Finite Differences Method on PDEs

→ The problem is handling the linear part of PDEs such as

$$\frac{\partial u}{\partial t} = f + \nu \frac{\partial^2 u}{\partial x^2} \qquad \text{(diffusion equation)}$$

$$\frac{\partial u}{\partial t} = f + a \frac{\partial u}{\partial t} \qquad \text{(advection equation)}$$

$$\frac{\partial u}{\partial t} = f + a \frac{\partial u}{\partial t} + \nu \frac{\partial^2 u}{\partial x^2} \qquad \text{(advection - diffusion equation)}$$

when spectral methods are not available.

→ Example : The diffusion equation

To solve the diffusion equation for $u(x,t)$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

We divide the domain into "gridpoints" and define

$$u_m^n = u(m\Delta x, n\Delta t)$$

and use one of the following schemes:

① Forward Time Central Space scheme:

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = v \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2} + f_m^n$$

② Backward Time Central Space scheme

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = v \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2} + f_m^n$$

③ Crank-Nicholson scheme

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = \frac{v}{2} \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2} + \frac{v}{2} \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_m^{n+1}}{\Delta x^2} + \frac{1}{2}\left(f_n + f_{n+1}\right).$$

④ Leapfrog scheme

$$\frac{u_m^{n+1} - u_m^{n-1}}{2\Delta t} = v \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta t^2} + f_m^n$$

Nomenclature : Differential operators

▶ Spatial Difference Operators

$$\left.\begin{aligned}
\mu_+ u_m^n &= (1/2)(u_{m+1}^n + u_m^n) \\
\mu_- u_m^n &= (1/2)(u_{m-1} + u_m^n) \\
\mu_0 u_m^n &= (1/2)(u_{m-1}^n + u_{m+1}^n)
\end{aligned}\right\} \text{Averaging operators}$$

$$\left.\begin{aligned}
\delta_+ u_m^n &= (1/\Delta x)(u_{m+1}^n - u_m^n) \\
\delta_- u_m^n &= (1/\Delta x)(u_m^n - u_{m-1}^n) \\
\delta_0 u_m^n &= (1/2\Delta x)(u_{m+1}^n - u_{m-1}^n) \\
\delta_x u_m^n &= (1/\Delta x^2)(u_{m+1}^n - 2u_m^n + u_{m-1}^n)
\end{aligned}\right\} \text{Difference operators}$$

▶ Temporal Difference operators

$$\mu^+ u_m^n = (1/2)(u_m^{n+1} + u_m^n)$$

$$\mu^- u_m^n = (1/2)(u_m^{n-1} + u_m^n)$$

$$\mu^0 u_m^n = (1/2)(u_m^{n-1} + u_m^{n+1})$$

$$\delta^+ u_m^n = (1/\Delta t)(u_m^{n+1} - u_m^n)$$

$$\delta^- u_m^n = (1/\Delta t)(u_m^n - u_m^{n-1})$$

$$\delta^0 u_m^n = (1/2\Delta t)(u_m^{n+1} - u_m^{n-1})$$

$$\delta^x u_m^n = (1/\Delta t^2)(u_m^{n+1} - 2u_m^n + u_m^{n-1})$$

<u>example</u> : Using the above notation, the schemes for the diffusion problem read

1) FTCS: $\delta^+ u_m^n = \nu \delta_x u_m^n$

2) BTCS: $\delta^+ u_m^n = \nu Z \delta_x u_m^n$

3) CN: $\delta^+ u_m^n = \nu \mu^+ \delta_x u_m^n$

4) LF: $\delta_0 u_m^n = \nu \delta_x u_m^n$

General Theory of Finite Differences

Consider a linear PDE : $Pu = f$ (1)

example : $\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2}\right) u = f.$

Consider its approximation via the finite difference scheme

$$P(\Delta t, \Delta x)\, u = R(\Delta t, \Delta x)\, f. \qquad (2)$$

with $P(\Delta t, \Delta x)$, $R(\Delta t, \Delta x)$ finite difference operators.

