EIGENVALUES AND EIGENVECTORS

Eigenvalues and Eigenvectors

- Let $A \in M_{nn}(\mathbb{R})$ be a square matrix. We say that $\lambda \in \mathbb{C}$ is an eigenvalue of A with eigenvector $x \in M_{n1}(\mathbb{C})$ if and only if

$$Ax = \lambda x .$$

with $x \neq 0$

- With every eigenvalue λ we associate an eigenvector space $E_\lambda(A)$ which consists of all vectors x that are eigenvectors to the eigenvalue λ. Thus:

$$E_\lambda(A) = \{ x \in M_{n1}(\mathbb{C}) \mid Ax = \lambda x \}$$

How to find the eigenvalues

- $\boxed{\lambda \text{ eigenvalue of } A \Leftrightarrow \det(A - \lambda I) = 0}$

Proof

Note that

$$Ax = \lambda x \Leftrightarrow Ax - \lambda x = 0 \Leftrightarrow Ax - \lambda I x = 0$$
$$\Leftrightarrow (A - \lambda I)x = 0 \qquad (1)$$

Eq. (1) has an obvious solution $x = \mathbf{0}$ and if $\det(A - \lambda I) \neq 0$, then this solution is unique. It follows that

λ eigenvalue of $A \Leftrightarrow$

$\Leftrightarrow$ Eq. (1) has a solution $x \neq 0$

$\Leftrightarrow$ $x = \mathbf{0}$ is NOT a unique solution

$\Leftrightarrow$ $\det(A - \lambda I) = 0$.

example

Find the eigenvalues of

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) =$$

$$= \begin{vmatrix} 2-\lambda & -1 & 1 \\ 0 & 3-\lambda & -1 \\ 2 & 1 & 3-\lambda \end{vmatrix} =$$

$$3-\lambda$$

$$= \begin{vmatrix} 2-\lambda & 2-\lambda & 1 \\ 0 & 0 & -1 \\ 2 & (3-\lambda)^2+1 & 3-\lambda \end{vmatrix} \rightarrow =$$

$$= -(-1) \begin{vmatrix} 2-\lambda & 2-\lambda \\ 2 & (3-\lambda)^2+1 \end{vmatrix} = (2-\lambda) \begin{vmatrix} 1 & 1 \\ 2 & (3-\lambda)^2+1 \end{vmatrix}$$

$$= (2-\lambda)\left[(3-\lambda)^2+1-2\right] = (2-\lambda)\left[(3-\lambda)^2-1\right]$$

$$= (2-\lambda)(3-\lambda-1)(3-\lambda+1) = (2-\lambda)(2-\lambda)(4-\lambda).$$

λ eigenvalue of $A \Leftrightarrow \det(A-\lambda I)=0 \Leftrightarrow$

$\Leftrightarrow (2-\lambda)(2-\lambda)(4-\lambda)=0$

$\Leftrightarrow \underline{\lambda = 2 \text{ or } \lambda = 4}$

$\lambda = 2$: <u>double</u> eigenvalue

How to find the eigenvectors

For each eigenvalue λ we use Gaussian Elimination to solve the equation

$$(A-\lambda I)x=0$$

The solution space of this equation coincides with the eigenvector space $E_\lambda(A)$.

example

In the previous example:

For $\lambda=2$

$$M \sim \left[\begin{array}{ccc|c} 2-\lambda & -1 & 1 & 0 \\ 0 & 3-\lambda & -1 & 0 \\ 2 & 1 & 3-\lambda & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & +1 & 0 \end{array}\right] (+1)$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] (-1) \sim$$

$$\sim \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \cdot (1/2) \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \Leftrightarrow \begin{cases} x + z = 0 \\ y - z = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = -z \\ y = z \end{cases} \Leftrightarrow (x, y, z) = (-z, z, z) = z(-1, 1, 1)$$

thus, the corresponding eigenvector space is given by $E_2(A) = \{z(-1,1,1) \mid z \in \mathbb{R}\}$.

