TECHNIQUES OF INTEGRATION

General Remarks

(1) During the evaluation of an indefinite integral, if we use two different techniques, it is possible to arrive to different results. These results will be correct if they are different only by a constant.

EXAMPLE

$$I = \int (x+1)\,dx = \frac{x^2}{2} + x + c$$

2nd method: Let $y = x+1 \Rightarrow dy = dx$, thus

$$I = \int y\,dy = \frac{y^2}{2} + c = \frac{(x+1)^2}{2} + c$$

$$= \frac{x^2+2x+1}{2} + c = \frac{x^2}{2} + x + \frac{1}{2} + c$$

(2) For integrals of the form $\rightarrow I = \int \frac{f'(x)}{f(x)}\,dx$

We use $\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c}$

EXAMPLE

$$I = \int \frac{4x-3}{2x^2-3x+1}\,dx = \int \frac{(2x^2-3x+1)'}{2x^2-3x+1}\,dx =$$
$$= \ln|2x^2-3x+1| + c$$

③ For integrals of the form $I = \int \frac{f'(x)}{\sqrt{f(x)}}\,dx$ we use the identity

$$\boxed{\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c}$$

EXAMPLE

$$I = \int_0^1 \frac{3x}{\sqrt{x^2+3}}\,dx = \frac{3}{2}\int_0^1 \frac{2x}{\sqrt{x^2+3}}\,dx =$$
$$= \frac{3}{2}\int_0^1 \frac{(x^2+3)'}{\sqrt{x^2+3}}\,dx = \frac{3}{2}\left[2\sqrt{x^2+3}\right]_0^1 =$$
$$= 3\left[\sqrt{x^2+3}\right]_0^1 = 3\sqrt{1^2+3} - 3\sqrt{0^2+3} =$$
$$= 3(\sqrt{4}-\sqrt{3}) = 3(2-\sqrt{3}).$$

EXERCISES

① Evaluate the following integrals

a) $I = \int \frac{2x+3}{x^2+3x+5}\,dx$

b) $I = \int_1^2 \frac{3x^2}{x^3+5}\,dx$

c) $I = \int_1^3 \frac{8x^3+2}{x^4+x}\,dx$

d) $I = \int_2^3 \frac{2x}{\sqrt{x^2+1}}\,dx$

e) $I = \int_{\sqrt{2}}^3 \frac{x^2}{\sqrt{x^3-1}}\,dx$

f) $I = \int_2^{\sqrt{5}} \frac{5x^4+5}{\sqrt{x^5+5x+1}}\,dx$

g) $I = \int \frac{2x+1}{x^2+1}\,dx$

h) $I = \int \frac{3x-1}{\sqrt{1-x^2}}\,dx$

② Use the method of substitution to evaluate the following integrals.

a) $I = \int_0^{\pi/3} 3^{\sin x} \cos x\,dx$

b) $I = \int_0^{\pi/4} \frac{\cos x}{1+\sin^2 x}\,dx$

c) $I = \int_{-\ln 4}^{-1} \frac{3e^{x+1}}{\sqrt{1-e^{2x}}}\,dx$

d) $I = \int \frac{x^2 \sin(x^3-\pi)}{\cos^3(x^3-\pi)}\,dx$

e) $I = \int \frac{\sin(\ln(2x^3))}{x}\,dx$

(3) Show that:

a) $\dfrac{1}{\cos x} = \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{1+\sin x}$

b) $\displaystyle\int \frac{dx}{\cos x} = \ln\left|\frac{1}{\cos x} + \tan x\right| + C$

(Hint: Use (a) to show (b)).

Integration by parts

This method is based on the following theorem:

Thm : Assume that

a) f, g differentiable at $[a,b]$

b) f', g' continuous at $[a,b]$

Then

$$\boxed{\int_a^b f'(x)\,g(x)\,dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b f(x)\,g'(x)\,dx}$$

Proof

$$\int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx =$$

$$= \int_a^b [f'(x)g(x) + f(x)g'(x)]\,dx =$$

$$= \int_a^b [f(x)g(x)]'\,dx = \Big[f(x)\,g(x)\Big]_a^b \Rightarrow$$

$$\Rightarrow \int_a^b f'(x)\,g(x)\,dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b f(x)\,g'(x)\,dx \quad \square$$

(1) $\boxed{I = \int P(x)\, e^{ax+b}\, dx}$

with $P(x)$ a polynomial.

We write $e^{ax+b} = \frac{1}{a}(e^{ax+b})'$ and apply integration by parts.

EXAMPLE

$$I = \int_0^1 (2x-3)e^{3x}\,dx = \frac{1}{3}\int_0^1 (2x-3)(e^{3x})'\,dx$$

$$= \frac{1}{3}\Big[(2x-3)e^{3x}\Big]_0^1 - \frac{1}{3}\int_0^1 (2x-3)'e^{3x}\,dx$$

$$= \frac{1}{3}\Big[(2\cdot 1-3)e^{3\cdot 1} - (2\cdot 0-3)e^{3\cdot 0}\Big] - \frac{1}{3}\int_0^1 2e^{3x}\,dx$$

$$= \frac{1}{3}\Big[-e^3 - (-3)\cdot 1\Big] - \frac{2}{3}\Big[\frac{e^{3x}}{3}\Big]_0^1 =$$

$$= \frac{3-e^3}{3} - \frac{2}{3}\,\frac{e^3-e^0}{3} =$$

$$= \frac{3-e^3}{3} - \frac{2(e^3-1)}{9} = \frac{3(3-e^3)-2(e^3-1)}{9} =$$

$$= \frac{9-3e^3-2e^3+2}{9} = \frac{11-5e^3}{9}$$

2 $\rightarrow$

$$\boxed{\begin{array}{l} I = \int P(x)\sin(ax+b)\,dx \\ \hline I = \int P(x)\cos(ax+b)\,dx \end{array}}$$

with $P(x)$ a polynomial.

We use: $\sin(ax+b) = \frac{-1}{a}\left[\cos(ax+b)\right]'$

$$\cos(ax+b) = \frac{1}{a}\left[\sin(ax+b)\right]'$$

and apply integration by parts.

