GENERALIZED FUNCTIONS

▼ Introduction - Motivation

In 1920, Paul Dirac introduced the Dirac delta function $\delta(x)$ for which he postulated the following properties

(a) $\forall x \in \mathbb{R} - \{0\}: \ \delta(x) = 0$,

(b) $\displaystyle\int_{-\infty}^{+\infty} \delta(x)\, dx = 1$, (c) $\displaystyle\int_{-\infty}^{+\infty} \delta(x-a) f(x) dx = f(a)$

No such function can be defined, but the idea was to introduce such exotic functions as a way of EXPANDING the space of available functions. Functions like $\delta(x)$ are called <u>generalized functions</u> or <u>distributions</u>. Rigorous theories formalizing the concept of distributions have been proposed by Schwarz, Mikusinski, Lighthill, and Sato. Below we will adopt and review the approach of Schwarz.

Generalized functions arise usually in the following contexts:

① <u>Probability theory</u>

Consider a <u>random variable</u> $x \in \mathbb{R}$ with <u>probability density function</u> $p(x)$ such that the probability $P(a \leq x \leq b)$ is given by

$$P(a \leq x \leq b) = \int_a^b p(x)\, dx$$

Obviously $p(x)$ will satisfy the normalization condition:

$$\int_{-\infty}^{+\infty} p(x)\,dx = P(x \in \mathbb{R}) = 1.$$

Note that x is continuously distributed on $\mathbb{R}$, therefore the probability that x is EXACTLY equal to some $a \in \mathbb{R}$ is zero:

$$P(x=a) = \int_a^a p(x)\,dx = 0$$

If we use the random variable x to evaluate $f(x)$, then the average value of $f(x)$ is given by:

$$\langle f(x) \rangle = \int_{-\infty}^{+\infty} f(x)\,p(x)\,dx$$

In this context, the Dirac delta function $\delta(x)$ can be thought of as the probability density function of a "random" variable such that $P(x=0)=1$. Then, indeed

$$\langle f(x) \rangle = \int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx = f(0).$$

In general, a discrete random variable with

$$\begin{cases} \forall k \in [n] : P(x=a_k) = p_k \\ \sum_{k\in[n]} p_k = 1 \end{cases}$$

can be represented with the probability density function

$$p(x) = \sum_{k\in[n]} p_k\,\delta(x-a_k)$$

such that the average of some evaluation $f(x)$ is:

$$\langle f(x)\rangle = \int_{-\infty}^{+\infty} f(x)\,p(x)\,dx = \int_{-\infty}^{+\infty} f(x)\left[\sum_{k\in[n]} p_k\,\delta(x-a_k)\right]dx =$$

$$= \sum_{k\in[n]}\left[p_k \int_{-\infty}^{+\infty} f(x)\,\delta(x-a_k)\,dx\right] =$$

$$= \sum_{k\in[n]} p_k\, f(a_k).$$

(2) Theory of Green's functions

Given a linear differential operator $L: C^n(\mathbb{R}) \to C^0(\mathbb{R})$, then the corresponding Green's function $G(x,t)$ can be found by solving the problem

$$L\,y(x) = \delta(x-t) \qquad (1)$$

Generalized functions are used to establish the theory of for calculating a particular solution $y_p(x)$ for the more general problem $Ly(x) = f(x)$, given the homogeneous solutions for the homogeneous problem $Ly(x) = 0$. This is explained in detail at the end of the lecture notes. chapter. The above problem given by Eq.(1) is the stepping stone for solving the general problem.

(3) Distributional derivatives

Functions with discontinuities or corner points cannot be differentiated in the usual sense. With the theory of distributions we can define a more general definition of the distributional derivative. Then non-differentiable functions will have a distributional derivative but it will be a generalized function, not a regular function.

For example, the function

$$f(x) = \begin{cases} 1 & , \text{ if } x \in (a, +\infty) \\ 0 & , \text{ if } x \in (-\infty, a] \end{cases}$$

is not differentiable at $x = a$, however it has a distributional derivative:

$$f'(x) = \delta(x-a)$$

▼ Schwarz definition of generalized functions

We define generalized functions via the following sequence of definitions.

<u>Def</u>: (<u>Compact support</u>)

Let $f: A \to \mathbb{R}$ be a function. We define the <u>support</u> $\mathrm{supp}(f)$ of A as:

$$\mathrm{supp}(f) = \{x \in A \mid f(x) \neq 0\}.$$

We say that

$$f \text{ has compact support} \Leftrightarrow \exists a, b \in A : \mathrm{supp}(f) \subseteq [a, b]$$

<u>Def</u>: (<u>Test functions</u>)

We define the space $\mathcal{D}(A)$ of <u>test functions</u> as:

$$\begin{aligned} \mathcal{D}(A) &= \{f \in C^{\infty}(A) \mid f \text{ has compact support}\} \\ &= \{f \in C^{\infty}(A) \mid \exists a, b \in \mathbb{R} : \mathrm{supp}(f) \subseteq [a, b]\} \end{aligned}$$

or equivalently, in terms of a belonging condition, as:

$$f \in \mathcal{D}(A) \Leftrightarrow \begin{cases} f \in C^{\infty}(A) \\ \exists a, b \in \mathbb{R} : \mathrm{supp}(f) \subseteq [a, b] \end{cases}$$

<u>Def</u>: (<u>Convergence in $\mathcal{D}(\mathbb{R})$</u>)

Consider a sequence $\varphi_1, \varphi_2, \ldots \in \mathcal{D}(\mathbb{R})$ of test functions and also a test function $\varphi \in \mathcal{D}(\mathbb{R})$. We say that

$$\varphi_n \xrightarrow{\mathcal{D}(\mathbb{R})} \varphi \Leftrightarrow \begin{cases} \exists a, b \in \mathbb{R} : (\mathrm{supp}(\varphi) \subseteq [a, b] \land \forall n \in \mathbb{N}^* : \mathrm{supp}(\varphi_n) \subseteq [a, b]) \\ \forall k \in \underline{\mathbb{N}} : \varphi_n^{(k)} \text{ converges uniformly to } \varphi^{(k)} \text{ on } [a, b]. \end{cases}$$

We recall from my Calculus 2 lecture notes that the definition of uniform convergence is:

$\varphi_n^{(k)}$ converges uniformly to $\varphi^{(k)}$ on $[a,b] \Leftrightarrow$

$\Leftrightarrow \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall x \in [a,b] : \forall n \in \mathbb{N}^* - [n_0] : |\varphi_n^{(k)}(x) - \varphi^{(k)}(x)| < \varepsilon$

For the next definitions we define:

a) $\mathrm{Seq}(A)$ as the set of all sequences $a : \mathbb{N}^* \to A$

b) $I(\mathbb{R})$ as the set of all <u>locally integrable</u> functions as follows:

$f \in I(\mathbb{R}) \Leftrightarrow \forall a,b \in \mathbb{R} : (a < b \Rightarrow f \text{ integrable on } [a,b])$

We can now give the formal definition for a generalized function (or distribution).

<u>Def</u>: (Generalized function or distribution)

A functional $F : \mathcal{X}(\mathbb{R}) \to \mathbb{C}$ is a <u>generalized function (or distribution)</u> if and only if it satisfies the following conditions:

(a) $\forall \lambda, \mu \in \mathbb{C} : \forall \varphi, \psi \in \mathcal{X}(\mathbb{R}) : F(\lambda\varphi + \mu\psi) = \lambda F(\varphi) + \mu F(\psi)$

(b) $\forall \varphi \in \mathrm{Seq}(\mathcal{X}(\mathbb{R})) : \forall \psi \in \mathcal{X}(\mathbb{R}) : (\varphi_n \xrightarrow{\mathcal{X}(\mathbb{R})} \psi \Rightarrow \lim\limits_{n \in \mathbb{N}} F(\varphi_n) = F(\psi))$

<u>notation:</u>

(a) $\mathcal{X}'(\mathbb{R})$ is the set of all distributions $F : \mathcal{X}(\mathbb{R}) \to \mathbb{C}$

(b) By convention, we write $(F, \varphi) = F(\varphi)$.

<u>Remark:</u>

Given any integrable function $f \in I(\mathbb{R})$ we define the distribution $F \in \mathcal{X}'(\mathbb{R})$ generated by f as:

$$\forall \varphi \in \mathcal{X}(\mathbb{R}) : (F, \varphi) = \int_{-\infty}^{+\infty} f(x)\varphi(x)\,dx$$

As a result, every locally integrable function f can be also thought of a distribution, and such trivial distributions are called <u>regular distributions</u>. One can prove that the distribution F defined above satisfies the formal definition of a distribution. (proof omitted). This motivates the following definition:

<u>Def</u>: Consider a distribution $F \in \mathcal{X}'(\mathbb{R})$. We say that

a) F is a regular distribution $\iff \exists f \in I(\mathbb{R}) : \forall \varphi \in \mathcal{X}(\mathbb{R}) : (F, \varphi) = \int_{-\infty}^{+\infty} f(x)\varphi(x)dx$

b) F is a singular distribution $\iff$ F is NOT a regular distribution

Singular distributions can be defined as limits of regular distributions. For example, given a sequence $f \in \mathrm{Seq}(I(\mathbb{R}))$ of locally integrable functions, we can define a possibly singular distribution $F \in \mathcal{X}'(\mathbb{R})$ according to:

$$\forall \varphi \in \mathcal{X}(\mathbb{R}) : (F, \varphi) = \lim_{n \in \mathbb{N}^*} \int_{-\infty}^{+\infty} f_n(x)\varphi(x)dx$$

as long as the limit exists. If F is indeed a singular distribution we may still introduce a fictitious function $F(x)$ and claim that

$$\forall \varphi \in \mathcal{X}(\mathbb{R}) : \int_{-\infty}^{+\infty} F(x)\varphi(x)dx \equiv (F, \varphi) = \lim_{n \in \mathbb{N}^*} \int_{-\infty}^{+\infty} f_n(x)\varphi(x)dx$$

$F(x)$ is not an actual function, in the usual sense, but it can be interpreted as a singular limit of the function sequence f_n.