EXAMPLES – THEORETICAL

a) Let $A \in M_n(\mathbb{R})$ be a matrix with $A^2 = 5A - 6I$. Show that

λ eigenvalue of $A \Rightarrow \lambda = 2 \vee \lambda = 3$.

Solution

Let $\lambda \in \sigma(A)$ be an eigenvalue of A with x a corresponding eigenvector. It follows that

$Ax = \lambda x$ and

$A^2x = (AA)x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$.

and therefore:

$$A^2 = 5A - 6I \Rightarrow A^2 - 5A + 6I = \mathbb{O} \Rightarrow$$

$$\Rightarrow (A^2 - 5A + 6I)x = \mathbb{O}x = \mathbb{O} \qquad (1)$$

and $(A^2 - 5A + 6I)x = A^2x - 5Ax + 6Ix = \lambda^2 x - 5\lambda x + 6x =$

$$= (\lambda^2 - 5\lambda + 6)x \qquad (2)$$

From (1) and (2):

$$(\lambda^2 - 5\lambda + 6)x = \mathbb{O} \Rightarrow \lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda - 2)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 2 \vee \lambda = 3$$

EXERCISES

① Find the eigenvalues and corresponding eigenvector spaces for the following matrices.

a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & -7 & 1 \end{bmatrix}$ $(\lambda = 1 \longrightarrow t(0,0,1))$

b) $B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{bmatrix}$ $(\lambda = 1 \longrightarrow t(1,-1,0)$
$\lambda = 2 \longrightarrow t(3,3,-1)$
$\lambda = 21 \longrightarrow t(1,1,6))$

c) $C = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ $(\lambda = 1 \longrightarrow t(3,-1,3)$
$\lambda = 2 \longrightarrow t(2,2,-1))$

[answers can be confirmed by matlab or octave]

(2) Let $A = \begin{bmatrix} 1 & 2a+1 \\ 2a-1 & 1 \end{bmatrix}$

For what values of a does A have only one eigenvalue?

(3) Rotation matrix

Let $R(\vartheta) = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$

Show that $R(\vartheta)$ has real eigenvalues if and only if $\sin\vartheta = 0$.

(4) Find the eigenvalues of the following matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(ans: $\lambda = +1, -1$)

(5) Let $A \in M_{nn}(\mathbb{R})$ be a matrix such that $A^2 = I$. Show that if λ is an eigenvalue of A, then $\lambda = 1$ or $\lambda = -1$.

(6) Let $A \in M_{nn}(\mathbb{R})$ be a non-singular matrix. Show that if $\lambda \neq 0$ is an eigenvalue of A, then $1/\lambda$ is an eigenvalue of A^{-1}.

7 Let $A \in M_n(\mathbb{R})$ with $A^2 + 3A = -2I$. Show that if:
λ eigenvalue of $A \Rightarrow \lambda = -1 \vee \lambda = -2$.

8 Let $A \in M_n(\mathbb{R})$ with $A^{-1} = 2I - A$. Show that:
λ eigenvalue of $A \Rightarrow \lambda = 1$.

Characteristic polynomial

> **Thm:** Let $A \in M_n(\mathbb{R})$ be a square matrix. Then the determinant $\det(A-\lambda I)$ simplifies to a polynomial of the form
> $$\det(A-\lambda I) = (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0$$

- We call the polynomial obtained by expanding $\det(A-\lambda I)$ the <u>characteristic polynomial</u> of A.

Proof

Let $\sigma_0 \in S_n$ be the do-nothing permutation such that $\forall a \in [n] : \sigma_0(a) = a$
Then, we have $\$(\sigma) = 1$, and therefore

$$\begin{aligned}
\det(A-\lambda I) &= \sum_{\sigma \in S_n} \left[\$(\sigma) \prod_{a=1}^{n} (A-\lambda I)_{a,\sigma(a)} \right] \\
&= \prod_{a=1}^{n} (A-\lambda I)_{a,\sigma_0(a)} + g(\lambda) \\
&= \prod_{a=1}^{n} (A-\lambda I)_{aa} + g(\lambda) \\
&= \prod_{a=1}^{n} (A_{aa} - \lambda) + g(\lambda) \qquad (1)
\end{aligned}$$

with

$$g(\lambda) = \sum_{\sigma \in S_n - \{\sigma_0\}} \left[\$(\sigma) \prod_{a=1}^{n} (A-\lambda I)_{a,\sigma(a)} \right] \qquad (2)$$