EXAMPLE

$$I = \int_0^{\pi/2} (2x-1)\cos(3x)\,dx = \frac{1}{3}\int_0^{\pi/2} (2x-1)\left(\sin(3x)\right)'dx =$$

$$= \frac{1}{3}\Big[(2x-1)\sin(3x)\Big]_0^{\pi/2} - \frac{1}{3}\int_0^{\pi/2} (2x-1)'\sin(3x)\,dx =$$

$$= \frac{1}{3}\left[\left(2\frac{\pi}{2}-1\right)\sin\left(\frac{3\pi}{2}\right) - (2\cdot 0-1)\sin 0\right] - \frac{1}{3}\int_0^{\pi/2} 2\sin(3x)\,dx$$

$$= \frac{1}{3}\left[(\pi-1)\cdot(-1) - 0\right] - \frac{2}{3}\left[\frac{-\cos(3x)}{3}\right]_0^{\pi/2} =$$

$$= \frac{1-\pi}{3} + \frac{2}{9}\left[\cos(3\pi/2) - \cos(0)\right]$$

$$= \frac{1-\pi}{3} + \frac{2}{9}(0-1) = \frac{1-\pi}{3} - \frac{2}{9} =$$

$$= \frac{3(1-\pi)-2}{9} = \frac{3-3\pi-2}{9} = \frac{1-3\pi}{9}$$

(3) $\rightarrow$ $\boxed{I = \int e^{ax+b} \sin(cx+d)\,dx}$

$\boxed{I = \int e^{ax+b} \cos(cx+d)\,dx}$

Apply integration by parts twice. This leads to an equation with I unknown, which we then solve for I.

EXAMPLE

$$I = \int_0^{\pi/2} e^{-x} \sin(2x)\,dx = -\int_0^{\pi/2} (e^{-x})' \sin(2x)\,dx =$$

$$= -\left[e^{-x}\sin(2x)\right]_0^{\pi/2} + \int_0^{\pi/2} e^{-x}\cdot(\sin(2x))'\,dx$$

$$= -\left[e^{-\pi/2}\sin(2\cdot(\pi/2)) - e^0 \sin 0\right] + 2\int_0^{\pi/2} e^{-x}\cos(2x)\,dx$$

$$= -e^{-\pi/2}\sin(\pi) - 2\int_0^{\pi/2} (e^{-x})' \cos(2x)\,dx =$$

$$= -e^{-\pi/2}\cdot 0 - 2\left[e^{-x}\cos(2x)\right]_0^{\pi/2} + 2\int_0^{\pi/2} e^{-x}\cdot(\cos(2x))'\,dx$$

$$= -2\left[e^{-\pi/2}\cos(2\cdot\pi/2) - e^0\cos 0\right] - 4\int_0^{\pi/2} e^{-x}\sin(2x)\,dx$$

$$= -2\left[e^{-\pi/2}\cos(\pi) - 1\right] - 4I = -2\left[e^{-\pi/2}\cdot(-1) - 1\right] - 4I$$

$$= 2e^{-\pi/2} + 2 - 4I \Leftrightarrow I + 4I = 2e^{-\pi/2} + 2 \Leftrightarrow$$

$$\Leftrightarrow 5I = 2(e^{-\pi/2} + 1) \Leftrightarrow I = \frac{2(e^{-\pi/2}+1)}{5}$$

4 $\rightarrow$ $\boxed{I = \int P(x) \ln(f(x))\, dx}$

Where $P(x)$ is a polynomial, and $f(x)$ is a function. The logarithm can be eliminated by writing $P(x) = Q'(x)$ and applying integration by parts:

EXAMPLE

a) $I = \int_1^2 (2x+1) \ln x \, dx = \int_1^2 (x^2+x)' \ln x \, dx =$

$= \left[(x^2+x) \ln x \right]_1^2 - \int_1^2 (x^2+x)(\ln x)' \, dx =$

$= \left[(2^2+2)\ln 2 - (1^2+1)\ln 1 \right] - \int_1^2 (x^2+x) \frac{1}{x} \, dx$

$= 6\ln 2 - 2\ln 1 - \int_1^2 (x+1)\, dx =$

$= 6\ln 2 - \left[\frac{x^2}{2} + x \right]_1^2 = 6\ln 2 - \left[\frac{2^2 - 1^2}{2} + (2-1) \right]$

$= 6\ln 2 - \frac{3}{2} - 1 = 6\ln 2 - \frac{5}{2}$

B) $I = \int_0^{1/2} \ln\left(\frac{1-x}{1+x}\right) dx = \int_0^{1/2} (x)' \ln\left(\frac{1-x}{1+x}\right) dx$

$$= \left[x \ln\left(\frac{1-x}{1+x}\right) \right]_0^{1/2} - \int_0^{1/2} x \left[\ln\left(\frac{1-x}{1+x}\right) \right]' dx \qquad (1)$$

We note that

$$\left[x \ln\left(\frac{1-x}{1+x}\right) \right]_0^{1/2} = \frac{1}{2} \ln\left(\frac{1-1/2}{1+1/2}\right) - 0 \ln\left(\frac{1-0}{1+0}\right)$$

$$= \frac{1}{2} \ln\left(\frac{1/2}{3/2}\right) = \frac{1}{2} \ln(1/3) =$$

$$= \frac{-\ln 3}{2} \qquad (2)$$

and

$$\left[\ln\left(\frac{1-x}{1+x}\right) \right]' = \frac{\left(\frac{1-x}{1+x}\right)'}{\frac{1-x}{1+x}} =$$

$$= \frac{1+x}{1-x} \frac{(1-x)'(1+x) - (1-x)(1+x)'}{(1+x)^2} =$$

$$= \frac{1+x}{1-x} \frac{(-1)(1+x) - (+1)(1-x)}{(1+x)^2} =$$

$$= \frac{1+x}{1-x} \frac{-1-x-1+x}{(1+x)^2} = \frac{1+x}{1-x} \frac{-2}{(1+x)^2} =$$

$$= \frac{-2}{(1-x)(1+x)} = \frac{2}{x^2-1} \qquad (3)$$

From Eq. (1), Eq. (2), Eq. (3), it follows that

$$I = \frac{-\ln 3}{2} - \int_0^{1/2} x \frac{2}{x^2-1}\,dx =$$

$$= \frac{-\ln 3}{2} - \int_0^{1/2} \frac{2x}{x^2-1}\,dx =$$

$$= \frac{-\ln 3}{2} - \int_0^{1/2} \frac{(x^2-1)'}{x^2-1}\,dx =$$

$$= \frac{-\ln 3}{2} - \Big[\ln|x^2-1| \Big]_0^{1/2} =$$

$$= \frac{-\ln 3}{2} - \Big[\ln|(1/2)^2-1| - \ln|0^2-1| \Big] =$$

$$= \frac{-\ln 3}{2} - \big(\ln(3/4) - \ln 1 \big) =$$

$$= \frac{-\ln 3}{2} - (\ln 3 - \ln 4) =$$

$$= \left(-\frac{1}{2} - 1\right) \ln 3 + 2\ln 2$$

$$= \frac{-3\ln 3}{2} + 2\ln 2 = \frac{4\ln 2 - 3\ln 3}{2}$$

Miscelleneous Integrals

Sometimes we get integrals that cannot be categorized and human creativity becomes necessary