Order of Accuracy

① The scheme (2) is consistent with (1) with order (p, q) iff

$$\forall \varphi : P(\Delta t, \Delta x)\varphi - R(\Delta t, \Delta x) P \varphi = O(\Delta t^p) + O(\Delta x^q)$$

② The scheme (2) with $\Delta t = \Lambda(\Delta x)$ is consistent with (1) with order r iff

$$\forall \varphi : P(\Delta t, \Delta x)\varphi - R(\Delta t, \Delta x) P \varphi = O(\Delta x^r)$$

Symbols of difference schemes

To determine order of accuracy we work with "symbols" of finite difference operators

<u>Def</u>: The symbol $S[P](s,\xi)$ of the finite difference operator is defined as:

$$P(\Delta t,\Delta x)\left[e^{sn\Delta t}\, e^{i\xi m\Delta x}\right] = S[P](s,\xi)\left[e^{sn\Delta t}\, e^{i\xi m\Delta x}\right]$$

<u>Def</u>: The symbol $S[p](s,\xi)$ of a linear differential operator P is defined by

$$P\left[e^{st} e^{i\xi x}\right] = p(s,\xi)\, e^{st} e^{i\xi x}.$$

The definitions for order of accuracy can now be rewritten in terms of:

$$\Delta \equiv S[P(\Delta t,\Delta x)] - S[R(\Delta t,\Delta x)]\, S[P].$$

Order (p,q): $\quad \Delta = O(\Delta t^{p}) + O(\Delta x^{q})$

Order r with $\Delta t = \Lambda(\Delta x)$: $\quad \Delta = O(\Delta x^{r})$.

example

For $P = \frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2}$ we have

$$P(e^{st} e^{i\xi x}) = [s - \nu (i\xi)^2] e^{st} e^{i\xi x}$$
$$= [s + \nu \xi^2] e^{st} e^{i\xi x}$$

thus

$$\$[P] = s + \nu \xi^2.$$

Consider the scheme ~~$[\delta^+ \; \mu^+ \delta$~~

$$[\delta^+ - \nu \mu^+ \delta_x] u_m^n = \mu^+ f_m^n .$$

Note that

$$\$[\delta^+] = \frac{e^{s\Delta t} - 1}{\Delta t}, \quad \$[\mu^+] = \frac{e^{s\Delta t} + 1}{\Delta t},$$

$$\$[\delta_x] = \frac{e^{i\xi \Delta x} - 2 + e^{-i\xi \Delta x}}{\Delta x^2} = \frac{2\cos(\xi \Delta x) - 2}{\Delta x^2}$$

$$= \frac{2}{\Delta x^2} \left[\cos(\xi \Delta x) - 1 \right]$$

therefore :

$$\$[\delta^+ - v\mu^+\delta_x] = \$[\delta^+] - v\,\$[\mu^+]\,\$[\delta_x]$$

$$= \frac{e^{s\Delta t}-1}{\Delta t} - v\,\frac{e^{s\Delta t}+1}{\Delta t}\,\frac{2}{\Delta x^2}\left[\cos(\xi\Delta x)-1\right]$$

Consequently

$$\Delta = \$[\delta^+ - v\mu^+\delta_x] - \$[\mu^+]\,\$[\partial/\partial t - v\,\partial^2/\partial x^2]$$

$$= \frac{e^{s\Delta t}-1}{\Delta t} - v\,\frac{e^{s\Delta t}+1}{\Delta t}\,\frac{2}{\Delta x^2}\left[\cos(\xi\Delta x)-1\right] - \frac{e^{s\Delta t}+1}{\Delta t}\,(s+v\xi^2)$$

$= O(\Delta t^2) + O(\Delta x^2) \Rightarrow (2,2)$ order accurate.

↳ In general, for operators P, Q the corresponding symbols satisfy

$$\$[PQ] = \$[P]\,\$[Q]$$

$$\$[P+Q] = \$[P] + \$[Q]$$

Taylor expansions can be messy and are usually done by software.

Von Neumann Stability analysis

Consider a finite differences scheme

$$P(\Delta t, \Delta x)\, u_m^n = R(\Delta t, \Delta x) f_m^n \qquad (1)$$

We define the amplification polynomial

$$\Phi(g, \vartheta) \equiv P(\Delta t, \Delta x)\, g^n e^{im\vartheta}$$

→ The scheme (1) is <u>Von-Neumann stable</u> iff $\forall \vartheta \in [0, 2\pi]$, the roots ρ of the equation $\Phi(g, \vartheta) = 0$ satisfy

a) $|\rho| \leq 1$

AND b) $|\rho| = 1 \Rightarrow \rho$ simple root.

To conduct stability analysis we use the following theory.

→ <u>Schur-Von Neumann polynomials</u>

Let $\varphi \in \mathbb{C}[z]$ be a polynomial with complex coefficients.