From Eq.(1), the highest-order term from the first contribution is $(-\lambda)^n = (-1)^n \lambda^n$. We also note that for $\sigma \neq \sigma_0$, the products that appear in $g(\lambda)$ involve at least two non-diagonal elements, since σ is at least one transposition away from σ_0, and therefore $\deg(g(\lambda)) \leq n-2$. It follows that $g(\lambda)$ does not contribute additional λ^n terms. The conclusion follows $\square$

- Recall that according to the fundamental theorem of algebra, a polynomial
$\forall x \in \mathbb{R}: \; p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
has zeroes $x_1, x_2, \dots, x_n \in \mathbb{C}$, and it can be factored as

$$p(x) = a_n (x - x_1)(x - x_2) \cdots (x - x_n) = a_n \prod_{b=1}^{n} (x - x_b)$$

- Combining the previous result with the fundamental theorem of algebra gives; the following theorem:

Thm: Let $A \in M_n(\mathbb{R})$ be a square matrix and let $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ be the eigenvalues of A. Then,

$$\det A = \lambda_1 \lambda_2 \cdots \lambda_n = \prod_{a \in [n]} \lambda_a$$

Proof

The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ are the n zeroes of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.

Using the fundamental theorem of algebra, it follows that

$$\det(A-\lambda I) = (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0$$

$$= (-1)^n \prod_{a\in[n]} (\lambda-\lambda_a) = \prod_{a\in[n]} (\lambda_a-\lambda) \Rightarrow$$

$$\Rightarrow \det A = \det(A-0I) = \prod_{a\in[n]} (\lambda_a - 0) = \prod_{a\in[n]} \lambda_a \qquad \square$$

▶ <u>Trace of a matrix</u>

> <u>Def:</u> Let $A \in M_n(\mathbb{R})$ be a square matrix. We define the trace of A as:
> $$tr(A) = \sum_{a=1}^{n} A_{aa} = A_{11} + A_{22} + \dots + A_{nn}$$

We can now show that:

> <u>Prop:</u> Let $A \in M_n(\mathbb{R})$ be a matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$. Then, we have:
> $$tr(A) = \sum_{a=1}^{n} \lambda_a = \lambda_1 + \lambda_2 + \dots + \lambda_n.$$

<u>Proof</u>

We note that

$$\det(A-\lambda I) = \prod_{a=1}^{n} (\lambda_a - \lambda) =$$

$$= (-1)^n \lambda^n + \left(\sum_{a=1}^{n} \lambda_a\right)(-\lambda)^{n-1} + \dots + c_1\lambda + c_0$$

$$= (-1)^n \lambda^n + (-1)^{n-1}\left(\sum_{a=1}^{n} \lambda_a\right)\lambda^{n-1} + \dots + c_1\lambda + c_0$$

thus the coefficient c_{n-1} of the λ^{n-1} term is

$$c_{n-1} = (-1)^{n-1}\left(\sum_{a=1}^{n} \lambda_a\right)$$

We also note, from Eq.(1) in the proof of the first theorem of this section that we have:

$$\det(A-\lambda I) = \prod_{a=1}^{n} (A_{aa}-\lambda) + g(\lambda) =$$

$$= (-1)^n\lambda^n + (-1)^{n-1}\left(\sum_{a=1}^{n} A_{aa}\right)\lambda^{n-1} + \dots + d_1\lambda + d_0 + g(\lambda)$$

$$= (-1)^n\lambda^n + (-1)^{n-1}\operatorname{tr}(A)\lambda^{n-1} + \dots + d_1\lambda + d_0 + g(\lambda)$$

with $\deg(g(\lambda)) \leq n-2$. It follows that $g(\lambda)$ does not contribute to the coefficent of λ^{n-1} and thus

$$c_{n-1} = (-1)^{n-1}\operatorname{tr}(A)$$

We conclude that

$$(-1)^{n-1}\operatorname{tr}(A) = (-1)^{n-1}\left(\sum_{a=1}^{n} \lambda_a\right) \Rightarrow \operatorname{tr}(A) = \sum_{a=1}^{n} \lambda_a \qquad \square$$

