Analysis and Curve sketching of a function

For the analysis and construction of the curve that represents the graph of a function we work as follows:

- 1 Find the domain A_f of f.
- 2 Calculate and FACTOR f' and f''
- 3 Construct a sign table for f'
  - a) Analyze monotonicity
  - b) Locate x AND y coordinates of min/max.
- 4 Construct a sign table for f''
  - a) Analyze convexity
  - b) Locate x AND y coordinates of inflection points.
- 5 Find all asymptotes (if they exist)
- 6 Find
  - a) The y-intercept $y_0 = f(0)$ (if it exists)
  - b) Solve $f(x) = 0 \Leftrightarrow \cdots x_0$ to find x-intercepts (if they exist)
- 7 Construct the graph of f.

examples

1) $f(x) = x^5 - 5x$

Function Analysis / Sketching

▶ Analyze and sketch $f(x)=\dfrac{x^2}{x+1}$

- Domain : $A=\mathbb{R}-\{-1\}=(-\infty,-1)\cup(-1,+\infty)$

- Intercepts:

  Solve $f(x)=0 \Leftrightarrow x^2=0 \Leftrightarrow \underline{x=0} \leftarrow$ x-axis intercept

  $y_0=f(0)=0 \longleftarrow$ y-axis intercept.

- Asymptotes

  $x=-1$ vertical asymptote.

  Expect a slant asymptote.

$$\alpha=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{x^2}{x(x+1)}=\lim_{x\to\pm\infty}\frac{x^2}{x^2}=1$$

$$\beta=\lim_{x\to\pm\infty}[f(x)-x]=\lim_{x\to\pm\infty}\left[\frac{x^2}{x+1}-x\right]$$

$$=\lim_{x\to\pm\infty}\frac{x^2-x(x+1)}{x+1}=\lim_{x\to\pm\infty}\frac{-x}{x+1}=\lim_{x\to\pm\infty}\frac{-x}{x}$$

$$=-1$$

Thus $y=x-1$ slant asymptote

- Derivatives

$$f'(x) = \left(\frac{x^2}{x+1}\right)' = \frac{(x^2)'(x+1) - x^2(x+1)'}{(x+1)^2} =$$

$$= \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{x[2x+2-x]}{(x+1)^2} =$$

$$= \frac{x(x+2)}{(x+1)^2}$$

$$f''(x) = \left[\frac{x^2+2x}{(x+1)^2}\right]' = \frac{(x^2+2x)'(x+1)^2 - (x^2+2x)\left[(x+1)^2\right]'}{(x+1)^4}$$

$$= \frac{(2x+2)(x+1)^2 - (x^2+2x)\cdot 2(x+1)(x+1)'}{(x+1)^4}$$

$$= \frac{2(x+1)\left[(x+1)^2 - (x^2+2x)\right]}{(x+1)^4} =$$

$$= \frac{2(x^2+2x+1-x^2-2x)}{(x+1)^3} = \frac{2}{(x+1)^3}$$

- Monotonicity

- Monotonicity

x		-2		-1		0	
x	$-$		$-$		$-$	0	$+$
$x+2$	$-$	0	$+$		$+$		$+$
$(x+1)^2$	$+$		$+$	0	$+$		$+$
f'	$+$	0	$-$	$\nparallel$	$-$	0	$+$
f	$\nearrow$	-4	$\searrow$	$\nparallel$	$\searrow$	0	$\nearrow$

max at $x=-2$: $f(-2) = \dfrac{(-2)^2}{-2+1} = -4$

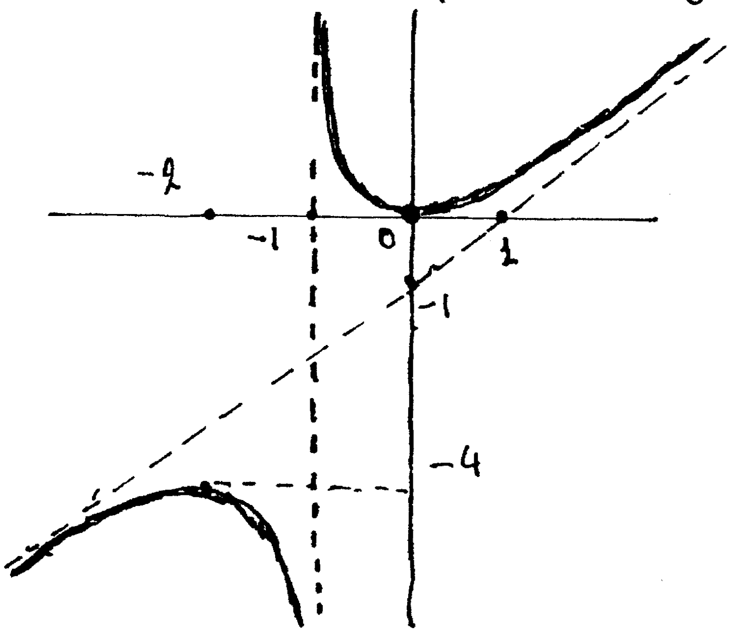
min at $x=0$: $f(0) = 0$.