EXAMPLE

a) $I = \int_0^{\pi/4} \frac{x}{\cos^2 x}\, dx = \int_0^{\pi/4} x\,(\tan x)'\, dx =$

$= \left[x \tan x\right]_0^{\pi/4} - \int_0^{\pi/4} (x)' \tan x\, dx =$

$= \left[\frac{\pi}{4}\tan\left(\frac{\pi}{4}\right) - 0 \tan 0\right] - \int_0^{\pi/4} \tan x\, dx =$

$= \frac{\pi}{4} - \int_0^{\pi/4} \frac{\sin x}{\cos x}\, dx = \frac{\pi}{4} + \int_0^{\pi/4} \frac{(\cos x)'}{\cos x}\, dx =$

$= \frac{\pi}{4} + \left[\ln|\cos x|\right]_0^{\pi/4} = \frac{\pi}{4} + \left[\ln|\cos(\pi/4)| - \ln|\cos 0|\right]$

$= \frac{\pi}{4} + \ln\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} + \ln\left(2^{-1/2}\right) =$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

b) $I = \int_0^1 \operatorname{Arctan} x \, dx = \int_0^1 (x)' \operatorname{Arctan} x \, dx =$

$= \left[x \operatorname{Arctan} x \right]_0^1 - \int_0^1 x (\operatorname{Arctan} x)' dx =$

$= 1 \operatorname{Arctan} 1 - 0 \cdot \cancel{\operatorname{Arctan} 0}^{0} - \int_0^1 x \cdot \frac{1}{1+x^2} dx$

$= \frac{\pi}{4} - \frac{1}{2} \int_0^1 \frac{2x}{1+x^2} dx =$

$= \frac{\pi}{4} - \frac{1}{2} \int_0^1 \frac{(1+x^2)'}{(1+x^2)} dx = \frac{\pi}{4} - \frac{1}{2} \left[\ln |1+x^2| \right]_0^1$

$= \frac{\pi}{4} - \frac{1}{2} \left[\ln (1+1^2) - \cancel{\ln (1+0^2)}^{0} \right] =$

$= \frac{\pi}{4} - \frac{1}{2} \ln 2$

EXERCISES

④ Evaluate the following integrals

a) $I = \int_0^2 (3x+1)e^{2x}\,dx$

b) $I = \int_1^3 x^2 e^{-x}\,dx$

c) $I = \int_0^3 x(x+1)e^{4x}\,dx$

d) $I = \int_1^{\sqrt{2}} (x-1)(x^2+x+1)e^{-2x}\,dx$

e) $I = \int_0^{\pi/4} x^2\cos(3x)\,dx$

f) $I = \int_{\pi/6}^{\pi/4} (2x-1)\sin(2x)\,dx$

g) $I = \int x^3 \sin x \cos x\,dx$

h) $I = \int_0^{\pi/6} x^2[\cos x\cos 2x - \sin x \sin 2x]\,dx$

⑤ Evaluate the following integrals.

a) $I = \int_0^{\pi/4} e^{-x}\sin(2x)\,dx$

b) $I = \int e^{2x}\cos^2 x\,dx$

c) $I = \int_0^{\pi/6} e^{-3x}\sin^2 x\,dx$

d) $I = \int \dfrac{\ln x}{x^2}\,dx$

e) $I = \int_1^3 \dfrac{\ln(3x^2)}{x^5}\,dx$

f) $I = \int (x^2-x)\ln(3x+1)\,dx$

g) $I = \int x^4 \ln(x^7)\,dx$

h) $I = \int \ln\left(\dfrac{2x-1}{2x+1}\right)dx$

i) $I = \int_1^{\sqrt{2}} \ln\left(\dfrac{3x-2}{3x+2}\right)dx$

⑥ Use integration by parts to show that

a) $\displaystyle\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}(a\sin(bx) - b\cos(bx))}{a^2+b^2} + C$

b) $\displaystyle\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}(a\cos(bx) + b\sin(bx))}{a^2+b^2} + C$

c) $\displaystyle\int x^a \ln x\,dx = \frac{x^{a+1}\left[(a+1)\ln x - 1\right]}{(a+1)^2} + C$, with $a \neq -1$.

d) $\displaystyle\int x^a (\ln x)^2\,dx = \frac{x^{a+1}\left[(a+1)^2(\ln x)^2 - 2(a+1)\ln x + 2\right]}{(a+1)^3} + C$, with $a \neq -1$.

⑦ Evaluate the following integrals.

a) $\displaystyle I = \int_{\pi/6}^{\pi/3} \frac{2x+1}{\sin^2 x}\,dx$

b) $\displaystyle I = \int_0^{\sqrt{3}} x \operatorname{Arctan}(x)\,dx$

c) $\displaystyle I = \int_0^{\sqrt{2}/2} \operatorname{Arcsin}(x)\,dx$

d) $\displaystyle I = \int_1^e \ln^3(x).\,dx$

⑧ We define the logarithmic integral function

$$Li(x) = \int_2^x \frac{dt}{\ln t}, \quad \forall x \in (1,+\infty)$$

and the incomplete gamma function

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\,dt, \quad \forall x \in \mathbb{R},$$

Show that:

a) $I = \int_e^a \ln(\ln(x))\,dx = a\ln(\ln a) - [Li(a) - Li(e)]$

b) $I = \int_2^a Li(x)\,dx = aLi(a) + 2[\gamma(0, 2\ln a) - \gamma(0, 2\ln 2)]$

↳ Hint: In both cases, use integration by parts. Question (b) will also require the Fundamental theorem of Calculus - Part I and a change of variables.