We may then say that $f_n \xmapsto{\mathcal{X}(\mathbb{R})} F$, in the sense of distributions. The precise definition of the above statement is:

Def: Let $f \in Seq(\mathcal{I}(\mathbb{R}))$ be a sequence of locally integrable functions and let $F \in \mathcal{X}'(\mathbb{R})$ be a distribution. We say that
$$f_n \xmapsto{\mathcal{X}(\mathbb{R})} F \Leftrightarrow \forall k \in \mathbb{N} : \forall \varphi \in \mathcal{X}(\mathbb{R}) : (F, \varphi^{(k)}) = \lim_{n \in \mathbb{N}^*} \int_{-\infty}^{+\infty} f_n(x) \varphi^{(k)}(x) dx$$

and we write:
$$\forall \varphi \in \mathcal{X}(\mathbb{R}) : \int_{-\infty}^{+\infty} F(x) \varphi(x) dx \equiv (F, \varphi)$$

Given F defined as $f_n \xmapsto{\mathcal{X}(\mathbb{R})} F$, we also define integrals over an $[a,b]$ interval with $a, b \in \mathbb{R}$ as follows:

$$\forall \varphi \in \mathcal{X}(\mathbb{R}) : \int_a^b F(x) \varphi(x) dx = \lim_{n \in \mathbb{N}^*} \int_a^b f_n(x) \varphi(x) dx$$

The Dirac delta function

The Dirac delta function is a singular distribution that is defined as a limit of regular distributions as follows:

- 1 We define a sequence of Gaussian distributions $\Delta_n(x)$ as:

$$\forall n \in \mathbb{N}^* : \forall x \in \mathbb{R} : \Delta_n(x) = \sqrt{\frac{n}{2\pi}} \exp(-nx^2)$$

- 2 The Dirac delta function is a singular distribution defined as

$$\Delta_n(x) \xrightarrow{\mathcal{D}(\mathbb{R})} \delta(x)$$

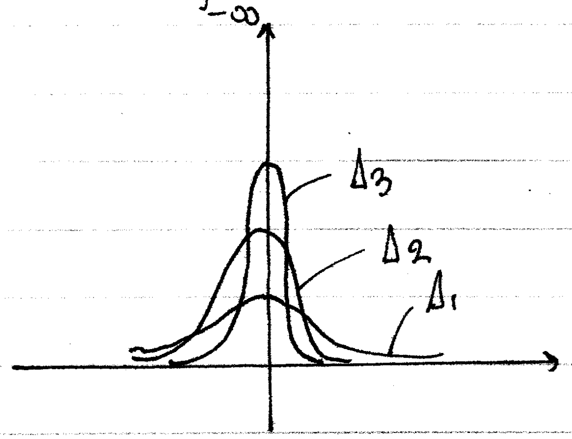
- 3 Now, we can show that

$$\forall \varphi \in \mathcal{D}(\mathbb{R}) : \int_{-\infty}^{+\infty} \delta(x) \varphi(x)\, dx = \lim_{n \in \mathbb{N}^*} \int_{-\infty}^{+\infty} \Delta_n(x) \varphi(x)\, dx = \varphi(0)$$

Geometric interpretation

The function $\Delta_n(x)$ is a bell-shaped function. With increasing n, the peak becomes taller and the graph becomes narrower such that the following constraint is always satisfied:

$$\forall n \in \mathbb{N}^* : \int_{-\infty}^{+\infty} \Delta_n(x)\, dx = 1$$



In probability theory these functions are known as Gaussian distributions. As $n \to +\infty$, we obtain the Dirac delta function that can be visualised as a spike located at $x=0$ with infinitesimal width and infinite height.

▸ Adjusted delta functions

We can likewise define the following singular distributions:

a) $\Delta_n(x-a) \xrightarrow{\mathcal{D}(\mathbb{R})} \delta(x-a)$ (shifting)

b) $\Delta_n(ax) \xrightarrow{\mathcal{D}(\mathbb{R})} \delta(ax)$ (dilation)

with $a \in \mathbb{R}-\{0\}$. Then, we can show that

$$\boxed{\begin{aligned} &\forall \varphi \in \mathcal{D}(\mathbb{R}): \int_{-\infty}^{+\infty} \delta(x-a)\varphi(x)\,dx = \varphi(a) \\ &\forall \varphi \in \mathcal{D}(\mathbb{R}): \int_{-\infty}^{+\infty} \delta(ax)\varphi(x)\,dx = \frac{\varphi(0)}{|a|} \end{aligned}}$$

Shifting and dilation can be combined to define $\delta(ax+b)$ with $a \in \mathbb{R}-\{0\}$ and $b \in \mathbb{R}$ via:

$$\Delta_n(ax+b) \xrightarrow{\mathcal{D}(\mathbb{R})} \delta(ax+b)$$

Then, it follows that

$$\boxed{\forall \varphi \in \mathcal{D}(\mathbb{R}): \int_{-\infty}^{+\infty} \delta(ax+b)\varphi(x)\,dx = \frac{\varphi(-b/a)}{|a|}}$$

EXAMPLES

a) Evaluate $I = \int_{-\infty}^{+\infty} \delta(6x-\pi)\cos^2(x+\pi/4)\,dx$

Solution

$$I = \int_{-\infty}^{+\infty} \delta(6x-\pi)\cos^2(x+\pi/4)\,dx = \frac{1}{|6|} \cos^2(\pi/6+\pi/4) =$$

$$= \frac{1}{6}\left[\cos(\pi/6)\cos(\pi/4) - \sin(\pi/6)\sin(\pi/4)\right]^2 =$$

$$= \frac{1}{6}\left[\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2}\right]^2 = \frac{1}{6}\left(\frac{\sqrt{2}}{2}\right)^2\left(\frac{\sqrt{3}-1}{2}\right)^2 =$$

$$= \frac{1}{6}\,\frac{1}{2}\,\frac{(\sqrt{3})^2 - 2\sqrt{3}+1}{4} = \frac{3-2\sqrt{3}+1}{48} = \frac{4-2\sqrt{3}}{48} =$$

$$= \frac{2-\sqrt{3}}{24}$$

b) Evaluate $I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; \cos x \cos y \, \delta(6x-\pi)\,\delta(4y-\pi)$

Solution

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; \cos x \cos y \, \delta(6x-\pi)\,\delta(4y-\pi) =$$

$$= \int_{-\infty}^{+\infty} dx \; \cos x \, \delta(6x-\pi)\left[\int_{-\infty}^{+\infty} dy \; \cos y \, \delta(4y-\pi)\right] =$$

$$= \left[\int_{-\infty}^{+\infty} dy \cos y \, \delta(4y-\pi) \right] \left[\int_{-\infty}^{+\infty} dx \cos x \, \delta(6x-\pi) \right]$$

$$= \left[\frac{\cos(\pi/4)}{|4|} \right] \left[\frac{\cos(\pi/6)}{|6|} \right] = \frac{1}{24} \cos(\pi/4) \cos(\pi/6) =$$

$$= \frac{1}{24} \frac{\sqrt{2}}{2} \frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{96}$$

EXERCISES

① Evaluate the following integrals.

a) $I = \int_{-\infty}^{+\infty} \cos^2 x \, \delta(x - \pi/6) \, dx$

b) $I = \int_{-\infty}^{+\infty} \delta(3x) \, \mathrm{Arccos}(x) \, dx$

c) $I = \int_{-\infty}^{+\infty} \delta(3x - \pi) \sin(x + \pi/4) \, dx$

d) $I = \int_{-\infty}^{+\infty} \delta(2 - 5x)(x^2 + 3x)^2 \, dx$

e) $I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; xy(x-y) \, \delta(2x+3) \, \delta(y-2)$

f) $I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; x^2(x+y) \, \delta(2x - a) \, \delta(3y + a - 2)$

Operations with distributions

Def: (Equality of distributions)
Let $F, G \in \mathcal{D}'(\mathbb{R})$ be two distributions. We say that
$F = G \iff \forall \varphi \in \mathcal{D}(\mathbb{R}) : (F, \varphi) = (G, \varphi)$

Given the definition of equality of distributions, we can now introduce algebra with distributions as follows:

Def: (Addition of distributions)
Let $F, G \in \mathcal{D}'(\mathbb{R})$ be two distributions. We say that
We define the distribution $(F+G) \in \mathcal{D}'(\mathbb{R})$ as:
$\forall \varphi \in \mathcal{D}(\mathbb{R}) : (F+G, \varphi) = (F, \varphi) + (G, \varphi)$

Note that there are many technical difficulties with respect to defining multiplication of distributions. However, we may define multiplication of a smooth function with a distribution

Def: (Function-distribution multiplication)
Let $F \in \mathcal{D}'(\mathbb{R})$ be a distribution and let $g \in C^\infty(\mathbb{R})$ be a smooth function. We define the distribution $(gF) \in \mathcal{D}'(\mathbb{R})$ as:
$\forall \varphi \in \mathcal{D}(\mathbb{R}) : (gF, \varphi) = (F, g\varphi)$

Writing F, G as generalized functions $F(x), G(x)$, the above definitions can be rewritten equivalently as:

(a) For ~~no~~ addition of distributions:

$$\forall \varphi \in \mathcal{D}(\mathbb{R}): (F+G, \varphi) = \int_{-\infty}^{+\infty} [F(x)+G(x)]\varphi(x)\,dx =$$

$$= \int_{-\infty}^{+\infty} F(x)\varphi(x)\,dx + \int_{-\infty}^{+\infty} G(x)\varphi(x)\,dx$$

$$= (F, \varphi) + (G, \varphi)$$

(b) For function-distribution multiplication

$$\forall \varphi \in \mathcal{D}(\mathbb{R}): (gF, \varphi) = \int_{-\infty}^{+\infty} [g(x)F(x)]\varphi(x)\,dx =$$

$$= \int_{-\infty}^{+\infty} F(x)[g(x)\varphi(x)]\,dx = (F, g\varphi)$$

- Multiplying a number $\lambda \in \mathbb{R}$ with a distribution $F \in \mathcal{D}'(\mathbb{R})$ is a special case of the function-distribution product gF using the function $\forall x \in \mathbb{R}: g(x) = \lambda$.

