{2-\sqrt{3}}} = \frac{6\sqrt{2-\sqrt{3}}}{2-\sqrt{3}} =$$

$$= \frac{6(2+\sqrt{3})\sqrt{2-\sqrt{3}}}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{6(2+\sqrt{3})\sqrt{2-\sqrt{3}}}{4-3} =$$

$$= 6(2+\sqrt{3})\sqrt{2-\sqrt{3}} \qquad (3)$$

From (3) and $b = 3$:

$$c^2 = a^2 - b^2 = \left[6(2+\sqrt{3})\sqrt{2-\sqrt{3}}\right]^2 - 3^2 =$$

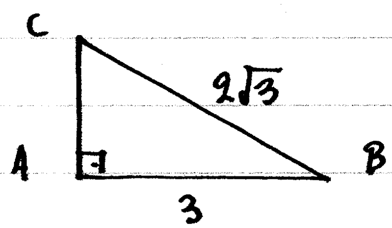
$$= 36(2+\sqrt{3})^2(2-\sqrt{3}) - 9 = 36(2+\sqrt{3})(4-3) - 9$$

$$= 36(2+\sqrt{3}) - 9 = 9\left[4(2+\sqrt{3}) - 1\right] = 9\left[7 + 4\sqrt{3}\right] \Rightarrow$$

$$\Rightarrow c = 3\sqrt{7 + 4\sqrt{3}}$$

## 3) Side + Hypotenuse

Given: $A = 90°$, $a = 2\sqrt{3}$, $c = 3$




$b^2 = a^2 - c^2 = (2\sqrt{3})^2 - 3^2 = 4 \cdot 3 - 9 =$

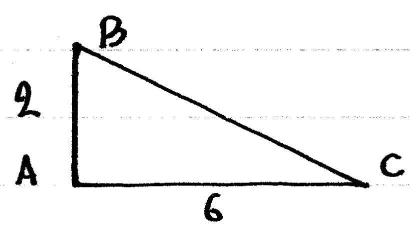
$= 12 - 9 = 3 \Rightarrow \underline{b = \sqrt{3}}$

$\cos B = \dfrac{AB}{BC} = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2} = \cos 30° \Rightarrow \underline{B = 30°}$

$\underline{C} = 90 - B = 90 - 30 = \underline{60°}$

## 4) Side + Side

Given: $A = 90°$, $b = 6$, $c = 2$




$a^2 = b^2 + c^2 = 6^2 + 2^2 = 36 + 4 = 40 \Rightarrow \underline{a = 2\sqrt{10}}$

$\tan B = \dfrac{AC}{AB} = \dfrac{b}{c} = \dfrac{6}{2} = 3 \Rightarrow \underline{B = \text{Arctan}(3)}.$

$\underline{C} = 90 - B = \underline{90 - \text{Arctan}(3)}.$

# EXERCISES

① Solve the following right triangles with $A = 90°$:

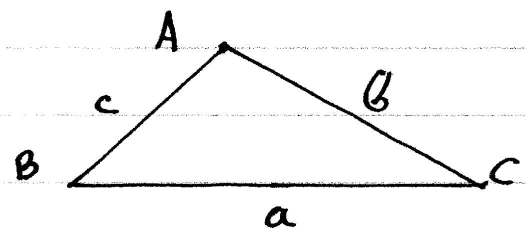
a) $b = 3$ , $c = 4$
b) $B = 60°$ , $a = 2$
c) $B = 45°$ , $b = 3$
d) $C = 15°$ , $a = 1$
e) $a = 2\sqrt{3}$ , $b = 3$
f) $b = 1 + \sqrt{2}$ , $a = \sqrt{6}$

↳ To check your answers, use a calculator to confirm that your results satisfy Mollweide's identity:

$$\boxed{\frac{b-c}{a} \cos\left(\frac{A}{2}\right) = \sin\left(\frac{B-C}{2}\right)}$$

## General Triangles

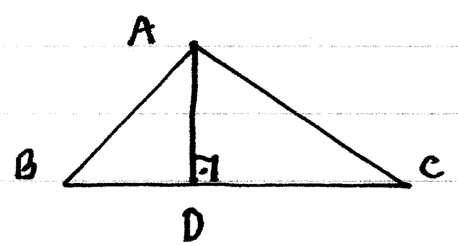
Consider an arbitrary triangle $A\hat{B}C$.