Evaluating integrals via recursion

An expedient method for evaluating integrals is by deriving and employing recurrence relations as follows:

EXAMPLE

Define $I_n = \int_0^{\pi/2} \sin^n x\,dx$. Show that

a) $I_n = \frac{n-1}{n} I_{n-2}, \quad \forall n \in \mathbb{N} : n \geq 3$

b) Evaluate $\int_0^{\pi/2} \sin^7 x\,dx$

Solution

a) $I_n = \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \sin^{n-1} x \cdot \sin x\,dx =$

$= \int_0^{\pi/2} \sin^{n-1} x \cdot (-\cos x)'\,dx =$

$= \left[-\sin^{n-1} x \cos x \right]_0^{\pi/2} - \int_0^{\pi/2} (\sin^{n-1} x)' (-\cos x)\,dx$

$= \left[-\sin^{n-1}(\pi/2)\cos(\pi/2) + \sin^{n-1}(0)\cos 0 \right]$

$+ \int_0^{\pi/2} (n-1)\sin^{n-2} x \cdot (\sin x)' \cos x\,dx =$

$$= \int_0^{\pi/2} (n-1) \sin^{n-2} x \cos^2 x \, dx =$$

$$= \int_0^{\pi/2} (n-1) \sin^{n-2} x \, (1 - \sin^2 x) \, dx =$$

$$= (n-1) \int_0^{\pi/2} \sin^{n-2} x \, dx - (n-1) \int_0^{\pi/2} \sin^n x \, dx =$$

$$= (n-1) I_{n-2} - (n-1) I_n \Leftrightarrow$$

$$\Leftrightarrow I_n + (n-1) I_n = (n-1) I_{n-2} \Leftrightarrow$$

$$\Leftrightarrow n I_n = (n-1) I_{n-2} \Leftrightarrow I_n = \frac{n-1}{n} I_{n-2}.$$

$$\text{β)} \int_0^{\pi/2} \sin^7 x \, dx = I_7 = \frac{7-1}{7} I_5 = \frac{6}{7} I_5 =$$

$$= \frac{6}{7} \frac{5-1}{5} I_3 = \frac{6}{7} \frac{4}{5} \frac{3-1}{3} I_1 =$$

$$= \frac{6 \cdot 4 \cdot 2}{7 \cdot 5 \cdot 3} \int_0^{\pi/2} \sin x \, dx = \frac{2 \cdot 4 \cdot 2}{7 \cdot 5} \Big[-\cos x \Big]_0^{\pi/2} =$$

$$= \frac{16}{35} \Big[-\cos(\pi/2) + \cos 0 \Big] = \frac{16}{35} (-0 + 1)$$

$$= \frac{16}{35}.$$

EXERCISES

(9) Consider the function $f_n(x) = \int_0^x t^n e^t \, dt$, $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}$

a) Show that

$f_n(x) = x^n e^x - n f_{n-1}(x)$, $\forall x \in \mathbb{R}, \forall n \in \mathbb{N}^*$

b) Use the recurrence relation to evaluate the integral

$I = \int_0^{\ln 3} (x^2+2x-1)(x+1) e^x \, dx$

↳ Hint: First write I in terms of $f_n(x)$. Then evaluate the corresponding terms via the recurrence relation.

(10) Consider the functions:

$f_n(x) = \int_0^x t^n \sin t \, dt$, $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}$

$g_n(x) = \int_0^x t^n \cos t \, dt$, $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}$

a) Show that:

$f_n(x) = -x^n \cos x + n g_{n-1}(x)$, $\forall n \in \mathbb{N}^*, \forall x \in \mathbb{R}$

$g_n(x) = x^n \sin x - n f_{n-1}(x)$, $\forall n \in \mathbb{N}^*, \forall x \in \mathbb{R}$

b) Use the recurrence relations to evaluate

$I_1 = \int_0^{\pi/4} x^3 (2x-1) \cos x \, dx$

$I_2 = \int_0^{\pi/6} x^2 (x^2+2) \sin x \, dx$

(11) Consider the function $f_n(x) = \int_1^x (\ln t)^n \, dt$, $\forall n \in \mathbb{N}, \forall x \in (0, +\infty)$

a) Show that:

$$f_n(x) = x(\ln x)^n - n f_{n-1}(x) \quad , \quad \forall x \in (0, +\infty), \forall n \in \mathbb{N}^*$$

b) Use the recurrence relation to evaluate the integral

$$I = \int_1^{e^3} \ln x \, [\ln x - 2]^2 \, dx$$

(12) We define the sequence $I_n = \int_0^{\pi/2} e^x \cos^n x \, dx$

a) Show that

$$I_n = \frac{-1}{n^2+1} + \frac{n(n-1)}{n^2+1} I_{n-2} \quad , \quad \forall n \in \mathbb{N} - \{0, 1, 2\}$$

b) Use the recurrence relation to evaluate the integral

$$J = \int_0^{\pi/2} e^x \cos^5 x \, dx$$

(13) Use integration by parts and the fundamental theorem of Calculus - Part 1, to show that

$$\int_0^x dt \int_0^t dz \, f(z) = \int_0^x (x-t) f(t) \, dt$$

(14) Let $f: A \to \mathbb{R}$ be a twice differentiable equation with $a, b \in A$ and $f'(a) = f'(b) = 0$. Show that

$$\int_a^b f''(t) \, f(t) \, dt \leq 0$$ (Hint: Rewrite the integral in terms of $[f'(t)]^2$)

Integrals of rational functions

These are integrals of the form

$$I = \int \frac{P(x)}{Q(x)}\, dx$$

where P, Q are polynomials. In general, these integrals can always be evaluated, depending on the factorization of the denominator $Q(x)$.

1 → $\boxed{\text{$Q(x)$ as a product of distinct linear factors.}}$

Assume that $Q(x)$ can be written as

$$Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$$

with $a_1 \neq a_2 \neq a_3 \cdots \neq a_n \neq a_1$. We also assume that $\deg P(x) < \deg Q(x)$. We define $[n] = \{1, 2, 3, \ldots, n\}$. Then we can show that

$$\boxed{\begin{gathered} f(x) = \frac{P(x)}{\prod_{k \in [n]} (x - a_k)} = \sum_{k \in [n]} \frac{A_k}{x - a_k} \\ \text{with} \\ \forall k \in [n]:\ A_k = \left. \frac{P(x)}{\prod_{\ell \in [n] - \{k\}} (x - a_\ell)} \right|_{x = a_k} \end{gathered}}$$

<u>Mnemonic Rule</u> : To find A_k, delete the factor $x-a_k$ from $f(x)$ and the set $x=a_k$.

↳ This technique is known as the Parāvartya-Ŝūtra method and has been known since ancient times.