<u>Prop:</u> $\boxed{\forall \lambda \in \mathbb{R}: \forall \varphi \in \mathcal{D}(\mathbb{R}): (\lambda F, \varphi) = \lambda (F, \varphi)}$

<u>Proof</u>

Let $\lambda \in \mathbb{R}$ and $\varphi \in \mathcal{D}(\mathbb{R})$ be given. Then

$(\lambda F, \varphi) = (F, \lambda\varphi)$ [definition of function-distribution product]

$= \lambda (F, \varphi)$ [definition of distribution-linearity of F]

It follows that $\forall \lambda \in \mathbb{R}: \forall \varphi \in \mathcal{D}(\mathbb{R}): (\lambda F, \varphi) = \lambda (F, \varphi)$

EXAMPLES

a) Use the definition of $\delta(ax)$:

$$\Delta_n(ax) \xrightarrow{\mathcal{D}(\mathbb{R})} \delta(ax)$$

to show that $\delta(ax) = \frac{1}{|a|}\delta(x)$, for $a \neq 0$.

Solution

▸ It is sufficient to show that

$$\forall \varphi \in \mathcal{D}(\mathbb{R}): \int_{-\infty}^{+\infty} \delta(ax)\varphi(x)\,dx = \int_{-\infty}^{+\infty} \left[\frac{1}{|a|}\delta(x)\right]\varphi(x)\,dx$$

Let $\varphi \in \mathcal{D}(\mathbb{R})$ and let $n \in \mathbb{N}^*$ be given. Then:

$$(\Delta_n(ax), \varphi) = \int_{-\infty}^{+\infty} \Delta_n(ax)\varphi(x)\,dx = \int_{-\infty}^{+\infty} \Delta_n(|a|x)\varphi(x)\,dx$$

because

$$\Delta_n(ax) = \sqrt{\frac{n}{2\pi}}\exp(-n(ax)^2) = \sqrt{\frac{n}{2\pi}}\exp(-na^2x^2)$$
$$= \sqrt{\frac{n}{2\pi}}\exp(-n|a|^2x^2) = \sqrt{\frac{n}{2\pi}}\exp(-n(|a|x)^2)$$
$$= \Delta_n(|a|x),\ \forall x \in \mathbb{R}.$$

Define $y = |a|x$. Then $dy = |a|dx \Leftrightarrow dx = (1/|a|)dy$ and

$x \to -\infty \Rightarrow y \to -\infty$

$x \to +\infty \Rightarrow y \to +\infty$

and with change of variables:

$$(\Delta_n(ax), \varphi) = \int_{-\infty}^{+\infty} \Delta_n(y)\varphi\left(\frac{y}{|a|}\right)\frac{1}{|a|}\,dy = \frac{1}{|a|}\int_{-\infty}^{+\infty} \Delta_n(y)\varphi\left(\frac{y}{|a|}\right)dy$$

and therefore:

$$\int_{-\infty}^{+\infty} \delta(ax)\varphi(x)\,dx = \lim_{n\in\mathbb{N}^*} (\Delta_n(ax),\varphi) = \lim_{n\in\mathbb{N}^*}\left[\frac{1}{|a|}\int_{-\infty}^{+\infty}\Delta_n(y)\varphi\left(\frac{y}{|a|}\right)dy\right]$$

$$= \frac{1}{|a|}\lim_{n\in\mathbb{N}^*}\int_{-\infty}^{+\infty}\Delta_n(y)\varphi\left(\frac{y}{|a|}\right)dy =$$

$$= \frac{1}{|a|}\int_{-\infty}^{+\infty}\delta(y)\varphi\left(\frac{y}{|a|}\right)dy = \frac{1}{|a|}\varphi\left(\frac{0}{|a|}\right)$$

$$= \frac{\varphi(0)}{|a|} = \frac{1}{|a|}\int_{-\infty}^{+\infty}\delta(x)\varphi(x)\,dx =$$

$$= \int_{-\infty}^{+\infty}\left[\frac{1}{|a|}\delta(x)\right]\varphi(x)\,dx$$

It follows that

$$\forall\varphi\in\mathcal{D}(\mathbb{R}): \int_{-\infty}^{+\infty}\delta(ax)\varphi(x)\,dx = \int_{-\infty}^{+\infty}\left[\frac{1}{|a|}\delta(x)\right]\varphi(x)\,dx$$

$$\Rightarrow \delta(ax) = \frac{1}{|a|}\delta(x).$$

Derivative of distributions

Following the example of the Dirac delta function, we introduce the following general concept of the derivative of a distribution as follows:

Def: Let $F \in \mathcal{D}'(\mathbb{R})$ be a distribution and let $k \in \mathbb{N}^*$. We define the k^{th} derivative $F^{(k)}$ of F as follows:
$$\forall \varphi \in \mathcal{D}(\mathbb{R}) : (F^{(k)}, \varphi) = (-1)^k (F, \varphi^{(k)})$$

Representing F in terms of a generalized function $F(x)$, the above definition can be equivalently be rewritten as

$$\forall \varphi \in \mathcal{D}(\mathbb{R}) : \int_{-\infty}^{+\infty} F^{(k)}(x) \varphi(x) dx = (-1)^k \int_{-\infty}^{+\infty} F(x) \varphi^{(k)}(x) dx$$

To ensure the self-consistency of this definition, we have to ensure that if F is a regular distribution, in which case $F(x)$ is an ordinary function, the above equation holds. Using integration by parts and proof by induction, we can show that indeed it holds. For singular distributions defined via a sequence of locally integrable functions, we can show that

Prop: Let $f \in \mathrm{Seq}(C^\infty(\mathbb{R}))$ be a sequence of locally integrable functions and let $F \in \mathcal{D}'(\mathbb{R})$ be a distribution. Then:
$$\forall k \in \mathbb{N}^* : \left(f_n \xrightarrow{\mathcal{D}(\mathbb{R})} F \Rightarrow f_n^{(k)} \xrightarrow{\mathcal{D}(\mathbb{R})} F^{(k)} \right)$$

This proposition ensures the self-consistency between the above definition of distributional derivative and the standard definition of the derivative of a function from Calculus.

▸ <u>Properties of distributional derivatives.</u>

Distributional derivatives continue to satisfy some standard differentiation rules: addition rule, product rule, scalar product rule:

$$\boxed{\begin{array}{l} \forall F, G \in \mathcal{D}'(\mathbb{R}) : (F(x)+G(x))' = F'(x)+G'(x) \\ \forall g \in C^{\infty}(\mathbb{R}) : \forall F \in \mathcal{D}'(\mathbb{R}) : (g(x)F(x))' = g'(x)F(x)+g(x)F'(x) \\ \forall \lambda \in \mathbb{R} : \forall F \in \mathcal{D}'(\mathbb{R}) : (\lambda F(x))' = \lambda F'(x). \end{array}}$$

<u>Proof</u>

a) Let $F, G \in \mathcal{D}'(\mathbb{R})$ be given, and let $\varphi \in \mathcal{D}(\mathbb{R})$ be given. Then,

$$\begin{aligned} ((F+G)', \varphi) &= (-1)(F+G, \varphi') = \\ &= (-1)[(F, \varphi') + (G, \varphi')] = \\ &= (-1)(F, \varphi') + (-1)(G, \varphi') = \\ &= (F', \varphi) + (G', \varphi) = (F'+G', \varphi), \ \forall \varphi \in \mathcal{D}(\mathbb{R}) \end{aligned}$$

It follows that $(F+G)' = F'+G'$.

b) Let $F \in \mathcal{D}'(\mathbb{R})$ and $g \in C^{\infty}(\mathbb{R})$ be given. Then

$$\begin{aligned} \forall \varphi \in \mathcal{D}(\mathbb{R}) : ((gF)', \varphi) &= (-1)(gF, \varphi') = (-1)(F, g\varphi') = \\ &= (-1)(F, (g\varphi)' - g'\varphi) = \end{aligned}$$

$$= (-1)(F, (g\varphi)') - (-1)(F, g'\varphi) =$$
$$= (F', g\varphi) - (-1)(g'F, \varphi) =$$
$$= (gF', \varphi) + (g'F, \varphi) = (gF' + g'F, \varphi)$$
$$\Rightarrow (gF)' = g'F + gF'.$$

c) Let $\lambda \in \mathbb{R}$ and $F \in \mathcal{D}'(\mathbb{R})$ be given. Then

$$((\lambda F)', \varphi) = (-1)(\lambda F, \varphi') = (-1)(F, \lambda\varphi') = (-1)\lambda(F, \varphi')$$
$$= \lambda(F', \varphi) = (F', \lambda\varphi) = (\lambda F', \varphi), \ \forall \varphi \in \mathcal{D}(\mathbb{R}) \Rightarrow$$
$$\Rightarrow (\lambda F)' = \lambda F'.$$

Derivatives of Dirac delta functions

Derivatives of the Dirac delta function are defined via the previously stated distributional derivative definition which immediately yields:

$$\boxed{\begin{aligned}\forall k \in \mathbb{N}^* : \forall \varphi \in \mathcal{D}(\mathbb{R}) : (\delta^{(k)}(x), \varphi(x)) &= \int_{-\infty}^{+\infty} \delta^{(k)}(x)\varphi(x)\,dx \\ &= (-1)^k \int_{-\infty}^{+\infty} \delta(x)\varphi^{(k)}(x)\,dx = (-1)^k \varphi^{(k)}(0)\end{aligned}}$$