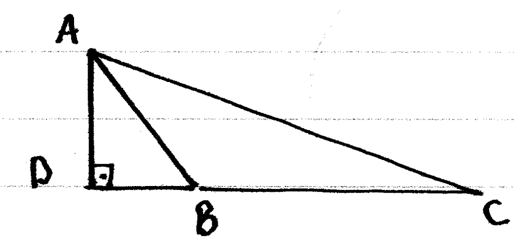



$a = BC$, $b = CA$, $c = AB$.

### ① Law of sines $\longrightarrow$ $\boxed{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}}$

**Proof**




- Bring the height $AD$ with $AD \perp BC$.

From $A\overset{\Delta}{D}C$ : $AD = AC \cdot \sin C = b \sin C$ (1)

From $A\overset{\Delta}{D}B$ : $AD = AB \cdot \sin B = c \sin B$ (2)

From (1) and (2):

$$b \sin C = c \sin B \Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C}$$

Similarly we get $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$. □

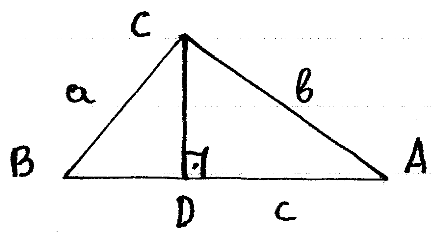
- We now use the projection laws to prove the law of cosines.

## Projection laws

$$\boxed{\begin{aligned} c &= a\cos B + b\cos A \\ a &= b\cos C + c\cos B \\ b &= c\cos A + a\cos C \end{aligned}}$$

**Proof**

**Case 1:** $B \leq \pi/2$ (acute angle)




▸ Bring the height $CD \perp AB$ with $D \in AB$.

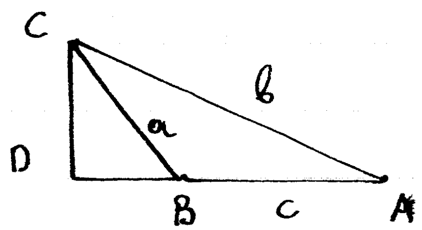
From $B\hat{D}C$: $BD = BC\cos B = a\cos B$. (1)

From $C\hat{D}A$: $AD = AC\cos A = b\cos A$ (2)

From (1) and (2):

$c = AB = BD + AD = a\cos B + b\cos A$.

**Case 2:** $B > \pi/2$




▸ Bring the height $CD \perp AB$ with $B \in AD$.

From $B\hat{D}C$: $BD = BC\cos(C\hat{B}D) = a\cos(\pi - B) =$

$= -a\cos B$. (3)

From $C\hat{D}A$: $AD = AC\cos A = b\cos A$ (4)

From (3) and (4):

$c = AB = AD - BD = b\cos A - (-a\cos B) =$

$= a\cos B + b\cos A$.

Repeat argument for the other two equations.

② Law of cosines

| |
|---|
| $a^2 = b^2 + c^2 - 2bc\cos A$ |
| $b^2 = c^2 + a^2 - 2ca\cos B$ |
| $c^2 = a^2 + b^2 - 2ab\cos C$ |

$\longrightarrow$

| |
|---|
| $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ |
| $\cos B = \dfrac{c^2+a^2-b^2}{2ca}$ |
| $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$ |

Proof

$$
\begin{aligned}
b^2+c^2 &= b(c\cos A + a\cos C) + c(a\cos B + b\cos A)\\
&= bc\cos A + ab\cos C + ac\cos B + bc\cos A\\
&= 2bc\cos A + (ab\cos C + ac\cos B) =\\
&= 2bc\cos A + a(b\cos C + c\cos B)\\
&= 2bc\cos A + a^2 \Rightarrow
\end{aligned}
$$

$$\Rightarrow a^2 = b^2 + c^2 - 2bc\cdot\cos A$$

Repeat argument for the other two equations.

## Solving general triangles

- We use the <u>law of sines</u> when given:
  a) 1 side + 2 angles
  b) 2 sides + angle not between them
- We use the <u>law of cosines</u> when given
  c) 3 sides
  d) 2 sides + angle between them.
- We also note that for any triangle angle, $A$, we have $0 < A < \pi$, and therefore:
  $\sin A = x \Leftrightarrow A = \text{Arcsin}(x) \vee A = \pi - \text{Arcsin}(x)$
  $\cos A = x \Leftrightarrow A = \text{Arccos}(x)$.
- When solving $\sin A = x$ we use the following triangle property to accept or reject solutions.
  $a < b \Leftrightarrow A < B$
  $b < c \Leftrightarrow B < C$
  $c < a \Leftrightarrow C < A$ etc.