EXAMPLES

a) Let $A = \begin{bmatrix} a+3 & 1 \\ 2a & 2 \end{bmatrix}$ and let $\lambda_1, \lambda_2 \in \mathbb{C}$ be the eigenvalues of A. Find all $a \in \mathbb{R}$ such that $\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} = 1$

Solution

$$\lambda_1 \lambda_2 = \det A = \begin{vmatrix} a+3 & 1 \\ 2a & 2 \end{vmatrix} = 2(a+3) - 1 \cdot 2a = 2a+6-2a = 6$$

and

$$\lambda_1 + \lambda_2 = \operatorname{tr} A = (a+3) + 2 = a+5$$

so it follows that

$$\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} = \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 \lambda_2^2} = \frac{(\lambda_1+\lambda_2)^2 - 2(\lambda_1\lambda_2)}{(\lambda_1\lambda_2)^2} =$$

$$= \frac{(\operatorname{tr} A)^2 - 2\det A}{(\det A)^2} = \frac{(a+5)^2 - 2\cdot 6}{6^2} =$$

$$= \frac{(a+5)^2 - 12}{36}.$$

and therefore:

$$\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} = 1 \Leftrightarrow \frac{(a+5)^2 - 12}{36} = 1 \Leftrightarrow (a+5)^2 - 12 = 36$$

$$\Leftrightarrow (a+5)^2 = 12+36 \Leftrightarrow (a+5)^2 = 48 = 16 \cdot 3 = 4^2 \cdot 3$$

$$\Leftrightarrow a+5 = 4\sqrt{3} \vee a+5 = -4\sqrt{3} \Leftrightarrow$$

$$\Leftrightarrow a = -5+4\sqrt{3} \vee a = -5-4\sqrt{3}$$

EXERCISES

(9) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2x+3 & 1 \\ 1 & 1 & 2x-1 \end{bmatrix}$$

If $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A, find $x \in \mathbb{C}$ such that

a) $\lambda_1 \lambda_2 \lambda_3 = 1$

b) $\lambda_1 + \lambda_2 + \lambda_3 = 4$

(10) Let $A = \begin{bmatrix} x+1 & 2x \\ 3 & x-1 \end{bmatrix}$

with eigenvalues λ_1, λ_2. Find $x \in \mathbb{C}$ such that

a) $\lambda_1 + \lambda_2 = 3$

b) $\dfrac{1}{\lambda_1} + \dfrac{1}{\lambda_2} = 1$

c) $\lambda_1^2 + \lambda_2^2 = 1$ (Hint: $a^2+b^2 = (a+b)^2 - 2ab$)

(11) Let $A = \begin{bmatrix} a & 1 \\ 1 & 1 \end{bmatrix}$

with eigenvalues λ_1, λ_2. Find $a \in \mathbb{C}$ such that $\lambda_1^3 + \lambda_2^3 = 0$.

(Hint: $(a+b)^3 = (a^3+b^3) + 3ab(a+b)$)

The following problems use
a) $\det(AB) = \det(A)\det(B)$
b) $\det(I) = 1$.

(12) Let $A \in M_{nn}(\mathbb{R})$ be a matrix and let $B = P^{-1}AP$ with $P \in M_{nn}(\mathbb{R})$ a non-singular matrix. Show that A, B have the same eigenvalues.

(13) Let $A, B \in M_{nn}(\mathbb{R})$ with A non-singular. Show that AB and BA have the same eigenvalues.