The resulting integrals are calculated by the following integration formula:

$$\int \frac{dx}{x-a} = \ln|x-a| + C$$

EXAMPLE

Evaluate $I = \int_0^1 \frac{3x-2}{x^3+x^2-4x-4}\,dx$

Solution

We note that

$$Q(x) = x^3+x^2-4x-4 = x^2(x+1) - 4(x+1) =$$
$$= (x+1)(x^2-4) = (x+1)(x-2)(x+2)$$

and therefore

$$f(x) = \frac{3x-2}{(x-2)(x+2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x+1}$$

with

$$A = \left.\frac{3x-2}{(x+2)(x+1)}\right|_{x=2} = \frac{3\cdot 2-2}{(2+2)(2+1)} = \frac{6-2}{4\cdot 3} =$$
$$= \frac{4}{4\cdot 3} = \frac{1}{3}$$

$$B = \left.\frac{3x-2}{(x-2)(x+1)}\right|_{x=-2} = \frac{3(-2)-2}{[(-2)-2][(-2)+1]} =$$
$$= \frac{-6-2}{(-4)(-1)} = \frac{-8}{4} = -2$$

$$C = \left.\frac{3x-2}{(x-2)(x+2)}\right|_{x=-1} = \frac{3(-1)-2}{[(-1)-2][(-1)+2]} =$$
$$= \frac{-3-2}{(-3)(+1)} = \frac{-5}{-3} = \frac{5}{3}$$

and therefore,

$$I = \int_0^1 \frac{3x-2}{(x-2)(x+2)(x+1)}\,dx =$$

$$= \frac{1}{3}\int_0^1 \frac{dx}{x-2} - 2\int_0^1 \frac{dx}{x+2} + \frac{5}{3}\int_0^1 \frac{dx}{x+1} =$$

$$= \frac{1}{3}\Big[\ln|x-2|\Big]_0^1 - 2\Big[\ln|x+2|\Big]_0^1 + \frac{5}{3}\Big[\ln|x+1|\Big]_0^1$$

$$= (1/3)(\ln|1-2| - \ln|0-2|) - 2(\ln|1+2| - \ln|0+2|)$$
$$+ (5/3)(\ln|1+1| - \ln|0+1|) =$$

$$= (1/3)(\ln 1 - \ln 2) - 2(\ln 3 - \ln 2) + (5/3)(\ln 2 - \ln 1)$$

$$= (1/3)(-\ln 2) - 2(\ln 3 - \ln 2) + (5/3)\ln 2$$

$$= [-(1/3) + 2 + (5/3)]\ln 2 - 2\ln 3$$

$$= ((-1+6+5)/3)\ln 2 - 2\ln 3$$

$$= (10/3)\ln 2 - 2\ln 3.$$

EXERCISES

(16) Evaluate the following integrals:

a) $I = \int \frac{3x+1}{x^2+2x}\,dx$

b) $I = \int \frac{3x-2}{x^2+5x+6}\,dx$

c) $I = \int \frac{x^2+3x-1}{x^3-4x^2-4x+16}\,dx$

d) $I = \int \frac{2x^2+5x-1}{(x-1)(x-2)(x+3)}\,dx$

e) $I = \int \frac{x^2+4}{x^3-9x^2}\,dx$

f) $I = \int \frac{2x^2-x-1}{x^3-4x}\,dx$

g) $I = \int \frac{x^2+3x+5}{x^3-2x}\,dx$

h) $I = \int \frac{x^3+x^2-x+2}{3x^4-4x^3-12x^2+16}\,dx$

(17) Similarly for the following more challenging integrals.

a) $I = \int \frac{2x^2+3x-1}{x^3+7x^2+14x+8}\,dx$

b) $I = \int \frac{x^2+3x+8}{x^3+7x^2+7x-15}\,dx$

↳ To factor the denominator, use polynomial division as well as the results:

a) If $P(x)$ is a polynomial then:

$(x-a) \mid P(x) \Leftrightarrow P(a)=0$

b) If $P(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ then

$(P(p)=0 \wedge p \in \mathbb{Q}) \Rightarrow p \mid a_0$.

c) $I = \int \frac{2x^2+3}{(x^2-1)(x^2+1)} dx$

↳ Because the integral is exclusively dependent on x^2, we can use the Parāvartya-Sutra method to expand:

$$\frac{2y+3}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1}$$

and then set $y = x^2$.

2 → **Q(x) is a product of linear factors some of which are repeated.**

In general, it can be shown that

$$f(x) = \frac{P(x)}{(x-a)^n Q(x)} = \sum_{k=1}^{n} \frac{A_k}{(x-a)^k} + \frac{P_1(x)}{Q(x)}$$

if we assume that $\deg P(x) < n + \deg Q(x)$.
The highest order A_n can be found by:

$$\boxed{A_n = \left.\frac{P(x)}{Q(x)}\right|_{x=a} = \frac{P(a)}{Q(a)}}$$

The technique of the previous case can still be applied on all coefficients associated with linear non-repeating factors. There are multiple techniques for finding the coefficients $A_1, A_2, \dots, A_{n-1}$:

- 1 In general:

$$\boxed{\forall k \in [n]: \quad A_k = \frac{1}{(n-k)!} \frac{d^{n-k}}{dx^{n-k}} \left(\frac{P(x)}{Q(x)} \right) \Bigg|_{x=a}}$$

will give all the coefficients, but calculating the derivatives can be a lot of work.
We define $0! = 1$ and $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

• 2 For the special case $n=2$, we first find A_2. To find A_1, we multiply both sides of the decomposition with x and take the limit $x \to +\infty$. Consider

$$f(x) = \frac{P(x)}{(x-a)^2 Q(x)} = \frac{A_2}{(x-a)^2} + \frac{A_1}{(x-a)} + \frac{P_1(x)}{Q(x)}$$

Then, it follows that

$$\lim_{x \to +\infty} \frac{x P(x)}{(x-a)^2 Q(x)} = \lim_{x \to +\infty} \frac{A_2 x}{(x-a)^2} + \lim_{x \to +\infty} \frac{A_1 x}{x-a} + \lim_{x \to +\infty} \frac{x P_1(x)}{Q(x)} \Leftrightarrow$$

$$\Leftrightarrow L_1 = 0 + A_1 + L_2 \Leftrightarrow A_1 = L_1 - L_2$$

with

$$L_1 = \lim_{x \to +\infty} \frac{x P(x)}{(x-a)^2 Q(x)} \quad \text{and} \quad L_2 = \lim_{x \to +\infty} \frac{x P_1(x)}{Q(x)}$$

Note that A_1 can be calculated without knowing A_2! For $n>2$, this technique can still be used to find A_1. However, it will not work conveniently on the other coefficients. So, for $n=3$ we can easily find A_1, A_3 but need to do something else to find A_2. Same consideration applies for $n = 4, 5, 6, \ldots$

• 3 The above techniques can be augmented by subtracting the known terms in the partial fraction decomposition from the original function $f(x)$ and then attempting to decompose the resulting difference to obtain subleading terms.