For the shifted delta function k^{th} derivative $\delta^{(k)}(x-a)$ we have:

$$\boxed{\begin{aligned}\forall k \in \mathbb{N}^* : \forall \varphi \in \mathcal{D}(\mathbb{R}) : (\delta^{(k)}(x-a), \varphi(x)) &= \int_{-\infty}^{+\infty} \delta^{(k)}(x-a)\varphi(x)\,dx \\ &= (-1)^k \int_{-\infty}^{+\infty} \delta(x-a)\varphi^{(k)}(x)\,dx \\ &= (-1)^k \varphi^{(k)}(a)\end{aligned}}$$

Using the previously defined Gaussian distributions and the theory of distributional derivatives, in general we have:

$$\boxed{\forall k \in \mathbb{N}^* : \Delta_n^{(k)}(ax+b) \xrightarrow{\mathcal{D}(\mathbb{R})} \delta^{(k)}(ax+b)}$$

with $a \in \mathbb{R} - \{0\}$ and $b \in \mathbb{R}$.
Since we have previously shown that

$$\delta(ax+b) = \frac{1}{|a|} \delta\left(x + \frac{b}{a}\right)$$

differentiating both sides with a distributional derivative and using the scalar-multiplication rule gives the following more general result:

$$\boxed{\forall k \in \mathbb{N}: \ \delta^{(k)}(ax+b) = \frac{1}{|a|} \delta^{(k)}\left(x + \frac{b}{a}\right)}$$

EXAMPLE

Evaluate the integral $I = \int_{-\infty}^{+\infty} x^2 e^x \delta''(x-1)\,dx$

Solution

$$I = \int_{-\infty}^{+\infty} x^2 e^x \delta''(x-1)\,dx = (-1)^2 \int_{-\infty}^{+\infty} (x^2 e^x)'' \delta(x-1)\,dx$$

We note that

$$\begin{aligned}(x^2e^x)'' &= \left((x^2)'e^x + x^2(e^x)'\right)' = \left(2xe^x + x^2e^x\right)' \\ &= \left[e^x(2x+x^2)\right]' = (e^x)'(2x+x^2) + e^x(2x+x^2)' = \\ &= e^x(2x+x^2) + e^x(2+2x) = \\ &= e^x(2x+x^2+2+2x) = e^x(x^2+4x+2).\end{aligned}$$

and therefore

$$I = \int_{-\infty}^{+\infty} e^x(x^2+4x+2)\,\delta(x-1)\,dx = e^1(1^2+4\cdot 1+2) = 7e.$$

EXERCISES

② Evaluate the following integrals:

a) $I = \int_{-\infty}^{+\infty} x(x^2-1)^3 \delta''(x-1)\,dx$

b) $I = \int_{-\infty}^{+\infty} x^2 \exp(-x^2)\left[\delta'(x-2)+\delta''(x-2)\right]dx$

c) $I = \int_{-\infty}^{+\infty} \mathrm{Arctan}(x)\,\delta''(x-\sqrt{2})\,dx$

d) $I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \;\; xy\exp(xy)\,\delta'(x-1)\,\delta'(y-1)$

e) $I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \;\; \sin(xy)\,\delta''(x-\pi/4)\left[\delta'(y-1)+\delta''(y-1)\right]$

▼ Algebra with delta functions

Expressions with function-distribution products involving the Dirac delta functions or their derivatives can be simplified using the following fundamental properties.

•1 We begin by showing that:

$$\boxed{\forall f \in C^\infty(\mathbb{R}): \ f(x)\delta(x-a) = f(a)\delta(x-a)}$$

•2 Taking the distributional derivative on both sides and using the distributional differentiation rules gives the following identities:

$$\boxed{\begin{aligned}
&\forall f \in C^\infty(\mathbb{R}): \ f(x)\delta'(x-a) = f(a)\delta'(x-a) - f'(a)\delta(x-a)\\
&\forall f \in C^\infty(\mathbb{R}): \ f(x)\delta''(x-a) = f(a)\delta''(x-a) - 2f'(a)\delta'(x-a) + f''(a)\delta(x-a)\\
&\forall f \in C^\infty(\mathbb{R}): \ f(x)\delta'''(x-a) = f(a)\delta'''(x-a) - 3f'(a)\delta''(x-a)\\
&\qquad\qquad\qquad\qquad\qquad + 3f''(a)\delta'(x-a) - f'''(a)\delta(x-a)
\end{aligned}}$$

•3 The general result, established using proof by induction, is given by

$$\boxed{\begin{aligned}
&\forall f \in C^\infty(\mathbb{R}): \ f(x)\,\delta^{(n)}(x-a) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f^{(k)}(a)\,\delta^{(n-k)}(x-a)\\
&\text{with } \forall n,k \in \mathbb{N}: \ \binom{n}{k} = \frac{n!}{k!\,(n-k)!}
\end{aligned}}$$

Recall that the ~~Pascal~~ binomial coefficients satisfy the Pascal identities:

$$\boxed{\begin{array}{l} \forall n \in \mathbb{N}^* : \binom{n}{0} = \binom{n}{n} = 1 \\ \forall n, k \in \mathbb{N}^* : \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \end{array}}$$

and can be calculated via the Pascal triangle, where each coefficient is equal to the sum of the coefficient directly above it plus the coefficient located above and one step to the left.

$n=1$:	1	1				
$n=2$:	1	2	1			
$n=3$:	1	3	3	1		
$n=4$:	1	4	6	4	1	
$n=5$:	1	5	10	10	5	1

We now state the proof for the main result.

<u>Proof</u>

We use proof by induction. For $n=0$, let $f \in C^\infty(\mathbb{R})$ and $\varphi \in \mathcal{D}(\mathbb{R})$ be given. Then:

$$\begin{aligned} (f(x)\delta(x-a), \varphi(x)) &= (\delta(x-a), f(x)\varphi(x)) = f(a)\varphi(a) = \\ &= f(a)(\delta(x-a), \varphi(x)) = (\delta(x-a), f(a)\varphi(x)) \\ &= (f(a)\delta(x-a), \varphi(x)) \end{aligned}$$

It follows that

$$\forall f \in C^\infty(\mathbb{R}) : \forall \varphi \in \mathcal{D}(\mathbb{R}) : (f(x)\delta(x-a), \varphi(x)) = (f(a)\delta(x-a), \varphi(x))$$

$$\Rightarrow \forall f \in C^\infty(\mathbb{R}) : f(x)\delta(x-a) = f(a)\delta(x-a).$$

For $n=m$, we assume that

$$f(x)\,\delta^{(m)}(x-a) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f^{(k)}(a)\,\delta^{(m-k)}(x-a)$$

For $n=m+1$, we have:

$$f(x)\,\delta^{(m+1)}(x-a) = (d/dx)[f(x)\,\delta^{(m)}(x-a)] - f'(x)\,\delta^{(m)}(x-a) =$$

$$= \frac{d}{dx}\left[\sum_{k=0}^{m} (-1)^k \binom{m}{k} f^{(k)}(a)\,\delta^{(m-k)}(x-a)\right]$$

$$- \sum_{k=0}^{m} (-1)^k \binom{m}{k} f^{(k+1)}(a)\,\delta^{(m-k)}(x-a) =$$

$$= \sum_{k=0}^{m} (-1)^k \binom{m}{k} f^{(k)}(a)\,\delta^{(m-k+1)}(x-a)$$

$$- \sum_{k=0}^{m} (-1)^k \binom{m}{k} f^{(k+1)}(a)\,\delta^{(m-k)}(x-a) =$$

$$= f(a)\,\delta^{(m+1)}(x-a) + \sum_{k=1}^{m} (-1)^k \binom{m}{k} f^{(k)}(a)\,\delta^{(m-k+1)}(x-a)$$

$$- \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} f^{(k+1)}(a)\,\delta^{(m-k)}(x-a) - (-1)^m f^{(m+1)}(a)\,\delta(x-a)$$

$$= f(a)\,\delta^{(m+1)}(x-a) + \sum_{k=1}^{m} (-1)^k \binom{m}{k} f^{(k)}(a)\,\delta^{(m-k+1)}(x-a)$$

$$- \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k-1} f^{(k)}(a)\,\delta^{(m-k+1)}(x-a) - (-1)^m f^{(m+1)}(a)\,\delta(x-a)$$

$$= f(a)\,\delta^{(m+1)}(x-a) + \sum_{k=1}^{m} (-1)^k \left[\binom{m}{k} + \binom{m}{k-1}\right] f^{(k)}(a)\,\delta^{(m+1-k)}(x-a)$$

$$- (-1)^m f^{(m+1)}(a)\,\delta(x-a) =$$

$$= f(a)\,\delta^{(m+1)}(x-a) + \sum_{k=1}^{m} (-1)^k \binom{m+1}{k} f^{(k)}(a)\,\delta^{(m+1-k)}(x-a)$$

$$+ (-1)^m f^{(m+1)}(a)\,\delta(x-a) =$$

$$= \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} f^{(k)}(a)\,\delta^{(m+1-k)}(x-a)$$

