INTEGRALS - REVIEW FROM CALCULUS I

We summarize the Riemann definition of the integral and the main theorems from Calculus 1.

▼ Riemann definition of the integral

Let $f: A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$ and $[a,b] \subseteq A$.
We define:

$$x_k = a + (b-a)(k/n), \quad \forall k \in [n] \cup \{0\}$$

$$m_k(f \mid a,b,n) = \min_{x \in [x_{k-1}, x_k]} f(x), \quad \forall k \in [n]$$

$$M_k(f \mid a,b,n) = \max_{x \in [x_{k-1}, x_k]} f(x), \quad \forall k \in [n]$$

$$L_n(f \mid a,b) = \sum_{k=1}^{n} m_k(f \mid a,b,n)(x_k - x_{k-1})$$

$$U_n(f \mid a,b) = \sum_{k=1}^{n} M_k(f \mid a,b,n)(x_k - x_{k-1})$$

We say that:

$$\boxed{f \text{ integrable on } [a,b] \Longleftrightarrow \exists \ell \in \mathbb{R} : \lim_{n \in \mathbb{N}^*} L_n(f \mid a,b) = \lim_{n \in \mathbb{N}^*} U_n(f \mid a,b) = \ell}$$

If f is integrable on $[a,b]$, then we define the definite integral as follows:

$$\boxed{\begin{gathered}\int_a^b f(x)\,dx = \lim_{n\in\mathbb{N}^*} L_n(f|a,b) = \lim_{n\in\mathbb{N}^*} U_n(f|a,b) \\ \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \quad \text{and} \quad \int_a^a f(x)\,dx = 0\end{gathered}}$$

Basic properties of integrals

1) $\boxed{f \text{ continuous on } [a,b] \Rightarrow f \text{ integrable on } [a,b]}$

2) Let $f: A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$ and assume that f integrable on $[a,b]$. Then

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\in\mathbb{N}^*} \left[\frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\,\frac{b-a}{n}\right)\right]}$$

3) Let f,g be integrable on $[a,b]$. Then:

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\forall \lambda \in \mathbb{R}: \int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$$

$$\left.\begin{array}{l}\forall k \in [n]: f_k \text{ integrable on } [a,b] \\ \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}\end{array}\right\} \Rightarrow$$

$$\Rightarrow \int_a^b \left[\sum_{k=1}^{n} \lambda_k f_k(x)\right] dx = \sum_{k=1}^{n} \lambda_k \int_a^b f(x)\,dx$$

3) Chasles theorem

$$\left.\begin{array}{l} f \text{ integrable at } I \\ I \text{ interval} \\ a, b, \gamma \in I \end{array}\right\} \Rightarrow \int_a^b f(x)dx = \int_a^\gamma f(x)dx + \int_\gamma^b f(x)dx$$

By induction, Chasles theorem can be generalized to give:

$$\left.\begin{array}{l} f \text{ integrable on } I \\ I \text{ interval} \\ a, b, \gamma_1, \gamma_2, \dots, \gamma_n \in I \end{array}\right\} \Rightarrow \int_a^b f(x)dx = \int_a^{\gamma_1} f(x)dx + \sum_{k=1}^{n-1} \int_{\gamma_k}^{\gamma_{k+1}} f(x)dx + \int_{\gamma_n}^b f(x)dx$$

4) Integral bounding / comparison

$$\left.\begin{array}{l} f \text{ integrable on } [a,b] \\ \forall x \in [a,b]: f(x) \geq m \end{array}\right\} \Rightarrow \int_a^b f(x)dx \geq m(b-a)$$

$$\left.\begin{array}{l} f \text{ integrable on } [a,b] \\ \forall x \in [a,b]: f(x) \leq m \end{array}\right\} \Rightarrow \int_a^b f(x)dx \leq m(b-a)$$

← Note the implicit requirement that $\underline{\underline{a < b}}$!!