- Convexity

x		-1	
$(x+1)^3$	$-$	0	$+$
$f''(x)$	$-$	$\nparallel$	$+$
$f(x)$	$\cap$	$\nparallel$	$\cup$

- Variation Table

x	$-\infty$		-2		-1		0		$+\infty$
$f'(x)$		$+$	0	$-$	‖	$-$	0	$+$	
$f''(x)$		$-$		$-$	‖	$+$		$+$	
$f(x)$	$-\infty$	↗	-4	$+\infty$ ↘	‖	$+\infty$ ↘	0	↗	$+\infty$



Graph

This example is meant to demonstrate notation and the level of detail needed for this type of problem. The sketch is not meant to be exact, but it should be consistent with the variation table and all points on the plot that are noteworthy (i.e. min/max/intercepts/inflection points) MUST have their coordinates on the axes labeled.

Curve Sketching (Radicals)

example : Sketch $f(x) = x\sqrt{2-x^2}$

- Domain:

Need $2 - x^2 \geq 0 \Leftrightarrow x \in [-\sqrt{2}, \sqrt{2}]$

x		$-\sqrt{2}$		$+\sqrt{2}$	
$2-x^2$	$-$	0	$+$	0	$-$

Thus $A_f = [-\sqrt{2}, \sqrt{2}]$

- Intercepts

$y_0 = f(0) = 0 \Rightarrow (0,0)$ y-intercept

$f(x) = 0 \Leftrightarrow x\sqrt{2-x^2} = 0 \Leftrightarrow$

$\Leftrightarrow x = 0 \vee \sqrt{2-x^2} = 0$

$\Leftrightarrow x = 0 \vee 2 - x^2 = 0$

$\Leftrightarrow x = 0 \vee \sqrt{2} - x = 0 \vee \sqrt{2} + x = 0$

$\Leftrightarrow x \in \{0, \sqrt{2}, -\sqrt{2}\}$

Thus $(0,0), (\sqrt{2},0), (-\sqrt{2},0)$ are x-intercepts.

- Asymptotes

f continuous in $A_f \Rightarrow$ no vertical asymptotes

$\pm\infty$ not accessible in $A_f \Rightarrow$ no horizontal / slant asymptotes.

- Derivatives

$$f'(x) = \left[x\sqrt{2-x^2}\right]' = (x)'\sqrt{2-x^2} + x\left(\sqrt{2-x^2}\right)'$$

$$= \sqrt{2-x^2} + x\,\frac{(2-x^2)'}{2\sqrt{2-x^2}} =$$

$$= \sqrt{2-x^2} + \frac{x(-2x)}{2\sqrt{2-x^2}} = \frac{2(2-x^2)+x(-2x)}{2\sqrt{2-x^2}} =$$

$$= \frac{4-2x^2-2x^2}{2\sqrt{2-x^2}} = \frac{2(1-x^2)}{\sqrt{2-x^2}}.$$

$$f''(x) = \frac{1}{\left(\sqrt{2-x^2}\right)^2}\left[2(1-x^2)'\sqrt{2-x^2} - 2(1-x^2)\left(\sqrt{2-x^2}\right)'\right]$$

$$= \frac{1}{2-x^2}\left[2(-2x)\sqrt{2-x^2} - \cancel{2}(1-x^2)\frac{(2-x^2)'}{\cancel{2}\sqrt{2-x^2}}\right]$$

$$= \frac{1}{2-x^2}\left[-4x\sqrt{2-x^2} - \frac{-2x(1-x^2)}{\sqrt{2-x^2}}\right]$$

$$= \frac{1}{2-x^2}\left[\frac{-4x(2-x^2)+2x(1-x^2)}{\sqrt{2-x^2}}\right]$$

$$= \frac{1}{2-x^2}\left[\frac{-8x+4x^3+2x-2x^3}{\sqrt{2-x^2}}\right]$$

$$= \frac{1}{2-x^2}\left[\frac{2x^3-6x}{\sqrt{2-x^2}}\right] = \frac{2x(x^2-3)}{(2-x^2)\sqrt{2-x^2}}$$