Let $\mathcal{S}[\varphi]$ be the solution set of the polynomial φ defined as:

$$\mathcal{S}[\varphi] = \{ z \in \mathbb{C} \mid \varphi(z) = 0 \}.$$

Definition : The polynomial $\varphi \in \mathbb{C}[z]$ is:

a) Schur $\Leftrightarrow \forall \rho \in \mathcal{S}[\varphi] : |\rho| < 1$

b) von Neumann $\Leftrightarrow \forall \rho \in \mathcal{S}[\varphi] : |\rho| \leq 1$

c) simple von Neumann $\Leftrightarrow \forall \rho \in \mathcal{S}[\varphi] : \begin{cases} |\rho| \leq 1 \\ |\rho| = 1 \Rightarrow \rho \text{ simple} \end{cases}$

d) conservative $\Leftrightarrow \forall \rho \in \mathcal{S}[\varphi] : |\rho| = 1.$

- The stability condition on the amplification polynomial for $\Phi(g, \vartheta)$ is:

$$\boxed{\forall \vartheta \in [0, 2\pi] : \varphi(g) = \Phi(g, \vartheta) \text{ is simple von Neumann}}$$

Notation

1) Let $\varphi \in \mathbb{C}[z]$ be a polynomial

$\varphi \in N_d \Leftrightarrow \deg \varphi = d \wedge \varphi$ simple von Neumann

$\varphi \in \mathcal{S}_d \Leftrightarrow \deg \varphi = d \wedge \varphi$ Schur

2) Let $\varphi \in \mathbb{C}[z]$ with $\deg \varphi = d$ and

$$\varphi(z) = \sum_{a=1}^{d} c_a z^a$$

We define

$$\varphi^*(z) = \sum_{a=1}^{d} \overline{c_{d-a}}\, z^a = \overline{\varphi((\bar{z})^{-1})}\, z^d$$

and

$$\psi(z) = \frac{\varphi^*(0)\,\varphi(z) - \varphi(0)\,\varphi^*(z)}{z}$$

Note that $\deg \psi = \deg \varphi - 1$

To establish von Neumann stability we use the following theorem:

$$\boxed{\begin{array}{l} \varphi \in S_d \Leftrightarrow |\varphi(0)| < |\varphi^*(0)| \wedge \psi \in S_{d-1} \\ \varphi \in N_d \Leftrightarrow (|\varphi(0)| < |\varphi^*(0)| \wedge \psi \in N_{d-1}) \vee \\ \qquad\qquad (\psi = 0 \wedge \varphi' \in S_{d-1}) \end{array}}$$

with φ' = the derivative of φ.

example : Leapfrog scheme

$$\frac{u_m^{n+1} - u_m^{n-1}}{2\Delta t} = \nu \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2} + f_m^n .$$

• 1 Substitute $u_m^n = g^n e^{im\vartheta}$

$$\frac{g^2 - 1}{2\Delta t} = \nu g \frac{e^{i\vartheta} - 2 + e^{-i\vartheta}}{\Delta x^2}$$

$$= \nu g \frac{2\cos\vartheta - 2}{\Delta x^2} \Leftrightarrow$$

$$g^2 + \nu \frac{2\Delta t}{\Delta x^2} 2(1 - \cos\vartheta) g - 1 = 0.$$

Define $\mu = \Delta t / \Delta x^2$. The amplification polynomial is given by

$$\Phi(g) = g^2 + 4\mu\nu(1 - \cos\vartheta) g - 1 \equiv \varphi_0(g)$$

$$\varphi_0^*(g) = -g^2 + 4\mu\nu(1 - \cos\vartheta) g + 1$$

$$\varphi_1(g) = \frac{\varphi_0^*(0)\varphi_0(g) - \varphi_0(0)\varphi_0^*(g)}{g} =$$

$$= \frac{(1) \cdot [\cancel{g^2} + 4\mu\nu(1 - \cos\vartheta) g \cancel{-1}] - (-1)[\cancel{-g^2} + 4\mu\nu(1 - \cos\vartheta) g \cancel{+1}]}{g}$$

$$= 8\mu\nu(1 - \cos\vartheta)$$

- Check $|\varphi_0(0)| < |\varphi_0^*(0)|$ (1)

Since $\left.\begin{array}{l}|\varphi(0)| = |-1| = 1 \\ |\varphi^*(0)| = |+1| = 1\end{array}\right\} \Rightarrow$ (1) not true.