By induction, this concludes the argument. □

EXAMPLE

Simplify the generalized function

$f(x) = x^3 e^x [\delta''(x-1) + 3\delta'(x-1)]$

Solution

Define $g(x) = x^3 e^x$ and note that

$$g'(x) = (x^3)' e^x + x^3 (e^x)' = 3x^2 e^x + x^3 e^x = (x^3 + 3x^2) e^x$$

$$\begin{aligned} g''(x) &= (x^3+3x^2)' e^x + (x^3+3x^2)(e^x)' = \\ &= (3x^2+6x) e^x + (x^3+3x^2) e^x = \\ &= (3x^2+6x+x^3+3x^2) e^x = (x^3+6x^2+6x) e^x \end{aligned}$$

and it follows that

$$g(1) = 1^3 e^1 = e$$

$$g'(1) = (1^3 + 3\cdot 1^2) e^1 = (1+3)e = 4e$$

$$g''(1) = (1^3 + 6\cdot 1^2 + 6\cdot 1) e^1 = (1+6+6)e = 13e$$

Consequently, $f(x)$ simplifies to:

$$\begin{aligned} f(x) &= g(x)\delta''(x-1) + 3g(x)\delta'(x-1) \\ &= [g(1)\delta''(x-1) - 2g'(1)\delta'(x-1) + g''(1)\delta(x-1)] + 3[g(1)\delta'(x-1) - g'(1)\delta(x-1)] \\ &= g(1)\delta''(x-1) + [-2g'(1) + 3g(1)]\delta'(x-1) + [g''(1) - 3g'(1)]\delta(x-1) \\ &= e\delta''(x-1) + [-2(4e)+3e]\delta'(x-1) + [13e - 3(4e)]\delta(x-1) \\ &= e\delta''(x-1) + (-8e+3e)\delta'(x-1) + (13e-12e)\delta(x-1) \\ &= e\delta''(x-1) - 5e\delta'(x-1) + e\delta(x-1). \end{aligned}$$

EXERCISE

3 Simplify the following generalized functions.

a) $f(x) = (x^2 \sin x)\,\delta(x-\pi/4)$

b) $f(x) = \sin^3(x)\,\delta'(x-\pi/3)$

c) $f(x) = x^2 e^x\,\delta'(x-1)$

d) $f(x) = (x+1)^2 e^x\,\delta''(x)$

e) $f(x) = (x-2)^3 (2x-1)^2\,\delta''(x-3)$

f) $f(x) = 3x\sqrt{x^2+1}\,[2\delta'(x-1)+\delta''(x-2)]$

g) $f(x) = x\,\mathrm{Arctan}(x)\,[2\delta'(x)-\delta(x-1)]$

h) $f(x) = x^n\,\delta^{(m)}(x)$ with $n, m \in \mathbb{N}^*$

i) $f(x) = (\sin x)\,\delta^{(n)}(x)$

The Heaviside distribution

The Heaviside distribution is an example of a regular distribution. Given the previously defined sequence of gaussian distributions $\Delta_n(x)$ we define

$$\forall x \in \mathbb{R}: \; E_n(x) = \int_{-\infty}^{x} \Delta_n(t)\,dt = \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{x} \exp(-nt^2)\,dt$$

The Heaviside function $H(x)$ is a regular distribution defined as:

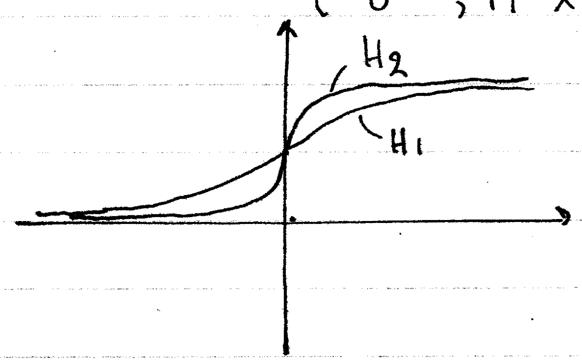
$$E_n(x) \xrightarrow{\mathcal{D}(\mathbb{R})} H(x)$$

and we can show that

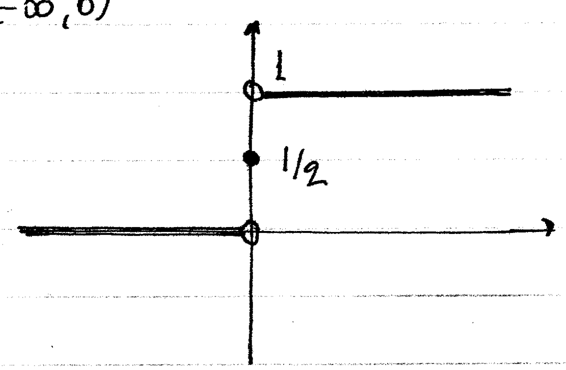
$$\forall \varphi \in \mathcal{D}(\mathbb{R}): \quad (H(x), \varphi(x)) = \int_{-\infty}^{+\infty} H(x)\varphi(x)\,dx = \int_{0}^{+\infty} \varphi(x)\,dx$$

with the representation function $H(x)$ given by

$$\forall x \in \mathbb{R}: \; H(x) = \begin{cases} 1 & , \text{if } x \in (0, +\infty) \\ 1/2 & , \text{if } x = 0 \\ 0 & , \text{if } x \in (-\infty, 0) \end{cases}$$



Graph of $E_1(x)$, $E_2(x)$, ...



Graph of $H(x)$.

- $H(x)$ is not differentiable on $x=0$. However, since $(d/dx)\,En(x) = \Delta n(x)$, it follows that, in the sense of distributions, the derivative of the distribution generated by $H(x)$ satisfies:

$$\boxed{(d/dx)\,H(x) = \delta(x)}$$

- Shifted Heaviside distributions can be defined via

$$En(x-a) \xrightarrow{\mathcal{X}(\mathbb{R})} H(x-a)\ , \quad \forall a \in \mathbb{R}$$

Applied on a test function, the Heaviside distribution gives:

$$\boxed{\forall \varphi \in \mathcal{A}(\mathbb{R}) : (H(x-a), \varphi(x)) = \int_a^{+\infty} \varphi(x)\,dx}$$

The distributional derivative of $H(x-a)$ is given by:

$$\boxed{(d/dx)\,H(x-a) = \delta(x-a)}$$

- <u>Distributional derivative of piecewise discontinuous functions</u>

We consider a potentially piecewise discontinuous function of the form

$$f(x) = \begin{cases} f_0(x) & , \text{ if } x < a_1 \\ f_1(x) & , \text{ if } a_1 < x < a_2 \\ f_2(x) & , \text{ if } a_2 < x < a_3 \\ \vdots & \\ f_n(x) & , \text{ if } a_n < x \end{cases}$$

This function may not necessarily be differentiable or even continuous at the points $a_1, a_2, \ldots, a_n$. However, it induces a distribution that can be written in terms of the Heaviside distribution as:

$$
\begin{aligned}
f(x) &= f_0(x) + [f_1(x) - f_0(x)]H(x-a_1) + [f_2(x) - f_1(x)]H(x-a_2) + \\
&\qquad + \cdots + [f_n(x) - f_{n-1}(x)]H(x-a_n) \\
&= f_0(x) + \sum_{k=1}^{n} [f_k(x) - f_{k-1}(x)]H(x-a_k)
\end{aligned}
$$

It should be noted that the values of the original function at the points $a_1, a_2, \ldots, a_n$ have no effect in the above result. Although the original functions will be different functions if they disagree at the isolated points $a_1, a_2, \ldots, a_n$, all such functions will induce a unique regular distribution, given by the above equation. Consequently, the distributional derivative of $f(x)$ is given by

$$
\begin{aligned}
f'(x) &= f_0'(x) + \sum_{k=1}^{n} (d/dx)\{[f_k(x) - f_{k-1}(x)]H(x-a_k)\} \\
&= f_0'(x) + \sum_{k=1}^{n} [f_k'(x) - f_{k-1}'(x)]H(x-a_k) + \\
&\qquad + \sum_{k=1}^{n} [f_k(x) - f_{k-1}(x)]\delta(x-a_k) \\
&= f_0'(x) + \sum_{k=1}^{n} [f_k'(x) - f_{k-1}'(x)]H(x-a_k) + \\
&\qquad + \sum_{k=1}^{n} [f_k(a_k) - f_{k-1}(a_k)]\delta(x-a_k)
\end{aligned}
$$

Higher distributional derivatives can be taken that will result in additional terms involving derivatives of delta functions. All delta terms can be and should be simplified so that they have integer coefficients, as shown in the example below.

EXAMPLES

Find the distributional derivatives $f'(x), f''(x)$ of the distribution induced by:

$$f(x) = \begin{cases} x^2+x & , \text{ if } x<1 \\ x^2+3x & , \text{ if } 1<x<2 \\ x^3+x^2 & , \text{ if } 2<x \end{cases}$$

Solution

We note that

$$f(x) = (x^2+x)+\left[(x^2+3x)-(x^2+x)\right]H(x-1)+\left[(x^3+x^2)-(x^2+3x)\right]H(x-2)$$

$$= (x^2+x)+(x^2+3x-x^2-x)H(x-1)+(x^3+x^2-x^2-3x)H(x-2)$$

$$= (x^2+x)+2xH(x-1)+(x^3-3x)H(x-2)$$

and therefore:

$$f'(x) = (x^2+x)'+(2x)'H(x-1)+2x\delta(x-1)+(x^3-3x)'H(x-2)+(x^3-3x)\delta(x-2)$$

$$= 2x+1+2H(x-1)+2\cdot1\delta(x-1)+(3x^2-3)H(x-2)+(2^3-3\cdot2)\delta(x-2)$$

$$= (2x+1)+2H(x-1)+(3x^2-3)H(x-2)+2\delta(x-1)+2\delta(x-2)$$

and

$$f''(x) = (2x+1)'+2\delta(x-1)+(3x^2-3)'H(x-2)+(3x^2-3)\delta(x-2)$$
$$+2\delta'(x-1)+2\delta'(x-2)$$

$$= 2+6xH(x-2)+2\delta(x-1)+(3\cdot2^2-3)\delta(x-2)+2\delta'(x-1)+$$
$$+2\delta'(x-2)$$

$$= 2+6xH(x-2)+2\delta(x-1)+9\delta(x-2)+2\delta'(x-1)+2\delta'(x-2).$$

EXERCISES

④ Evaluate the distributional derivatives $f'(x), f''(x), f'''(x)$ for the following discontinuous functions.

