# Ordinary Differential Equations

## Definitions

An ordinary differential equation is an equation of the form

$$\frac{\partial u}{\partial t} = f(u(t), t)$$

with $u : \mathbb{R} \to \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$
$n$ = dimension of ODE.

### Special cases

1) If $f(u(t), t) = A(t)u(t) + f(t)$
   then the ODE is linear
   If $f(t) = 0$, then ODE is linear and homogeneous.
2) If $f(u(t), t) = g(u(t))$,
   then the ODE is autonomous

- An ODE combined with the condition $u(0) = u_0$ is an initial value problem.

→ <u>Exact methods</u> are available when

1) The ODE is linear and autonomous:

$$\frac{\partial u}{\partial t} = Au + f \longrightarrow \text{Matrix Exponential Method}$$

2) The ODE is linear and $A(t)$ is periodic

$$\frac{\partial u}{\partial t} = A(t)u + f \longrightarrow \text{Floquet theory}$$

$$A(t+T) = A(t), \forall t \geq 0$$

3) The ODE is reduced to integral.
Then, use numerical integration
(see Bender-Orzag textbook)

4) Solution reduced to convergent series expansion.
Add expansion numerically.

5) Otherwise use timestepping.

## Existence and Uniqueness (Cauchy 1824)

The initial value problem

$$\frac{\partial u}{\partial t} = f(u(t), t)$$

has a unique solution $u(t)$, $\forall t \in [0,T]$ if

a) $f(u,t)$ continuous wrt $t$ in $[0,T]$

b) $f(u,t)$ is uniform Lipschitz continuous wrt $u$

i.e.

$$\exists L > 0 : \forall u, v \in \mathbb{R}^n : \forall t \in [0,T] : \|f(u,t) - f(v,t)\| \leq L\|u - v\|$$

with $\|\cdot\|$ some p-norm.

## Higher Order ODEs

An ODE of order $p \in \mathbb{N}^*$ is an equation of the form

$$\frac{\partial^p u}{\partial t^p} = f\left(u(t), \frac{\partial u(t)}{\partial t}, \frac{\partial^2 u(t)}{\partial t^2}, \ldots, \frac{\partial^{p-1} u(t)}{\partial t^{p-1}}, t\right)$$

Such ODEs can be reduced analytically to ODEs of order 1.

Let $u_a(t) = \dfrac{\partial^a u(t)}{\partial t^a}$. Then we have

$$\begin{cases} \dfrac{\partial u_a(t)}{\partial t} = u_{a+1}(t), & \forall a \in \{0, 1, \ldots, p-1) \\ \dfrac{\partial u_p(t)}{\partial t} = f(u_0(t), u_1(t), u_2(t), \ldots, u_{p-1}(t), t) \end{cases}$$

which is a first order ODE of higher dimension!

## Time-stepping algorithms

1) Linear Multistep Methods
2) Runge-Kutta Methods.