Thus

$\varphi_0 \in N_2 \Leftrightarrow \varphi_1 = 0 \wedge \varphi_0' \in S$ (2)

However $\varphi_1(g) = 8\mu v(1-\cos\vartheta) \neq 0$
for at least one ϑ.

Thus (2) implies that φ_0 not simple von Neumann.

Thus leapfrog unconditionally unstable.

example : Crank-Nicholson

$$\delta^+ u_m^n = \nu \mu^+ \delta_x u_m^n + \mu^+ f_m^n$$

Let $u_m^n = g^n e^{im\varphi}$. It follows that

$$\delta^+ u_m^n = \frac{g-1}{\Delta t} g^n e^{im\varphi}$$

$$\mu^+ \delta_x u_m^n = \mu^+ \frac{e^{i\varphi} - 2 + e^{-i\varphi}}{\Delta x^2} g^n e^{im\varphi}$$

$$= \frac{e^{i\varphi} - 2 + e^{-i\varphi}}{\Delta x^2} \mu^+ (g^n e^{im\varphi})$$

$$= \frac{e^{i\varphi} - 2 + e^{-i\varphi}}{\Delta x^2} \frac{g+1}{2} g^n e^{im\varphi}.$$

and consequently

$$[\delta^+ - \mu^+ \delta_x] u_m^n = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{g-1}{\Delta t} - \frac{e^{i\varphi} - 2 + e^{-i\varphi}}{\Delta x^2} \frac{g+1}{2} = 0$$

$$\Leftrightarrow g - 1 - \frac{\Delta t}{2\Delta x^2} (g+1)(2\cos\vartheta - 2) = 0$$

so the amplification polynomial is:

with $\mu = \Delta t / (2 \Delta x^2)$

$$\Phi(g) = g - 1 - \mu (g+1)(-4 \sin^2(\vartheta/2))$$

$$= [1 + 4 \mu \sin^2(\vartheta/2)] g + [-1 + 4\mu \sin^2(\vartheta/2)] = 0$$

$$\Leftrightarrow g = \frac{1 - 4\mu \sin^2(\vartheta/2)}{1 + 4\mu \sin^2(\vartheta/2)} \leftarrow \text{unique solution.}$$

$$\forall \mu, \vartheta \in \mathbb{R} : |1 - 4\mu \sin^2(\vartheta/2)| < |1 + 4\mu \sin^2(\vartheta/2)| \Rightarrow$$

$\Rightarrow |g| \leqslant 1 \Rightarrow \Phi(g)$ simple von Neumann, $\forall \mu, \vartheta \in \mathbb{R}$

$\Rightarrow$ scheme is unconditionally stable!!

<u>example</u> : FTCS scheme

$$\delta^+ u_m^n = \nu \delta_x u_m^n \qquad (1)$$

Let $u_m^n = g^n e^{im\vartheta}$. Then

$$\delta^+ u_m^n = \frac{g-1}{\Delta t} g^n e^{im\vartheta} =$$

$$\nu \delta_x u_m^n = \nu \frac{e^{i\vartheta} - 2 + e^{i\vartheta}}{\Delta x^2} g^n e^{im\vartheta}$$

$$= \nu \frac{2\cos\vartheta - 2}{\Delta x^2} g^n e^{im\vartheta}$$

$$= \nu \frac{-\cancel{2}4 \sin^2(\vartheta/2)}{\Delta x^2} g^n e^{im\vartheta}$$

Thus

$$(1) \Leftrightarrow \frac{g-1}{\Delta t} + 4\nu \frac{\sin^2(\vartheta/2)}{\Delta x^2} = 0$$

$$\Leftrightarrow \quad g - 1 + 4\mu\nu \sin^2(\vartheta/2) = 0$$

$$\Leftrightarrow \quad g = 1 - 4\mu\nu \sin^2(\vartheta/2)$$

with $\mu = \Delta t/\Delta x^2$.

Note that

$$|g| \leqslant 1 \Leftrightarrow -1 \leqslant g \leqslant 1 \Leftrightarrow -1 \leqslant 1 - 4\mu\nu \sin^2(\vartheta/2) \leqslant 1$$

$$\Leftrightarrow 4\mu\nu \sin^2(\vartheta/2) \leqslant 2, \quad \forall \vartheta \in \mathbb{R}$$

$$\Leftrightarrow 4\mu\nu \leqslant 2 \Leftrightarrow \nu \frac{\Delta t}{\Delta x^2} \leqslant 2$$

$$\Leftrightarrow \boxed{\Delta t \leqslant 2\nu \Delta x^2}$$