## EXAMPLES

a) <u>2 sides + angle not between them.</u> (one solution)
Given: $a = 5$, $b = 6$, $B = 60°$.
Since: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} \Rightarrow$

$$\Rightarrow \sin A = \frac{a \sin B}{b} = \frac{5 \sin 60°}{6} = \frac{5 \cdot (\sqrt{3}/2)}{6} = \frac{5\sqrt{3}}{12} \Rightarrow$$

$\Rightarrow A = \text{Arcsin}\left(\frac{5\sqrt{3}}{12}\right) \vee A = \pi - \text{Arcsin}\left(\frac{5\sqrt{3}}{12}\right) \Rightarrow$

$\Rightarrow A \cong 46° \vee A \approx 180 - 46 = 134.$ (1)

Since $\left.\begin{array}{l} a < b \Rightarrow A < B \\ B = 60° \end{array}\right\} \Rightarrow \underline{A \cong 46°}$ (one solution)

$\underline{C} = 180 - A - B \cong 180 - 46 - 60 = \underline{74°}$

Finally:

$\frac{c}{\sin C} = \frac{b}{\sin B} \Rightarrow$

$\Rightarrow c = \frac{b \sin C}{\sin B} = \frac{6 \sin 74°}{\sin 60°} \approx 6.66$

Thus:

| | |
|---|---|
| $a = 5$ | $A \cong 46°$ |
| $b = 6$ | $B = 60°$ |
| $c \cong 6.66$ | $C \cong 74°$. |

b) <u>2 sides + angle not between them</u> (2 solutions)

Given: $A = 30°$, $a = 2$, $b = 3$

$\frac{b}{\sin B} = \frac{a}{\sin A} \Rightarrow$

$\Rightarrow \sin B = \frac{b \sin A}{a} = \frac{3 \sin 30°}{2} = \frac{3 \cdot (1/2)}{2} = \frac{3}{4} \Rightarrow$

$\Rightarrow B = \text{Arcsin}(3/4) \cong 48° \vee B \cong 180° - 48° = 132°$

Since $a < b \Rightarrow A < B \Rightarrow 30 < B$

Both solution for B are valid, thus there are two possible triangles.

c) <u>2 sides + angle not between them</u> (no solution)

Given: $A = 45^0$, $a = 2$, $b = 8$

$$\frac{b}{\sin B} = \frac{a}{\sin A} \Rightarrow$$

$$\Rightarrow \sin B = \frac{b \sin A}{a} = \frac{8 \sin 45^0}{2} = 4 \cdot \frac{\sqrt{2}}{2} = 2\sqrt{2} > 1$$

thus no triangle is possible.

d) <u>3 sides</u>

Given $a = \sqrt{3}/2$, $b = \sqrt{2}/2$, $c = (\sqrt{6} + \sqrt{2})/4$

Note that:

$a^2 = 3/4$ and $b^2 = 2/4 = 1/2$ and

$$c^2 = \frac{(\sqrt{6} + \sqrt{2})^2}{16} = \frac{6 + 2\sqrt{12} + 2}{16} = \frac{8 + 4\sqrt{3}}{16} =$$

$$= \frac{2 + \sqrt{3}}{4}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{\frac{1}{2} + \frac{2+\sqrt{3}}{4} - \frac{3}{4}}{\cancel{2} \cdot \frac{\sqrt{2}}{\cancel{2}} \cdot \frac{\sqrt{6} + \sqrt{2}}{4}} =$$

$$= \frac{2 + 2 + \sqrt{3} - 3}{\sqrt{2}(\sqrt{6} + \sqrt{2})} = \frac{1 + \sqrt{3}}{2 + \sqrt{12}} = \frac{1 + \sqrt{3}}{2 + 2\sqrt{3}} =$$

$$= \frac{1 + \sqrt{3}}{2(1 + \sqrt{3})} = \frac{1}{2} = \cos 60^0 \Rightarrow \underline{A = 60^0}.$$

$$\cos B = \frac{c^2+a^2-b^2}{2ac} = \frac{\frac{2+\sqrt{3}}{4}+\frac{3}{4}-\frac{2}{4}}{\cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}}\cdot\frac{\sqrt{6}+\sqrt{2}}{4}} =$$

$$= \frac{2+\sqrt{3}+3-2}{\sqrt{3}\,(\sqrt{6}+\sqrt{2})} = \frac{3+\sqrt{3}}{\sqrt{3}\,(\sqrt{6}+\sqrt{2})} = \frac{\sqrt{3}\,(\sqrt{3}+1)}{\sqrt{3}\sqrt{2}\,(\sqrt{3}+1)} =$$