a) $f(x) = \begin{cases} x^3 + 2x^2 - 1 & , \; x < 1 \\ x^4 + x + 1 & , \; x > 1 \end{cases}$

b) $f(x) = \begin{cases} x^2 e^x & , \; x < 2 \\ x^3 e^x & , \; x > 2 \end{cases}$

c) $f(x) = \begin{cases} \exp(-x^2) & , \; x < 0 \\ \exp(-x^3) & , \; x > 0 \end{cases}$

d) $f(x) = \begin{cases} \mathrm{Arctan}\, x & , \; x < \sqrt{3} \\ \mathrm{Arctan}\,(1/x) & , \; x > \sqrt{3} \end{cases}$

e) $f(x) = \begin{cases} \sin x + \cos x & , \; x < \pi/6 \\ \cos^2 x & , \; \pi/6 < x < \pi/3 \\ \sin^2 x & , \; \pi/3 < x \end{cases}$

f) $f(x) = \begin{cases} x^2 - \sin(\pi x) & , \; x < 1/3 \\ x \cos(\pi x/2) & , \; 1/3 < x < 1 \\ x \sin(\pi x/2) & , \; 1 < x \end{cases}$

Side limit evaluation of generalized functions

In general, in spite of the notation, a distribution $F(x)$ cannot be evaluated for specific values of x. However, if we restrict the space of distributions $\mathcal{D}'(\mathbb{R})$ to a smaller subspace, then we can assign to them side-values x^+, x^- as follows:

Def: We define the space $\Delta^\infty(\mathbb{R})$ of distributions $F \in \mathcal{D}'(\mathbb{R})$ that can be written as:

$$F(x) = f(x) + \sum_{n\in A} g_n(x) H(x-p_n) + \sum_{n\in B} a_n \delta^{(b_n)}(x-q_n)$$

with $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$ being finite or countable sets such that

$$\begin{cases} f \in C^\infty(\mathbb{R}) \\ \forall n \in A : (g_n \in C^\infty(\mathbb{R}) \land p_n \in \mathbb{R}) \\ \forall n \in B : (a_n, q_n \in \mathbb{R} \land b_n \in \mathbb{N}) \end{cases}$$

It can be shown that $\Delta^\infty(\mathbb{R})$ is closed with respect to most operations of interest:

Prop: $\forall F, G \in \Delta^\infty(\mathbb{R}) : (F+G) \in \Delta^\infty(\mathbb{R})$
$\forall F \in \Delta^\infty(\mathbb{R}) : \forall g \in C^\infty(\mathbb{R}) : gF \in \Delta^\infty(\mathbb{R})$
$\forall F \in \Delta^\infty(\mathbb{R}) : F' \in \Delta^\infty(\mathbb{R})$

Remark: If $\max_{n\in B} b_n = N-1$ with $N \in \mathbb{N}^*$, then we say that F is N^{th}-order singular and denote $\Delta^N(\mathbb{R})$ as the subset of all N^{th}-order singular distributions of $\Delta^\infty(\mathbb{R})$. Likewise, if $B = \varnothing$, then F will be a regular distribution, we say that it is 0th-order singular and we denote the space of all 0th-order singular distributions of $\Delta^\infty(\mathbb{R})$ as $\Delta^0(\mathbb{R})$. It is important to emphasize that regretably, the notations $\Delta^0(\mathbb{R})$, $\Delta^N(\mathbb{R})$, $\Delta^\infty(\mathbb{R})$ are not standard.

Side-values to distributions in $\Delta^\infty(\mathbb{R})$

Given $x \in \mathbb{R}$, we assign the values x^+ and x^- to distributions in $\Delta^\infty(\mathbb{R})$ according to the following rules:

a) $\forall f \in C^\infty(\mathbb{R}): f(x^+) = f(x^-) = f(x)$

b) $\begin{cases} \forall x \in (-\infty, 0): H(x^+) = H(x^-) = 0 \\ \forall x \in (0, +\infty): H(x^+) = H(x^-) = 1 \\ H(0^+) = 1 \wedge H(0^-) = 0 \end{cases}$

c) $\forall x \in \mathbb{R}: \delta(x^+) = \delta(x^-) = 0$

d) $\forall x \in \mathbb{R}: \forall n \in \mathbb{N}^*: \delta^{(n)}(x^+) = \delta^{(n)}(x^-) = 0$

Given a distribution $F \in \Delta^\infty(\mathbb{R})$, the expansion of $F(x)$ in conjunction with the above definitions uniquely defines $F(x^+)$ and $F(x^-)$ for all $x \in \mathbb{R}$.

Generalized integrals on $\Delta^\infty(\mathbb{R})$

In general, with distributions from $\mathcal{D}'(\mathbb{R})$ all integrals are defined on the $(-\infty,+\infty)$ interval. A generalized definition of the integral for distributions on $\Delta^\infty(\mathbb{R})$ is possible as follows:

- 1 Let $a, p \in \mathbb{R}$ be given. We define:

$$a \in (p,+\infty) \Rightarrow \int_{p^-}^{a^-} \delta(x-p)\,dx = \int_{p^-}^{a^+} \delta(x-p)\,dx = \int_{p^-}^{+\infty} \delta(x-p)\,dx = 1$$

$$a \in (p,+\infty) \Rightarrow \int_{p^+}^{a^-} \delta(x-p)\,dx = \int_{p^+}^{a^+} \delta(x-p)\,dx = \int_{p^+}^{+\infty} \delta(x-p)\,dx = 0$$

$$a \in (-\infty,p) \Rightarrow \int_{a^+}^{p^+} \delta(x-p)\,dx = \int_{a^-}^{p^+} \delta(x-p)\,dx = \int_{-\infty}^{p^+} \delta(x-p)\,dx = 1$$

$$a \in (-\infty,p) \Rightarrow \int_{a^+}^{p^-} \delta(x-p)\,dx = \int_{a^-}^{p^-} \delta(x-p)\,dx = \int_{-\infty}^{p^-} \delta(x-p)\,dx = 0$$

$$\int_{p^-}^{p^+} \delta(x-p)\,dx = 1$$

- 2 For derivatives of the Dirac delta functions such as $\delta'(x-p), \delta''(x-p), \ldots, \delta^{(n)}(x-p), \ldots$, all of the above integrals are zero.
- 3 Integrals involving H-terms can be evaluated as basic Riemann integrals.
- 4 For a general distribution $F \in \Delta^\infty(\mathbb{R})$ of the form

$$F(x) = f(x) + \sum_{n\in A} g_n(x)\,H(x-p_n) + \sum_{n\in B} a_n\,\delta^{(b_n)}(x-q_n)$$

integrals can be defined as linear combinations of the above cases.

A remarkable result about this generalized integral is that it satisfies the following generalized fundamental theorem of Calculus.

Thm: Let $a, b \in \mathbb{R}$ with $a < b$. Then:

$$\forall F \in \Delta^\infty(\mathbb{R}) : \int_{a^+}^{b^+} F'(x)\,dx = F(b^+) - F(a^+)$$

$$\forall F \in \Delta^\infty(\mathbb{R}) : \int_{a^+}^{b^-} F'(x)\,dx = F(b^-) - F(a^+)$$

$$\forall F \in \Delta^\infty(\mathbb{R}) : \int_{a^-}^{b^+} F'(x)\,dx = F(b^+) - F(a^-)$$

$$\forall F \in \Delta^\infty(\mathbb{R}) : \int_{a^-}^{b^-} F'(x)\,dx = F(b^-) - F(a^-)$$

▸ Extension to improper integrals

We can extend the above theorem to improper integrals by defining:

$$\begin{cases} H(+\infty) = 1 \wedge H(-\infty) = 0 \\ \delta(+\infty) = \delta(-\infty) = 0 \\ \forall n \in \mathbb{N}^* : \delta^{(n)}(+\infty) = \delta^{(n)}(-\infty) = 0 \\ \forall f \in C^\infty(\mathbb{R}) : \left(f(+\infty) = \lim\limits_{x \to +\infty} f(x) \wedge f(-\infty) = \lim\limits_{x \to -\infty} f(x) \right) \end{cases}$$

assuming that the limits exist. Note that the convergence of improper integrals is not assured and should be investigated on a case by case basis.

EXAMPLE

Evaluate the integral $I = \int_{0^-}^{+\infty} (x^2+1)^2 \delta''(x)\,dx$

Solution

Define $g(x) = (x^2+1)^2$, $\forall x \in \mathbb{R}$. Then:

$$g'(x) = [(x^2+1)^2]' = 2(x^2+1)(x^2+1)' = 2(x^2+1)(2x) = 4x(x^2+1)$$
$$= 4x^3 + 4x \;, \forall x \in \mathbb{R}$$

and

$$g''(x) = (4x^3+4x)' = 12x^2+4$$

and it follows that

$$I = \int_{0^-}^{+\infty} (x^2+1)^2 \delta''(x)\,dx =$$

$$= \int_{0^-}^{+\infty} [g(0)\delta''(x) - 2g'(0)\delta'(x) + g''(0)\delta(x)]\,dx =$$

$$= g(0)\int_{0^-}^{+\infty} \delta''(x)\,dx - 2g'(0)\int_{0^-}^{+\infty} \delta'(x)\,dx + g''(0)\int_{0^-}^{+\infty} \delta(x)\,dx$$

$$= g(0)\cdot 0 - 2g'(0)\cdot 0 + g''(0)\cdot 1 = g''(0) = 12\cdot 0^2 + 4 = 4.$$

EXERCISES

(5) Evaluate the following integrals by first simplifying the integrands.