(14) Let $A \in M_{22}(\mathbb{R})$ be a 2×2 matrix. If A is non-singular, show that

$$tr(A^{-1}) = \frac{tr(A)}{\det(A)}$$

(see exercise (6))

• (15) Let $A \in M_{33}(\mathbb{R})$ be a 3×3 non-singular matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ that satisfy

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$$

Show that

$$tr(A^{-1}) = \frac{(tr(A)+1)(tr(A)-1)}{2\det A}$$

▼ Cayley–Hamilton theorem

Thm: Let $A \in M_n(\mathbb{R})$ be a square matrix ~~we~~ with characteristic polynomial

$$\det(A-\lambda I) = (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

Then A satisfies

$$(-1)^n A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = 0$$

▸ **Method:** The Cayley–Hamilton theorem provides a second method for calculating the matrix inverse A^{-1} as shown in the following example

EXAMPLE

Given the matrix

$$A = \begin{bmatrix} 5 & 4 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

use the Cayley–Hamilton theorem to write A^{-1} in terms of A.

Solution

Since,

$$\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & 4 & 0 \\ 1 & 2-\lambda & 0 \\ 1 & 2 & 2-\lambda \end{vmatrix} =$$

$$= (2-\lambda)\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} =$$

$$= (2-\lambda)[(5-\lambda)(2-\lambda) - 4\cdot 1] =$$

$$= (2-\lambda)(10 - 5\lambda - 2\lambda + \lambda^2 - 4) =$$

$$= (2-\lambda)(\lambda^2 - 7\lambda + 6) =$$

$$= 2\lambda^2 - 14\lambda + 12 - \lambda^3 + 7\lambda^2 - 6\lambda =$$

$$= -\lambda^3 + (2+7)\lambda^2 + (-14-6)\lambda + 12$$

$$= -\lambda^3 + 9\lambda^2 - 20\lambda + 12 \Rightarrow$$

$$\Rightarrow -A^3 + 9A^2 - 20A + 12I = 0 \Rightarrow$$

$$\Rightarrow A^3 - 9A^2 + 20A = 12I \Rightarrow$$

$$\Rightarrow A(A^2 - 9A + 20I) = 12I \Rightarrow$$

$$\Rightarrow A\left[\frac{1}{12}(A^2 - 9A + 20I)\right] = I \Rightarrow$$

$$\Rightarrow A^{-1} = \frac{1}{12}(A^2 - 9A + 20I).$$

► Method: We can also use the Cayley-Hamilton theorem to write higher powers A^n in terms of a few powers of A.

EXAMPLE

Given the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Show that $\forall n \in \mathbb{N} - \{0,1\}: A^n = nA - (n-1)I$.

Solution

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = \lambda^2 - 2\lambda + 1 \Rightarrow$$

$\Rightarrow A^2 - 2A + I = 0 \Rightarrow A^2 = 2A - I.$

Assume that for $n = k$: $A^k = kA - (k-1)I$

For $n = k+1$, we will show that : $A^{k+1} = (k+1)A - kI$

We have:

$$\begin{aligned} A^{k+1} &= A^k A = [kA - (k-1)I]A = \\ &= kA^2 - (k-1)A = k(2A - I) - (k-1)A = \\ &= 2kA - kI - kA + A = (2k-k+1)A - kI = \\ &= (k+1)A - kI. \end{aligned}$$

EXERCISES

(16) For the following matrices, write A^{-1} and A^3 in terms of A and I.

a) $A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ d) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(17) For the following matrices, write A^{-1} and A^4 in terms of I, A, A^2.

a) $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ b) $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

c) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ d) $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Linear systems of differential equations

- Recall that the solution to the differential equation

$$\frac{dy(t)}{dt} = a\,y(t) + f(t)$$

with initial condition

$$y(0) = y_0$$

is given by:

$$y(t) = e^{at} y_0 + e^{at} \int_0^t e^{-a\tau} f(\tau)\, d\tau$$

- We want the solution to the more general system of linear differential equations:

$$\begin{cases} \dfrac{dy_1(t)}{dt} = A_{11} y_1(t) + A_{12} y_2(t) + \dots + A_{1n} y_n(t) + f_1(t) \\ \dfrac{dy_2(t)}{dt} = A_{21} y_1(t) + A_{22} y_2(t) + \dots + A_{2n} y_n(t) + f_2(t) \\ \vdots \\ \dfrac{dy_n(t)}{dt} = A_{n1} y_1(t) + A_{n2} y_2(t) + \dots + A_{nn} y_n(t) + f_n(t) \end{cases}$$

with initial condition:

$$y_1(0) = b_1 \ , \ y_2(0) = b_2 \ , \ \dots , \ y_n(0) = b_n$$

If we define

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} , \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} , \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

then the system can be rewritten as:

$$\boxed{\begin{cases} \dfrac{dy(t)}{dt} = Ay(t) + f(t) \\ y(0) = b \end{cases}}$$

The solution to this system is based on the matrix exponential.

The matrix exponential

- Let $A \in M_{nn}(\mathbb{R})$ be a square matrix. The exponential of A is defined as

$$\boxed{\exp(A) = \sum_{n=0}^{+\infty} \frac{1}{n!} A^n}$$

with $0! = 1$

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

This generalizes the identity

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

- The matrix exponential $\exp(A)$ converges for all matrices A.

▸ <u>Properties</u>

a) $\dfrac{d}{dt}\exp(tA) = A\exp(tA) = \exp(tA)A$

b) $[\exp(A)]^{-1} = \exp(-A)$

c) $AB = BA \Rightarrow \exp(A+B) = \exp(A)\exp(B)$

d) $\dfrac{dy(t)}{dt} = Ay(t) \rightarrow y(t) = \exp(tA)y(0)$

e) $\exp((t_1+t_2)A) = \exp(t_1A)\exp(t_2A)$

Solution to linear system of ODEs

The solution to the ODE system

$$\frac{dy(t)}{dt} = Ay(t) + f(t)$$

is given by

$$\boxed{y(t) = \exp(tA)\,y(0) + \exp(tA)\int_0^t \exp(-\tau A) f(\tau)\,d\tau}$$

How to calculate the matrix exponential

To calculate $\exp(tA)$ we work as follows:

- 1 Let $A \in M_{nn}(\mathbb{R})$. From the Cayley-Hamilton theorem we conclude that there are coefficients $c_0, c_1, \dots, c_{n-1}$ such that

$$\exp(tA) = c_{n-1}A^{n-1}t^{n-1} + \dots + a_1 At + a_0 I$$

  Before we find $c_0, \dots, c_{n-1}$ we simplify the right-hand side of the expression above.

- 2 We find the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the matrix A.

•3 Let

$f(x) = c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$

To find the coefficients $c_0, c_1, \ldots, c_{n-1}$:

a) If λ_k is an eigenvalue of $\underline{\underline{tA}}$, then

$$e^{\lambda_k} = f(\lambda_k)$$

b) If λ_k is an eigenvalue of A with multiplicity m, then we also have:

$$\begin{cases} e^{\lambda_k} = f'(\lambda_k) \\ e^{\lambda_k} = f''(\lambda_k) \\ \vdots \\ e^{\lambda_k} = f^{(m-1)}(\lambda_k) \end{cases}$$

Thus we get a system of n equations from which we find $c_0, \ldots, c_{n-1}$.

•4 Knowing the coefficients $c_0, \ldots, c_{n-1}$ we now calculate the exponential $\exp(tA)$.

example

For $A = \begin{bmatrix} 1 & 1 \\ 9 & 1 \end{bmatrix}$

we have

$$\exp(tA) = c_1 t A + c_0 I = c_1 t \begin{bmatrix} 1 & 1 \\ 9 & 1 \end{bmatrix} + c_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 t + c_0 & c_1 t \\ 9c_1 t & c_1 t + c_0 \end{bmatrix}$$