EXAMPLE

Evaluate $I = \int \frac{3x^2 - 16x + 21}{(x-1)^2 (x+3)} dx$

Solution

We note that

$$f(x) = \frac{3x^2 - 16x + 21}{(x-1)^2 (x+3)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+3}$$

with

$$A = \left. \frac{3x^2 - 16x + 21}{x+3} \right|_{x=1} = \frac{3 \cdot 1^2 - 16 \cdot 1 + 21}{1+3} = \frac{3 - 16 + 21}{4} =$$

$$= \frac{8}{4} = 2$$

$$C = \left. \frac{3x^2 - 16x + 21}{(x-1)^2} \right|_{x=-3} = \frac{3(-3)^2 - 16(-3) + 21}{((-3)-1)^2} =$$

$$= \frac{27 + 48 + 21}{16} = \frac{96}{16} = 6$$

$$B = \left. \frac{1}{(2-1)!} \frac{d}{dx} \left(\frac{3x^2 - 16x + 21}{x+3} \right) \right|_{x=1} =$$

$$= \left. \frac{(3x^2 - 16x + 21)'(x+3) - (3x^2 - 16x + 21)(x+3)'}{(x+3)^2} \right|_{x=1} =$$

$$= \left. \frac{(6x - 16)(x+3) - (3x^2 - 16x + 21)}{(x+3)^2} \right|_{x=1}$$

$$= \frac{(6 \cdot 1 - 16)(1+3) - (3 \cdot 1^2 - 16 \cdot 1 + 21)}{(1+3)^2} =$$

$$= \frac{(6-16)4-(3-16+21)}{4^2} = \frac{(-10)4-8}{16} =$$

$$= \frac{-40-8}{16} = \frac{-48}{16} = -3$$

and therefore

$$I = \int \frac{3x^2-16x+21}{(x-1)^2(x+3)}\,dx =$$

$$= \int \left[\frac{2}{(x-1)^2} + \frac{-3}{x-1} + \frac{6}{x+3} \right] dx =$$

$$= \frac{2}{-1}\,\frac{1}{x-1} - 3\ln|x-1| + 6\ln|x+3| + C_1$$

$$= \frac{-2}{x-1} - 3\ln|x-1| + 6\ln|x+3| + C_1$$

Alternate methods for determining B.

<u>2nd method</u>: Given A, C:

$$\frac{3x^2-16x+21}{(x-1)^2(x+3)} = \frac{2}{(x-1)^2} + \frac{B}{x-1} + \frac{6}{x+3} \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} \frac{x(3x^2-16x+21)}{(x-1)^2(x+3)} = \lim_{x\to+\infty} \frac{2x}{(x-1)^2} + \lim_{x\to+\infty} \frac{Bx}{x-1}$$

$$+ \lim_{x\to+\infty} \frac{6x}{x+3} \Rightarrow$$

$$\Rightarrow 3 = 0 + B + 6 \Rightarrow B = 3-6 = -3.$$

3rd method : Given A, G we have

$$\frac{3x^2-16x+21}{(x-1)^2(x+3)} = \frac{2}{(x-1)^2} + \frac{B}{x-1} + \frac{6}{x+3} \Rightarrow$$

$$\Rightarrow \frac{B}{x-1} = \frac{3x^2-16x+21}{(x-1)^2(x+3)} - \frac{2}{(x-1)^2} - \frac{6}{x+3} =$$

$$= \frac{(3x^2-16x+21)-2(x+3)-6(x-1)^2}{(x-1)^2(x+3)} =$$

$$= \frac{3x^2-16x+21-2x-6-6(x^2-2x+1)}{(x-1)^2(x+3)}$$

$$= \frac{3x^2-16x+21-2x-6-6x^2+12x-6}{(x-1)^2(x+3)}$$

$$= \frac{(3-6)x^2+(-16-2+12)x+(21-6-6)}{(x-1)^2(x+3)}$$

$$= \frac{-3x^2-6x+9}{(x-1)^2(x+3)} = \frac{-3(x^2+2x-3)}{(x-1)^2(x+3)} =$$

$$= \frac{-3(x-1)(x+3)}{(x-1)^2(x+3)} = \frac{-3}{x-1} \Rightarrow B = -3.$$

EXERCISES

(18) Evaluate the following integrals.

a) $I = \int \frac{x^2+3x-1}{(x-2)^2(x+1)}\,dx$ b) $I = \int \frac{2x^2+7x-1}{(x+1)^2(2x-1)}\,dx$

c) $I = \int \frac{x^3+3x+1}{(x+2)^3(x-1)}\,dx$

↳ To find the partial function decomposition, we note that

$$R(x) = \frac{x^3+3x+1}{(x+2)^3(x-1)} = \frac{A}{(x+2)^3} + \frac{B}{(x+2)^2} + \frac{C}{x+2} + \frac{D}{x-1}$$

and find A, D via Parāvartya-Sūtra method and then simplify:

$$R(x) - \frac{A}{(x+2)^3} - \frac{D}{x-1} = \dots = \frac{B_1 x + C_1}{(x+2)^2}$$

which does not really require further decomposition.

d) $I = \int \frac{x^2+3x+5}{(3x+2)^2(x-1)}\,dx$

↳ Rewrite $(3x+2)^2 = 9(x+2/3)^2$ and then apply the Parāvartya-Sutra method.

e) $I = \int \frac{x^3+5x+1}{(x-4)^2(x-2)^2}\,dx$

↳ To find the partial fraction decomposition, we note that

$$R(x) = \frac{x^3+5x+1}{(x-4)^2(x-2)^2} = \frac{A}{(x-4)^2} + \frac{B}{x-4} + \frac{C}{(x-2)^2} + \frac{D}{x-2}$$

and use the Parāvartya-Sūtra method to find A, C.
We the simplify

$$R(x) - \frac{A}{(x-4)^2} - \frac{C}{(x-2)^2} = \cdots = \frac{B}{x-4} + \frac{D}{x-2}$$

and use a second iteration of the Parāvartya-Sūtra method to find B, D.

f) $$I = \int \frac{3x^3+2x^2+5}{(x+1)^2(x+3)^2-(x+1)(x+3)^3}\,dx$$

↳ You need to factor the denominator in order to find the partial fraction decomposition and evaluate the integral.