a) $I = \int_{-\infty}^{1^+} e^x \left[\delta(x) + \delta(x-1)\right] dx$

b) $I = \int_{1^+}^{3^-} x^2 e^x \left[\delta'(x-1) + \delta'(x) + \delta'(x-2)\right] dx$

c) $I = \int_{0^-}^{\pi/4^+} \cos^2 x \left[\delta''(x) + 3\delta''(x-\pi/4)\right] dx$

d) $I = \int_{0^-}^{0^+} (x+3)^2 (2x-1)^3 \delta''(x)\, dx$

e) $I = \int_{0^+}^{(\pi/2)^+} \sin x \left[1-\cos x\right] \left[\delta''(x) + \delta''(x-\pi/4) + \delta''(x-\pi/2)\right] dx$

f) $I = \int_{0^+}^{+\infty} \exp(-x^2)\, \delta'''(x-1)\, dx$

g) $I = \int_{-1^-}^{+1^-} x^2 (x-2)^3 \left[\delta''(x-1) + \delta''(x+1)\right] dx$

h) $I = \int_{0^-}^{1^+} x^2 \sqrt{3x+1} \left[\delta'(x) + \delta'(x-1)\right] dx$

i) $I = \int_{0^+}^{1^-} (x+1)\sqrt{2-x^2}\, \delta'(x-1/2)\, dx$

Distributions and Green's Functions

Integrodifferential form

Consider an inhomogeneous linear differential equation of the form

$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = f(x) \qquad (1)$$

with $a_0, a_1, \ldots, a_n \in C^0(I)$ and $f \in C^0(I)$ with $I \subseteq \mathbb{R}$ some interval. Using the Dirac delta function and its derivatives, we define a generalized function $L(x,t)$ via:

$$L(x,t) = \sum_{k=0}^{n-1} (-1)^k a_k(x)\, \delta^{(k)}(t-x) \qquad (2)$$

Then we may rewrite the ODE above as

$$\int_I L(x,t)\, y(t)\, dt = f(x) \qquad (3)$$

Note the similarity in structure to a linear system of equations from linear algebra. Here, the generalized function $L(x,t)$ represents the linear differential operator $L: C^n(I) \to C^0(I)$ associated with Eq. (1), and it is called the <u>kernel</u> of the linear ODE. Eq. (3) is the <u>integrodifferential form</u> of the linear ODE given by Eq. (1).

EXAMPLE

Consider the linear ODE:

$$y''(x) + x y'(x) - (x^2 - 1)\, y(x) = f(x)$$

The corresponding kernel $L(x,t)$ is given by

$$L(x,t) = (-1)^2 \delta''(t-x) + (-1)^1 x \delta'(t-x) - (x^2-1)\delta(t-x)$$

$$= \delta''(t-x) - x\delta'(t-x) - (x^2-1)\delta(t-x)$$

and the integro differential form of the ODE is given by

$$\int_{-\infty}^{+\infty} \left[\delta''(t-x) - x\delta'(t-x) - (x^2-1)\delta(t-x)\right] y(t)\, dt = f(x)$$

General theory of Green's functions

Def: Consider an ODE of the form

$$\int_I L(x,t)\, y(t)\, dt = f(x)$$

with $I \subseteq \mathbb{R}$ some interval. A Green's function $G(x,\xi)$ is any function that satisfies the equation

$$\int_I L(x,t)\, G(t,\xi)\, dt = \delta(x-\xi)$$

in the sense of distributions.

Remark: The Green's function does not have to be unique. Usually, it will have free parameters that can be determined by applying an initial condition or some boundary conditions. Using a linear algebra analogy, $G(x,\xi)$ can be thought of as representing an "inverse" of the operator represented by $L(x,t)$, except that due to a null space of homogeneous solutions, the inverse operation is not unique.

Thm: Let $G(x,\xi)$ be a Green's function of a linear ODE $Ly = f$ defined on some interval $I \subseteq \mathbb{R}$. Then a particular solution of the ODE is given by:

$$y_p(x) = \int_I G(x,\xi)\, f(\xi)\, d\xi$$

Proof

Since:

$$\int_I L(x,t)\,y_p(t)\,dt = \int_I L(x,t)\left[\int_I G(t,\xi)f(\xi)\,d\xi\right]dt =$$

$$= \int_I dt \int_I d\xi\; L(x,t)\,G(t,\xi)\,f(\xi) =$$

$$= \int_I d\xi\; f(\xi)\left[\int_I dt\; L(x,t)\,G(t,\xi)\right]$$

$$= \int_I d\xi\; f(\xi)\,\delta(x-\xi) = f(x)$$

it follows that $y_p(x)$ is a solution of the linear ODE

$$\int_I L(x,t)\,y(t)\,dt = f(x) \qquad \square$$

- The Green's function can be found using the following theorem. Transform methods, e.g. the Laplace transform are an alternate technique which we shall discuss later.

Thm: Consider the linear ODE $Ly=f$ defined on an interval $I\subseteq\mathbb{R}$ with $L: C^n(I)\to C^0(I)$ defined as:
$\forall y\in C^n(I): Ly = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y$
with $a_0, a_1, \ldots, a_{n-1} \in C^0(I)$. Let $G(x,\xi)$ be a Green's function of the operator L. We assume that:

$$\operatorname{null}(L) = \operatorname{span}\{y_1, y_2, \ldots, y_n\}$$

Then it follows that for a given $\xi\in I$, $G(x,\xi)$ is given by

$$G(x,\xi) = \begin{cases} A_1(\xi)y_1(x) + A_2(\xi)y_2(x) + \cdots + A_n(\xi)y_n(x), & \text{if } x<\xi \\ B_1(\xi)y_1(x) + B_2(\xi)y_2(x) + \cdots + B_n(\xi)y_n(x), & \text{if } x>\xi \end{cases}$$

with $A_1, A_2, \ldots, A_n \in C^n(I)$ and $B_1, B_2, \ldots, B_n \in C^n(I)$ such that it satisfies the following conditions:

(a) $G(x,\xi), \partial G(x,\xi)/\partial x, \ldots, \partial^{n-2}G(x,\xi)/\partial x^{n-2}$ are continuous on $x=\xi$.

(b) $$\lim_{x\to\xi^+} \frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} - \lim_{x\to\xi^-} \frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} = 1$$

Proof

Since $G(x,\xi)$ is a Green's function of the linear operator L, it follows that $LG(x,\xi) = \delta(x-\xi)$. Localizing for $x<\xi$ and for $x>\xi$, we have:

For $x<\xi$:

$$LG(x,\xi) = 0 \Leftrightarrow G(x,\xi) = A_1(\xi)y_1(x) + A_2(\xi)y_2(x) + \cdots + A_n(\xi)y_n(x)$$

For $x>\xi$:

$$LG(x,\xi) = 0 \Leftrightarrow G(x,\xi) = B_1(\xi)y_1(x) + B_2(\xi)y_2(x) + \cdots + B_n(\xi)y_n(x)$$

It follows that:

$$G(x,\xi) = \begin{cases} A_1(\xi)\, y_1(x) + A_2(\xi)\, y_2(x) + \cdots + A_n(\xi)\, y_n(x), & \text{if } x < \xi \\ B_1(\xi)\, y_1(x) + B_2(\xi)\, y_2(x) + \cdots + B_n(\xi)\, y_n(x), & \text{if } x > \xi \end{cases}$$

To establish the conditions (a) and (b) we note that $G(x,\xi)$ satisfies the following equation, in the sense of distributions:

$$\frac{\partial^n G(x,\xi)}{\partial x^n} + a_{n-1}(x)\frac{\partial^{n-1} G(x,\xi)}{\partial x^{n-1}} + \cdots + a_1(x)\frac{\partial G(x,\xi)}{\partial x}$$

$$+ a_0(x)\, G(x,\xi) = \delta(x-\xi) \qquad (1)$$

(a) To show that $G(x,\xi), \partial G(x,\xi)/\partial x, \ldots, \partial^{n-2} G(x,\xi)/\partial x^{n-2}$ are continuous on $x=\xi$, we assume that one of them is not continuous on $x=\xi$. Then $\partial^{n-1} G(x,\xi)/\partial x^{n-1}$ is at least a 1st-order singular distribution and $\partial^n G(x,\xi)/\partial x^n$ is therefore at least a 2nd-order singular distribution. It follows that the left-hand-side of Eq.(1) is at least a 2nd-order singular distribution. This is a contradiction because the right-hand-side is a 1st-order singular distribution. Thus condition (a) is proved.

(b) We define

$$F(x,\xi) = \frac{\partial^n G(x,\xi)}{\partial x^n} - \delta(x-\xi) = \sum_{k=0}^{n-1} -a_k(x)\frac{\partial^k G(x,\xi)}{\partial x^k}$$

$$= -a_{n-1}(x)\frac{\partial^{n-1} G(x,\xi)}{\partial x^{n-1}} - \sum_{k=0}^{n-2} a_k(x)\frac{\partial^k G(x,\xi)}{\partial x^k}$$

From (a) we know that the 2nd term (the sum from $k=0$ to $k=n-2$) is continuous at $x=\xi$. We also know that

$$\frac{\partial^n G(x,\xi)}{\partial x^n} + a_{n-1}(x)\frac{\partial^{n-1} G(x,\xi)}{\partial x^{n-1}}$$

has to be a 1st-order singular distribution since the right-hand-side of Eq.(1) is 1st-order singular. It follows that:

$\partial^n G(x,\xi)/\partial x^n$ 1st-order singular at $x=\xi \Rightarrow$

$\Rightarrow \partial^{n-1}G(x,\xi)/\partial x^{n-1}$ not continuous and regular at $x=\xi \Rightarrow$

$\Rightarrow a_{n-1}(x)\partial^{n-1}G(x,\xi)/\partial x^{n-1}$ regular and not continuous at $x=\xi \Rightarrow$

$\Rightarrow F(x,\xi)$ regular and not continuous at $x=\xi$. (2)

We write:

$$F(x,\xi) = \frac{\partial^n G(x,\xi)}{\partial x^n} - \delta(x-\xi) = \frac{\partial}{\partial x}\left[\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} - H(x-\xi)\right]$$

$$= \frac{\partial f(x,\xi)}{\partial x}$$

with $f(x,\xi) = \dfrac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} - H(x-\xi)$

From Eq.(2) it follows that

$$(2) \Rightarrow f(x,\xi) \text{ continuous on } x=\xi \Rightarrow \lim_{x\to\xi^+} f(x,\xi) = \lim_{x\to\xi^-} f(x,\xi) \Rightarrow$$

$$\Rightarrow \lim_{x\to\xi^+}\left[\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} - H(x-\xi)\right] = \lim_{x\to\xi^-}\left[\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} - H(x-\xi)\right]$$

$$\Rightarrow \lim_{x\to\xi^+}\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} - 1 = \lim_{x\to\xi^-}\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} \Rightarrow$$

$$\Rightarrow \lim_{x\to\xi^+}\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} - \lim_{x\to\xi^-}\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} = 1 \qquad \square$$

Application to the initial value problem

We can now apply the Green's function theory to the initial value problem of a linear differential equation. To obtain a unique solution for the Green's function we make, in the proof below, the causality assumption that $G(x,\xi)=0$ for $x<\xi$ (i.e. the future has no effect on the past).