For $m = 0$, the above statement gives:

$$\left.\begin{array}{l} f \text{ integrable on } [a,b] \\ \forall x \in [a,b]: f(x) \geq 0 \end{array}\right\} \Rightarrow \int_a^b f(x)dx \geq 0$$

and an immediate corollary is that:

$$\left.\begin{array}{l} f, g \text{ integrable at } [a,b] \\ \forall x \in [a,b]: f(x) \geq g(x) \end{array}\right\} \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$$

5) Integral Mean-Value theorem

$$\boxed{\left.\begin{array}{l} f \text{ continuous on } [a,b] \\ g \text{ integrable on } [a,b] \\ \forall x \in [a,b] : g(x) \geq 0 \end{array}\right\} \Rightarrow \exists \xi \in [a,b] : \int_a^b f(x) g(x)\,dx = f(\xi) \int_a^b g(x)\,dx}$$

6) Absolute value of integral

$$\boxed{\left.\begin{array}{l} f \text{ integrable on } [a,b] \\ |f| \text{ integrable on } [a,b] \end{array}\right\} \Rightarrow \left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx}$$

Fundamental theorem of calculus

1) Fundamental Theorem I.

$$\left.\begin{array}{l} f \text{ continuous on } [a,b] \\ \forall x\in[a,b] : F(x) = \int_a^x f(t)\,dt \end{array}\right\} \Rightarrow \left\{\begin{array}{l} F \text{ differentiable on } [a,b] \\ \forall x\in[a,b] : F'(x) = f(x) \end{array}\right.$$

Immediate consequences:

$$\frac{d}{dx}\int_c^x f(t)\,dt = f(x)$$

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))\,b'(x) - f(a(x))\,a'(x)$$

2) Fundamental theorem II

$$\left.\begin{array}{l} F \text{ differentiable on } [a,b] \\ \forall x\in[a,b] : F'(x) = f(x) \\ f \text{ continuous on } [a,b] \end{array}\right\} \Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$$

↳ The FTC II motivates the definition of the indefinite integral.

$$\int f(x)\,dx = F(x) + c \Leftrightarrow \forall x\in \operatorname{dom}(F) : F'(x) = f(x)$$

F is the antiderivative of f.

3) Method of substitution theorem

$$\left.\begin{array}{l} g \text{ differentiable on } [a,b] \\ g' \text{ continuous on } [a,b] \\ f \text{ continuous on } g([a,b]) \end{array}\right\} \Rightarrow \int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(y)\,dy$$

Basic Integration formulas

1) $\int x^a dx = \begin{cases} \dfrac{x^{a+1}}{a+1} + c &, \text{ if } a \neq -1 \\ \ln|x| + c &, \text{ if } a = -1 \end{cases}$

2) $\int \sin(x)\,dx = -\cos x + c \longrightarrow \int \sin(ax+b)\,dx = \dfrac{-1}{a}\cos(ax+b) + c$

3) $\int \cos(x)\,dx = \sin x + c \longrightarrow \int \cos(ax+b)\,dx = \dfrac{1}{a}\sin(ax+b) + c$

4) $\int \dfrac{dx}{\cos^2 x} = \tan x + c \longrightarrow \int \dfrac{dx}{\cos^2(ax+b)} = \dfrac{1}{a}\tan(ax+b) + c$

5) $\int \dfrac{dx}{\sin^2 x} = -\cot x + c \longrightarrow \int \dfrac{dx}{\sin^2(ax+b)} = \dfrac{-1}{a}\cot(ax+b) + c$

6) $\int a^x dx = \dfrac{a^x}{\ln a} + c \longrightarrow \int a^{bx} dx = \dfrac{a^{bx}}{b\ln a} + c$

7) $\int \dfrac{dx}{1+x^2} = \operatorname{Arctan}(x) + c \longrightarrow \int \dfrac{dx}{x^2+a^2} = \dfrac{1}{a}\operatorname{Arctan}\left(\dfrac{x}{a}\right) + c$

8) $\int \dfrac{dx}{\sqrt{1-x^2}} = \operatorname{Arcsin}(x) + c \longrightarrow \int \dfrac{dx}{\sqrt{a^2-x^2}} = \operatorname{Arcsin}\left(\dfrac{x}{|a|}\right) + c$