- Monotonicity $\longrightarrow$ $f'(x) = \dfrac{2(1-x^2)}{\sqrt{2-x^2}}$

Zeroes: $\pm 1, \pm\sqrt{2}$

x		$-\sqrt{2}$		-1		1		$\sqrt{2}$	
$2(1-x^2)$	$-$		$-$	0	$+$	0	$-$		$-$
$\sqrt{2-x^2}$		0	$+$		$+$		$+$	0	
$f'(x)$		$\Vert$	$-$	0	$+$	0	$-$	$\Vert$	
$f(x)$		0	$\searrow$		$\nearrow$		$\searrow$	0	
		max		min		max		min	

- Convexity $\longrightarrow$ $f''(x) = \dfrac{2x(x^2-3)}{(2-x^2)\sqrt{2-x^2}}$

Zeroes: $0, \pm\sqrt{3}, \pm\sqrt{2}$

x		$-\sqrt{3}$		$-\sqrt{2}$		0		$\sqrt{2}$		$\sqrt{3}$	
$2x$	$-$		$-$		$-$	0	$+$		$+$		$+$
(x^2-3)	$+$	0	$-$		$-$		$-$		$-$	0	$+$
$(2-x^2)$	$-$		$-$	0	$+$		$+$	0	$-$		$-$
$\sqrt{2-x^2}$				0	$+$		$+$	0			
f''		0		$\Vert$	$+$	0	$-$	$\Vert$		0	
f				0	$\cup$		$\cap$	0			
						infl.					

- Variation Table

x		$-\sqrt{2}$		-1		0		1		$\sqrt{2}$	
f'		$\|\|$	$-$	0	$+$	$\|$	$+$	0	$-$	$\|\|$	
f''		$\|\|$	$+$	$\|$	$+$	0	$-$	$\|$	$-$	$\|\|$	
f		0	$\searrow$		$\nearrow$		$\nearrow$		$\searrow$	0	
		max		min		infl		max		min	

At $x = -\sqrt{2} \Rightarrow f(-\sqrt{2}) = 0$ (max)

$x = -1 \Rightarrow f(-1) = (-1)\sqrt{2-(-1)^2}$

$= (-1)\sqrt{1} = -1$ (min)

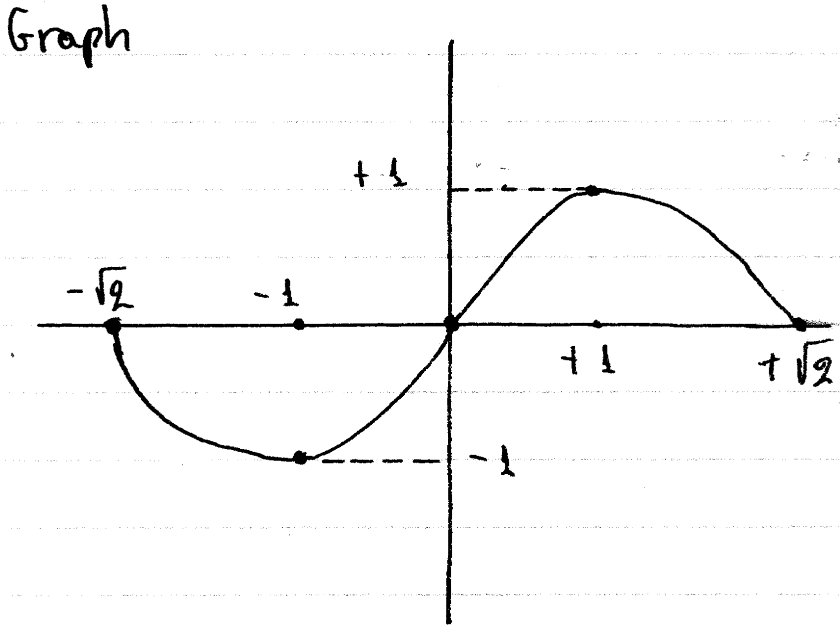
$x = 0 \Rightarrow f(0) = 0$ (infl.)

$x = 1 \Rightarrow f(1) = (+1)\sqrt{2-(+1)^2}$

$= (+1)\sqrt{1} = +1$ (max)

$x = \sqrt{2} \Rightarrow f(\sqrt{2}) = 0.$

- Graph



Curve Sketching (trig. functions)

example : Sketch $f(x) = 3\sin x - \sin^3 x$.