Thm: Consider the linear ODE $Ly=f$, defined on an interval $I=[a,b]\subseteq\mathbb{R}$ with $L: C^n(I)\to C^0(I)$ given by:
$\forall y\in C^n(I): Ly = y^{(n)}+a_{n-1}y^{(n-1)}+\dots+a_1y'+a_0y$
with $a_0,a_1,\dots,a_{n-1}\in C^0(I)$ and $f\in C^0(I)$.
We assume that $\mathrm{null}(L)=\mathrm{span}\{y_1,y_2,\dots,y_n\}$ for $y_1,y_2,\dots,y_n\in C^n(I)$. Then, the corresponding Green's function is given by

$$G(x,\xi)=\begin{cases}\sum_{k=1}^{n} B_k(\xi)y_k(x) & , \text{if } x\geq\xi \\ 0 & , \text{if } x<\xi\end{cases}$$

where $B_1,\dots,B_n$ are given by:

$$(B_1(\xi),B_2(\xi),\dots,B_n(\xi)) = W[y_1,y_2,\dots,y_n](\xi)^{-1}(0,0,\dots,0,1)$$

A corresponding particular solution is:

$$y_p(x)=\sum_{k=1}^{n} y_k(x)\left[\int_a^x f(t)B_k(t)\,dt\right], \quad \forall x\in[a,b].$$

Proof

From the previous theorem, the general form of the Green's function is:

$$G(x,\xi) = \begin{cases} \sum_{k=1}^{n} A_k(\xi) y_k(x), & \text{if } x<\xi \\ \sum_{k=1}^{n} B_k(\xi) y_k(x), & \text{if } x>\xi \end{cases}$$

We note that

$$\forall m \in [0,n-2]\cap\mathbb{N}: \partial^m G(x,\xi)/\partial x^m \text{ continuous at } x=\xi \Leftrightarrow$$

$$\Leftrightarrow \forall m\in[0,n-2]\cap\mathbb{N}: \lim_{x\to\xi^+} \frac{\partial^m G(x,\xi)}{\partial x^m} = \lim_{x\to\xi^-}\frac{\partial^m G(x,\xi)}{\partial x^m} \Leftrightarrow$$

$$\Leftrightarrow \forall m\in[0,n-2]\cap\mathbb{N}: \sum_{k=1}^{n} B_k(\xi) y_k^{(m)}(\xi) = \sum_{k=1}^{n} A_k(\xi) y_k^{(m)}(\xi) \Leftrightarrow$$

$$\Leftrightarrow \forall m\in[0,n-2]\cap\mathbb{N}: \sum_{k=1}^{n} [B_k(\xi)-A_k(\xi)] y_k^{(m)}(\xi) = 0 \quad (1)$$

and

$$\lim_{x\to\xi^+}\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} - \lim_{x\to\xi^-}\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} = 1 \Leftrightarrow$$

$$\Leftrightarrow \sum_{k=1}^{n} B_k(\xi) y_k^{(n-1)}(\xi) - \sum_{k=1}^{n} A_k(\xi) y_k^{(n-1)}(\xi) = 1 \Leftrightarrow$$

$$\Leftrightarrow \sum_{k=1}^{n} [B_k(\xi)-A_k(\xi)] y_k^{(n-1)}(\xi) = 1 \quad (2)$$

To enforce uniqueness we introduce the causality assumption that $\forall k\in[n]: A_k(\xi)=0$. Then, from Eq.(1) and Eq.(2), we have:

$$\begin{cases} \forall m\in[0,n-2]\cap\mathbb{N}: \sum_{k=1}^{n} B_k(\xi) y_k^{(m)}(\xi) = 0 \\ \sum_{k=1}^{n} B_k(\xi) y_k^{(n-1)}(\xi) = 1 \end{cases} \Leftrightarrow$$

$$\begin{bmatrix} y_1(\xi) & y_2(\xi) & \cdots & y_n(\xi) \\ y_1'(\xi) & y_2'(\xi) & \cdots & y_n'(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)}(\xi) & y_2^{(n-2)}(\xi) & \cdots & y_n^{(n-2)}(\xi) \\ y_1^{(n-1)}(\xi) & y_2^{(n-2)}(\xi) & \cdots & y_n^{(n-2)}(\xi) \end{bmatrix} \begin{bmatrix} B_1(\xi) \\ B_2(\xi) \\ \vdots \\ B_{n-1}(\xi) \\ B_n(\xi) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow W[y_1,\dots,y_n](\xi)(B_1(\xi),B_2(\xi),\dots,B_n(\xi)) = (0,0,\dots,0,1)$$

$$\Leftrightarrow (B_1(\xi),B_2(\xi),\dots,B_n(\xi)) = W[y_1,\dots,y_n](\xi)^{-1}(0,0,\dots,0,1)$$

The corresponding particular solution is:

$$y_p(x) = \int_a^b G(x,\xi)f(\xi)\,d\xi = \int_a^x \left[\sum_{k=1}^n B_k(\xi)y_k(x)\right] f(\xi)\,d\xi$$

$$= \sum_{k=1}^n \int_a^x B_k(\xi)y_k(x)f(\xi)\,d\xi =$$

$$= \sum_{k=1}^n y_k(x)\left[\int_a^x B_k(\xi)f(\xi)\,d\xi\right] \qquad \square$$

Remark: The particular solution $y_p(x)$ given above is the exact solution to the initial value problem

$$\begin{cases} \forall x\in[a,b] : (Ly)(x) = f(x) \\ y(a) = y'(a) = y''(a) = \cdots = y^{(n-1)}(a) = 0 \end{cases}$$

in which the system is initialized from an initial state of rest. The general solution is:

$$\forall x\in[a,b] : y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + \int_a^x G(x,\xi)f(\xi)\,d\xi$$

for more general initial conditions.

Remark: For the special case of a second-order linear ODE, the Green's function simplifies to

$$G(x,\xi) = \frac{\begin{vmatrix} y_1(\xi) & y_2(\xi) \\ y_1(x) & y_2(x) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} \quad \text{for } x \geq \xi$$

and $G(x,\xi) = 0$ for $x < \xi$, and the solution to the initial value problem

$$\begin{cases} y''(x) + a_1(x)\, y'(x) + a_0(x)\, y(x) = f(x) \\ y(a) = 0 \wedge y'(a) = 0 \end{cases}$$

is given by

$$y(x) = \int_a^x G(x,\xi) f(\xi)\, d\xi.$$

EXERCISES

(10) Find the kernel $L(x,t)$ for the following linear differential equations in order to rewrite them in the form: $\int_{-\infty}^{+\infty} L(x,t)y(t)\,dt = f(x)$.

a) $y''(x) + ay'(x) + by(x) = f(x)$

b) $xy''(x) + (x^2-1)y'(x) + x^2y(x) = f(x)$

c) $[(2x+1)y']' + x^3y(x) = f(x)$

d) $(x^2-1)y'''(x) + 3xy'(x) - y(x) = f(x)$

(11) Let $G(x,\xi)$ be the Green's function to the linear ODE $Ly = f$ with $L: C^n(\mathbb{R}) \to C^0(\mathbb{R})$ defined as
$\forall y \in C^n(\mathbb{R}): \; Ly = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y$
with $a_0, a_1, \dots, a_{n-1} \in C^0(\mathbb{R})$ corresponding to the initial condition $y(a) = y'(a) = y''(a) = \dots = y^{(n-1)}(a) = 0$ of initial rest.

a) If y_1 is the solution to $Ly_1 = f_1$ initialized from rest at $x = a$ and y_2 is the solution to $Ly_2 = f_2$ initialized from rest, then show that the unique solution to the initial value problem

$$\begin{cases} Ly = \lambda f_1 + \mu f_2 \\ y(a) = y'(a) = \dots = y^{(n-1)}(a) = 0 \end{cases}$$

is $y(x) = \lambda y_1(x) + \mu y_2(x)$.

b) Show that $G(x,x)=0$ and also that

$$\forall k \in [n-2]: \left.\frac{\partial^k G(x,\xi)}{\partial x^k}\right|_{\xi=x} = 0$$

(12) Find the Green's functions for the following linear ODEs satisfying the causality condition:

a) $y''(x)+3y'(x)+2y(x)=f(x)$

b) $y''(x)+6y'(x)+9y(x)=f(x)$

c) $x^2y''(x)+xy'(x)-2y(x)=f(x)$

d) $y'''(x)-y'(x)=f(x)$

e) $y'''(x)-y''(x)-y'(x)+y(x)=f(x)$

f) $y'''(x)-6y'(x)+5y(x)=f(x)$

g) $y^{(4)}(x)+y'''(x)+y''(x)+y'(x)=f(x)$