- Eigenvalues of A:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 9 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9 =$$

$$= (1-\lambda-3)(1-\lambda+3) =$$

$$= (-\lambda-2)(-\lambda+4) =$$

$$= (\lambda+2)(\lambda-4) \Rightarrow$$

$\Rightarrow$ The eigenvalues of A are $\lambda_1 = -2$ and $\lambda_2 = 4$

$\Rightarrow$ The eigenvalues of tA are $\lambda_1 = -2t$ and $\lambda_2 = 4t \Rightarrow$

$$\Rightarrow \begin{cases} e^{4t} = c_1(4t) + c_0 \\ e^{-2t} = c_1(-2t) + c_0 \end{cases} \Leftrightarrow \dots \Leftrightarrow \begin{cases} c_1 = \dfrac{1}{6t}\left(e^{4t} - e^{-2t}\right) \\ c_0 = \dfrac{1}{3}\left(e^{4t} + 2e^{-2t}\right) \end{cases}$$

It follows that

$$\exp(tA) = \cdots = \frac{1}{6}\begin{bmatrix} 3e^{4t}+3e^{-2t} & e^{4t}-e^{-2t} \\ 9e^{4t}-9e^{-2t} & 3e^{4t}+3e^{-2t} \end{bmatrix}$$

<u>example</u>

For $A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$

we have

$$\exp(tA) = c_1 t A + c_0 I = \cdots$$

$$= \begin{bmatrix} c_0 & c_1 t \\ -9c_1 t & 6c_1 t + c_0 \end{bmatrix}$$

Eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -9 & 6-\lambda \end{vmatrix} = -\lambda(6-\lambda) - (-9) =$$

$$= -6\lambda + \lambda^2 + 9 = (\lambda-3)^2 \Rightarrow$$

$\Rightarrow \lambda = 3$ double eigenvalue of A

$\Rightarrow \lambda = 3t$ double eigenvalue of tA

For $f(x) = c_1 x + c_0 \Rightarrow f'(x) = c_1$ thus

$$\begin{cases} e^{3t} = c_1(3t) + c_0 \\ e^{3t} = c_1 \end{cases} \Leftrightarrow \dots \Leftrightarrow \begin{cases} c_0 = e^{3t}(1-3t) \\ c_1 = e^{3t} \end{cases}$$

thus

$$\exp(tA) = \dots = \begin{bmatrix} (1-3t)e^{3t} & te^{3t} \\ -9te^{3t} & (1+3t)e^{3t} \end{bmatrix}$$

Now let us consider

$$\begin{cases} \dfrac{dy_1}{dt} = y_2 \\ \dfrac{dy_2}{dt} = -9y_1 + 6y_2 \end{cases} \Leftrightarrow \frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

It follows that

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \exp(tA)\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} =$$

$$= \begin{bmatrix} (1-3t)e^{3t} & te^{3t} \\ -9te^{3t} & (1+3t)e^{3t} \end{bmatrix}\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} y_1(t) = (1-3t)e^{3t}y_1(0) + te^{3t}y_2(0) \\ y_2(t) = -9te^{3t}y_1(0) + (1+3t)e^{3t}y_2(0) \end{cases}$$

2nd method

For a 2×2 matrix A with eigenvalues λ_1, λ_2, the matrix exponential is given by:

a) If $\lambda_1 \neq \lambda_2$ then

$$\boxed{\exp(tA) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} A}$$

b) If $\lambda_1 = \lambda_2 = \lambda$ then

$$\boxed{\exp(tA) = e^{\lambda t}(1 - \lambda t) I + t e^{\lambda t} A}$$

EXERCISES

(18) Use the matrix exponential to solve the following systems in terms of $y_1(0)$ and $y_2(0)$:

a) $\begin{cases} dy_1/dt = 4y_1 + y_2 \\ dy_2/dt = -2y_1 + y_2 \end{cases}$

b) $\begin{cases} dy_1/dt = -5y_1 - y_2 \\ dy_2/dt = y_1 - 3y_2 \end{cases}$

c) $\begin{cases} dy_1/dt = y_1 \\ dy_2/dt \quad y_1 + y_2 \end{cases}$

(19) Rotation matrix.
Show that the rotation matrix

$$R(\vartheta) = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$$

satisfies:

$$R(\vartheta) = \exp\left[\vartheta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right], \quad \vartheta \in \mathbb{R}.$$

↳ Use $e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$.