$$= \frac{1}{\sqrt{2}} = \cos 45^\circ \Rightarrow B = 45^\circ.$$

$$C = 180^\circ - A - B = 180^\circ - 60^\circ - 45^\circ = 75^\circ.$$

e) <u>2 sides + angle in between</u>

Given: $a=2$, $b=3$, $C=30^\circ$

$$c^2 = a^2+b^2-2ab\cos C = 2^2+3^2-2\cdot2\cdot3\cdot\cos 30^\circ =$$
$$= 4+9-12\cdot(\sqrt{3}/2) = 13-6\sqrt{3} \Rightarrow$$
$$\Rightarrow c = \sqrt{13-6\sqrt{3}}$$

$$\cos B = \frac{c^2+a^2-b^2}{2ac} = \frac{(13-6\sqrt{3})+2^2-3^2}{2\cdot2\cdot\sqrt{13-6\sqrt{3}}} =$$
$$= \frac{13-6\sqrt{3}+4-9}{4\sqrt{13-6\sqrt{3}}} = \frac{8-6\sqrt{3}}{4\sqrt{13-6\sqrt{3}}} =$$
$$= \frac{4-3\sqrt{3}}{2\sqrt{13-6\sqrt{3}}} \Rightarrow B = \operatorname{Arccos}\left(\frac{4-3\sqrt{3}}{2\sqrt{13-6\sqrt{3}}}\right)$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{3^2+(13-6\sqrt{3})-2^2}{2\cdot 3\cdot \sqrt{13-6\sqrt{3}}} =$$

$$= \frac{9+13-6\sqrt{3}-4}{6\sqrt{13-6\sqrt{3}}} = \frac{18-6\sqrt{3}}{6\sqrt{13-6\sqrt{3}}} =$$

$$= \frac{3-\sqrt{3}}{\sqrt{13-6\sqrt{3}}} \Rightarrow A = \operatorname{Arccos}\left(\frac{3-\sqrt{3}}{\sqrt{13-6\sqrt{3}}}\right)$$

# EXERCISES

(2) Solve the following general triangles $A\hat{B}C$:

a) $a=1$, $b=3$, $B=30°$
b) $a=2$, $b=1$, $B=75°$
c) $a=3$, $b=4$, $B=45°$
d) $a=3$, $b=4$, $c=5$
e) $a=2$, $b=\sqrt{6}$, $c=1+\sqrt{3}$
f) $A=60°$, $B=45°$, $a=5$
g) $a=3$, $b=\sqrt{2}$, $C=45°$
h) $a=1$, $b=\sqrt{3}$, $C=60°$

→ To confirm your answer use a calculator to verify that it satisfies the Mollweide identity:

$$\boxed{\frac{b-c}{a}\cos\left(\frac{A}{2}\right)=\sin\left(\frac{B-C}{2}\right)}$$

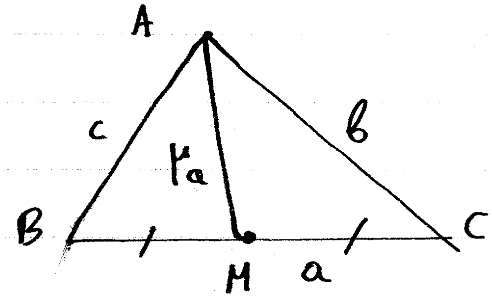
(3) Consider a triangle $A\hat{B}C$. Let $AD$ be the bisector of the angle $A$ with $D$ a point on $BC$. Show that

$$\frac{DB}{DC}=\frac{AB}{AC}$$

(Hint: Use the law of sines to calculate $DB$, $DC$

④ Let $\overset{\Delta}{ABC}$ be a triangle and let $AM$ be a median with $M$ on $BC$ such that $BM = CM$. If $\mu_a = AM$, show that

$$b^2 + c^2 = 2\mu_a^2 + \frac{a^2}{2}$$




(Hint: Use law of cosines to calculate $\mu_a$).

