Homework 04: Asymptotic methods and 1d dynamical systems

1. Consider the differential equation

$$x^2 u''(x) + 2bxu'(x) + [b(b-1) - x^{-a}]u(x) = 0$$

with $a \in (0, +\infty)$ and $b \in \mathbb{R}$.

   (a) Show that the substitution $y(x) = x^b u(x)$ reduces it to the equivalent equation $x^{a+2} y''(x) = y(x)$.

   (b) Find the leading order asymptotic solution for $u(x)$ in the limit $x \to 0^+$.

2. Show that:

   (a) $f(x) = \sin x$ is Lipschitz continuous on $\mathbb{R}$ using the definition.

   (b) $f(x) = x^a$ with $a \in (0, 1)$ is not Lipschitz continuous on $(0, +\infty)$.

   (*Hint*: use the inequality $\forall x \in \mathbb{R} : |\sin x| \leq |x|$.)

3. Consider the dynamical system

$$\begin{cases} \mathrm{d}p/\mathrm{d}t = f(p,q) \\ \mathrm{d}q/\mathrm{d}t = g(p,q) \end{cases}$$

with

$$f(p,q) = \frac{\partial H(p,q)}{\partial q} \quad \text{and} \quad g(p,q) = -\frac{\partial H(p,q)}{\partial p}$$

with $H(p,q)$ the *Hamiltonian* of the system. Assume that (p_0, q_0) is a fixed point of the dynamical system and assume that it satisfies the condition

$$\forall (p,q) \in A - \{(p_0, q_0)\} : H(p,q) > H(p_0, q_0) \tag{1}$$

with A an open set that contains (p_0, q_0). Then, show that the fixed point (p_0, q_0) is Lyapunov stable.

4. Find all fixed points of the dynamical system $\mathrm{d}x/\mathrm{d}t = 2\sin x + \sin(2x)$ and determine their stability.

5. Show that the dynamical system $\mathrm{d}x/\mathrm{d}t = \mu x - \ln(1 + x)$ undergoes a transcritical bifurcation at a unique bifurcation point.