(3) Integrals of the form $I = \int \frac{Ax+B}{ax^2+bx+c}\,dx$ with ax^2+bx+c an irreducible quadratic

(i.e. cannot be factored into a product of real linear factors).

To evaluate such integrals, we break the function to a contribution of the form

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

and a contribution of the form

$$\int \frac{dx}{ax^2+bx+c}$$

To evaluate the second integral, we complete the square on the denominator, using the identity

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}$$

with $\Delta = b^2-4ac$, and use the identities

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\operatorname{Arctan}\left(\frac{x}{a}\right) + C$$

$$\int_a^b f(x)\,dx = \int_{a-c}^{b-c} f(x+c)\,dx$$

to evaluate the integral.

EXAMPLE

Evaluate $I = \int_{-1}^{1} \frac{3x+2}{x^2+2x+6}\, dx$

Solution

We note that

$$I = \int_{-1}^{1} \frac{3x+2}{x^2+2x+6}\, dx = \int_{-1}^{1} \frac{(3/2)(2x+2)-3+2}{x^2+2x+6}\, dx =$$

$$= \frac{3}{2}\int_{-1}^{1} \frac{2x+2}{x^2+2x+6}\, dx - \int_{-1}^{1} \frac{dx}{x^2+2x+6}$$

$$= (3/2) I_1 - I_2$$

with

$$I_1 = \int_{-1}^{1} \frac{2x+2}{x^2+2x+6}\, dx = \int_{-1}^{1} \frac{(x^2+2x+6)'}{x^2+2x+6}\, dx =$$

$$= \Big[\ln|x^2+2x+6| \Big]_{-1}^{1} =$$

$$= \ln|1^2+2\cdot 1+6| - \ln|(-1)^2+2(-1)+6| =$$

$$= \ln|1+2+6| - \ln|1-2+6| = \ln 9 - \ln 5$$

and

$$I_2 = \int_{-1}^{1} \frac{dx}{x^2+2x+6} = \int_{-1}^{1} \frac{dx}{(x^2+2x+1)+5} =$$

$$= \int_{-1}^{1} \frac{dx}{(x+1)^2+5} = \int_{0}^{2} \frac{dx}{x^2+5} = \int_{0}^{2} \frac{dx}{x^2+(\sqrt{5})^2}$$

$$= \left[\frac{1}{\sqrt{5}} \operatorname{Arctan}\left(\frac{x}{\sqrt{5}}\right) \right]_{0}^{2} =$$

$$= \frac{1}{\sqrt{5}} \operatorname{Arctan}\left(\frac{2}{\sqrt{5}}\right) - \frac{1}{\sqrt{5}} \operatorname{Arctan}\left(\frac{0}{\sqrt{5}}\right) =$$

$$= \frac{1}{\sqrt{5}} \operatorname{Arctan}\left(\frac{2}{\sqrt{5}}\right)$$

and therefore

$$I = \frac{3I_1}{2} - I_2 = \frac{3(\ln 9 - \ln 5)}{2} - \frac{1}{\sqrt{5}} \operatorname{Arctan}\left(\frac{2}{\sqrt{5}}\right)$$

EXERCISES

(15) Evaluate the following integrals.

a) $I = \displaystyle\int_{1}^{4} \frac{dx}{x+3}$

b) $I = \displaystyle\int_{\sqrt{2}}^{\sqrt{3}} \frac{dx}{x-1}$

c) $I = \displaystyle\int_{3}^{3\sqrt{3}} \frac{dx}{x^2+9}$

d) $I = \displaystyle\int_{\sqrt{2}}^{\sqrt{6}} \frac{dx}{x^2+2}$

e) $I = \displaystyle\int_{4}^{5} \frac{3x\,dx}{x^2+1}$

f) $I = \displaystyle\int_{\sqrt{2}}^{\sqrt{5}} \frac{5x\,dx}{x^2+3}$

g) $I = \displaystyle\int_{\sqrt{3}}^{2} \frac{5x\,dx}{x^2-4}$

h) $I = \displaystyle\int_{1}^{1+\sqrt{2}} \frac{x\sqrt{2}}{x^2+2}\,dx$

i) $I = \displaystyle\int \frac{2x+3}{x^2+3x+1}\,dx$

j) $I = \displaystyle\int_{\sqrt{2}-1}^{\sqrt{2}+1} \frac{2x-1}{x^2-x+3}\,dx$

k) $I = \displaystyle\int \frac{3x+4}{x^2+5x+2}\,dx$

l) $I = \displaystyle\int \frac{5x-2}{x^2+4x+3}\,dx$

m) $I = \displaystyle\int_{1}^{3} \frac{x+1}{x^2+2x+3}\,dx$

n) $I = \displaystyle\int_{\sqrt{3}}^{\sqrt{7}} \frac{x\sqrt{2}+3}{x^2+7x+12}\,dx$

o) $I = \displaystyle\int \frac{(\sqrt{2}+\sqrt{3})x+1}{x^2+(\sqrt{2}+1)x+\sqrt{2}}\,dx$

p) $I = \displaystyle\int_{1}^{\sqrt{3}} \frac{x\sqrt{2}+\sqrt{3}}{x^2\sqrt{2}+(\sqrt{3}+1)x+2}\,dx$

q) $I = \displaystyle\int \frac{5-x}{x^2-x(\pi+4)+4\pi}\,dx$

④ → Integrals with non-repeating quadratic factors

Consider the integral

$$I = \int \frac{P(x)}{(x^2+ax+b)Q(x)}\,dx$$

with $\deg P(x) < 2 + \deg Q(x)$, where we assume that $Q(x)$ does not contain any additional factors x^2+ax+b. Then, the corresponding decomposition is

$$f(x) = \frac{Ax+B}{x^2+ax+b} + \frac{P_1(x)}{Q(x)}$$

If $z = p+qi$ is a zero of the equation $x^2+ax+b=0$ then we can calculate A,B via the equation

$$\boxed{A(p+qi)+B = \left.\frac{P(x)}{Q(x)}\right|_{x=p+qi}}$$

This reduces to a system of two equations via the property

$$\forall a_1,a_2,b_1,b_2 \in \mathbb{R} : (a_1+b_1 i = a_2+b_2 i \Leftrightarrow a_1=a_2 \wedge b_1=b_2)$$

that we can use to solve for A,B.