- Domain: $A = \mathbb{R}$
- Periodic with period 2π so sufficient to study the interval $[0, 2\pi]$.
- Intercepts:
  $y_0 = f(0) = 3\sin 0 - \sin^3 0 = 0 \Rightarrow (0,0)$ y intercept.

$$\begin{aligned}
f(x) = 0 &\Leftrightarrow 3\sin x - \sin^3 x = 0 \Leftrightarrow \\
&\Leftrightarrow \sin^2 x\,(3 - \sin x) = 0 \\
&\Leftrightarrow \sin^2 x = 0 \vee 3 - \sin x = 0 \\
&\Leftrightarrow \sin x = 0 \vee \sin x = 3 \leftarrow \text{2nd equation has no solutions.} \\
&\Leftrightarrow \sin x = 0 \\
&\Leftrightarrow x = 0 \vee x = \pi \vee x = 2\pi
\end{aligned}$$

Three x-intercepts: $(0,0)$, $(\pi, 0)$, $(2\pi, 0)$.

- Asymptotes

f continuous in $\mathbb{R} \Rightarrow$ no vertical asymptotes

$$\left|\frac{f(x)}{x}\right| = \left|\frac{3\sin x - \sin^3 x}{x}\right| \leq \frac{3|\sin x| + |\sin^3 x|}{|x|}$$

$$\left.\begin{aligned}
&\leq \frac{3+1}{|x|} = \frac{4}{|x|} \\
&\lim_{x \to \pm\infty} \frac{4}{|x|} = 0
\end{aligned}\right\} \Rightarrow \lim_{x \to \pm\infty} \frac{f(x)}{x} = 0$$

$\Rightarrow$ no slant asymptote.

The limit $\lim\limits_{x \to +\infty} f(x)$ does not exist because

for $a_k = 2k\pi \Rightarrow f(a_k) = 0$

$b_k = 2k\pi + \pi/2 \Rightarrow f(b_k) = 3 - 1 = 2$

and similarly the limit $\lim\limits_{x \to -\infty} f(x)$ does not exist.
Consequently no horizontal asymptotes either.

• Derivatives

$$f'(x) = [3\sin x - \sin^3 x]' = 3\cos x - 3\sin^2 x \cos x$$
$$= 3\cos x\,(1 - \sin^2 x)$$
$$= 3\cos x \cos^2 x = 3\cos^3 x.$$
$$f''(x) = 9\cos^2 x\,(\cos x)' = -9\cos^2 x \sin x.$$

• Monotonicity (Zeroes: $\pi/2$, $3\pi/2$)

x	0		$\pi/2$		$3\pi/2$		2π
$\cos x$		+	0	−	0	+	
$\cos^3 x$		+	0	−	0	+	
$f'(x)$		+	0	−	0	+	
$f(x)$		↗	max	↘	min	↗	

Convexity (Zeroes: $\underbrace{\pi/2, 3\pi/2}_{\cos^2 x}, \underbrace{0, \pi, 2\pi}_{\sin x}$)

x	0		$\pi/2$		π		$3\pi/2$		2π
$-2\cos^2 x$		$-$	0	$-$		$-$	0	$-$	
$\sin x$	0	$+$		$+$	0	$-$		$-$	0
$f''(x)$	0	$-$	0	$-$	0	$+$	0	$+$	0
$f(x)$		$\cap$		$\cap$	infl.	$\cup$		$\cup$	

Variation Table

x	0		$\pi/2$		π		$3\pi/2$		2π
f'		$+$	0	$-$		$-$	0	$+$	
f''	0	$-$	0	$-$	0	$+$	0	$+$	0
f		$\nearrow$		$\searrow$		$\searrow$		$\nearrow$	
	infl		max		infl		min		infl

Don't forget to check endpoints of $[0, 2\pi]$ for possible min/max/infl points!

Inflection at $x = 0 \Rightarrow f(0) = 0$

Max at $x = \pi/2 \Rightarrow f(\pi/2) = 3\sin(\pi/2) - \sin^3(\pi/2)$
$= 3 \cdot 1 - 1^3 = 2$.

Inflection at $x = \pi \Rightarrow f(\pi) = 3\sin(\pi) - \sin^3(\pi)$
$= 3 \cdot 0 - 0^3 = 0$

Min at $x = 3\pi/2 \Rightarrow f(3\pi/2) = 3\sin(3\pi/2) - \sin^3(3\pi/2)$

$$= 3\cdot(-1) - (-1)^3$$

$$= -3 + 1 = -2$$

Infl at $x = 2\pi \Rightarrow f(2\pi) = 0.$

Graph