⑤ Show the Mollweide identities; for any triangle

a) $$\frac{b-c}{a}\cos\left(\frac{A}{2}\right) = \sin\left(\frac{B-C}{2}\right)$$

b) $$\frac{b+c}{a}\sin\left(\frac{A}{2}\right) = \cos\left(\frac{B-C}{2}\right)$$

c) $$\frac{b-c}{b+c} = \tan\left(\frac{B-C}{2}\right)\tan\left(\frac{A}{2}\right)$$

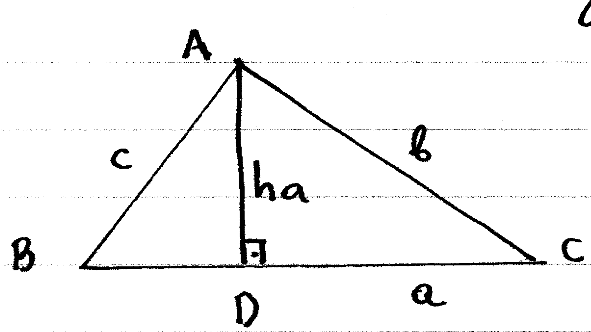
(Hint: Use law of sines to write $a, b, c$ in terms of $\sin A$, $\sin B$, $\sin C$. Then use the sum to product identities)

↳ The identity in 5c is the lesser-known <u>law of the tangents.</u>

d) $$\frac{b^2-c^2}{a^2}\sin A = \sin(B-C)$$

## Area of triangles

Let $\overset{\Delta}{ABC}$ be a triangle with heights $h_a, h_b, h_c$.




It is well-known that the area of $\overset{\Delta}{ABC}$ is given by

$$\boxed{A = \frac{a h_a}{2} = \frac{b h_b}{2} = \frac{c h_c}{2}}$$

We note that: $h_a = c \sin B$
$h_b = a \sin C$
$h_c = b \sin A$

and therefore

$$\boxed{A = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ac \sin B}$$

We will now show that

$$\boxed{\begin{array}{l} A = \sqrt{s(s-a)(s-b)(s-c)} \\ 2s = a+b+c \end{array}}$$ (Heron's formula)

Proof

$$A = \frac{1}{2} ac \sin B \Rightarrow$$

$$\Rightarrow A^2 = \frac{1}{4} a^2c^2 \sin^2 B = \frac{1}{4} a^2c^2 (1 - \cos^2 B) =$$

$$= \frac{1}{4} a^2c^2 (1-\cos B)(1+\cos B) =$$

$$= \frac{1}{4} a^2c^2 \left[1 - \frac{a^2+c^2-b^2}{2ac}\right]\left[1+\frac{a^2+c^2-b^2}{2ac}\right]$$

$$= \frac{1}{4} \frac{a^2c^2}{(2ac)^2} (2ac - a^2 - c^2 + b^2)(2ac + a^2 + c^2 - b^2)$$

$$= \frac{1}{16} [b^2 - (a-c)^2][(a+c)^2 - b^2] =$$

$$= \frac{1}{16} [b-a+c][b+a-c][a+c-b][a+c+b]$$

$$= \frac{(a+b+c)}{2} \frac{(-a+b+c)}{2} \frac{(a-b+c)}{2} \frac{(a+b-c)}{2}$$

Note that

$$s - a = \frac{a+b+c}{2} - a = \frac{a+b+c-2a}{2} = \frac{-a+b+c}{2}$$

$$s - b = \frac{a-b+c}{2} \quad \text{and} \quad s - c = \frac{a+b-c}{2}$$

thus $A^2 = s(s-a)(s-b)(s-c) \Rightarrow$

$$\Rightarrow A = \sqrt{s(s-a)(s-b)(s-c)} \qquad \square$$

# EXERCISES

⑥ Find the area of triangles with

a) $a=1$ , $b=2$ , $c=2$

b) $a=2$ , $b=4$ , $c=3$

c) $a=1$ , $b=2$ , $C=60°$

d) $a=2$ , $b=1$ , $C=45°$

⑦ Show that for any triangle $\hat{ABC}$:

a) $\sin\left(\frac{B}{2}\right)=\sqrt{\frac{(s-c)(s-a)}{ca}}$ (Hint: Use cos2a identities and the law of cosines)

b) $\cos\left(\frac{B}{2}\right)=\sqrt{\frac{s(s-b)}{ca}}$

⑧ Consider a triangle $\hat{ABC}$ and let AD be the bisector of the angle A with D on BC. Use the result of exercise 3 to show that

a) $DB=\frac{ac}{b+c}$ and $DC=\frac{bc}{b+c}$

b) $\delta_a = AD = \frac{ac}{b+c}\,\frac{2\sin(B/2)\cos(B/2)}{\sin(A/2)}$

(Hint: Use law of sines on $\hat{ABD}$)

c) $\delta_a = \frac{2\sqrt{bc}}{b+c}\cdot\sqrt{s(s-a)}$

(Hint: Use exercise 7)