EXAMPLE

Evaluate $I = \int \dfrac{3x}{(x^2+2)(x-1)}\,dx$

Solution

We note that

$$f(x) = \frac{3x}{(x^2+2)(x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x-1}$$

with

$$C = \left.\frac{3x}{x^2+2}\right|_{x=1} = \frac{3\cdot 1}{1^2+2} = \frac{3}{3} = 1$$

$$A(i\sqrt{2}) + B = \left.\frac{3x}{x-1}\right|_{x=i\sqrt{2}} = \frac{3\sqrt{2}\,i}{i\sqrt{2}-1} =$$

$$= \frac{3\sqrt{2}i\,(i\sqrt{2}+1)}{(i\sqrt{2}-1)(i\sqrt{2}+1)} = \frac{6i^2+3\sqrt{2}i}{2i^2-1} =$$

$$= \frac{-6+3\sqrt{2}i}{-2-1} = \frac{-6+3\sqrt{2}i}{-3} = 2-\sqrt{2}i \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} A\sqrt{2} = -\sqrt{2} \\ B = 2 \end{cases} \Leftrightarrow \begin{cases} A = -1 \\ B = 2 \end{cases}$$

and therefore

$$I = \int \frac{3x}{(x^2+2)(x-1)}\,dx =$$

$$= \int \left[\frac{-x+2}{x^2+2} + \frac{1}{x-1}\right] dx =$$

$$= \frac{-1}{2}\int \frac{2x\,dx}{x^2+2} + 2\int \frac{dx}{x^2+2} + \int \frac{dx}{x-1} =$$

$$= \frac{-1}{2}\int \frac{(x^2+2)'}{x^2+2}\,dx + 2\int \frac{dx}{x^2+(\sqrt{2})^2} + \ln|x-1| + C$$

$$= \frac{-1}{2}\ln|x^2+2| + 2\,\frac{1}{\sqrt{2}}\,\mathrm{Arctan}\left(\frac{x}{\sqrt{2}}\right) + \ln|x-1| + C$$

$$= \sqrt{2}\,\mathrm{Arctan}(x/\sqrt{2}) - (1/2)\ln(x^2+2) + \ln|x-1| + C$$

(19) Evaluate the following integrals.

a) $I = \int \frac{x^2-x+1}{(x^2+2x+4)(x+3)}\,dx$ b) $I = \int \frac{3x+1}{(x^2-x+1)(x+3)}\,dx$

c) $I = \int \frac{3x+1}{2x^3+16}\,dx$ d) $I = \int \frac{x^2+2x-1}{x^3-27}\,dx$

↳ For (c) and (d) the denominators can be factored using

$$a^3+b^3 = (a+b)(a^2-ab+b^2)$$
$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$

e) $I = \int \frac{3x}{x^4-1}\,dx$ f) $I = \int \frac{dx}{2x^4-32}$

↳ Note the factorization

$$x^4-1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$$

g) $I = \int \frac{3x-2}{x^4-2x^3-x+2}\,dx$ h) $I = \int \frac{x^2-3x+1}{x^3+3x^2+4x+12}\,dx$

5 → For all other more complex cases, such as
(a) Repeated quadratic factors
(b) More than one linear repeating factor

we can use the <u>method of undetermined coefficients</u>

- 1 Rewrite the partial fraction decomposition as a polynomial equation.
- 2 Set x equal to the zeroes of the denominators factors to obtain the coefficients of the leading terms of the partial fraction decomposition
- 3 The coefficients of the subleading terms can be obtained by trying other strategically chosen values of x.

EXAMPLE

$$I = \int \frac{6x^2 - 15x + 22}{(x+3)(x^2+2)^2}\, dx$$

<u>Solution</u>

Let $f(x) = \dfrac{6x^2 - 15x + 22}{(x+3)(x^2+2)^2} = \dfrac{A}{x+3} + \dfrac{Bx+C}{x^2+2} + \dfrac{Dx+E}{(x^2+2)^2} \Leftrightarrow$

$\Leftrightarrow A(x^2+2)^2 + (Bx+C)(x+3)(x^2+2) + (Dx+E)(x+3) = 6x^2 - 15x + 22, \forall x \in \mathbb{R}$ (1)

▸ We begin by trying the zeroes of the denominator:

For $x = -3$, from Eq. (1):

$A[(-3)^2 + 2]^2 + 0 + 0 = 6(-3)^2 - 15(-3) + 22 \Leftrightarrow$

$\Leftrightarrow A(9+2)^2 = 6\cdot 9+45+22 \Leftrightarrow 121A = 54+45+22 \Leftrightarrow$

$\Leftrightarrow 121A = 121 \Leftrightarrow A = 1.$

For $x = i\sqrt{2}$ (the root of x^2+2), from Eq.(1)

$0+0+[D(i\sqrt{2})+E](i\sqrt{2}+3) = 6(i\sqrt{2})^2 - 15(i\sqrt{2})+22 \Leftrightarrow$

$$\Leftrightarrow D(i\sqrt{2})+E = \frac{6(i\sqrt{2})^2-15(i\sqrt{2})+22}{i\sqrt{2}+3} =$$

$$= \frac{6(-2)-(15\sqrt{2})i+22}{i\sqrt{2}+3} = \frac{-12-(15\sqrt{2})i+22}{3+i\sqrt{2}}$$

$$= \frac{10-(15\sqrt{2})i}{3+i\sqrt{2}} = \frac{5[2-(3\sqrt{2})i]}{3+i\sqrt{2}}$$

$$= \frac{5[2-(3\sqrt{2})i][3-i\sqrt{2}]}{[3+i\sqrt{2}][3-i\sqrt{2}]} = \frac{5[6-(2\sqrt{2})i-(9\sqrt{2})i-6]}{9-(-2)}$$

$$= \frac{5[-(11\sqrt{2})i]}{11} = -(5\sqrt{2})i \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} D\sqrt{2} = -5\sqrt{2} \\ E = 0 \end{cases} \Leftrightarrow \begin{cases} D = -5 \\ E = 0 \end{cases}$$

↳ Note that we may indeed assume $D, E \in \mathbb{R}$ and then use the proposition:

$\forall a,b,c,d \in \mathbb{R} : (a+bi = c+di \Leftrightarrow (a=c \land b=d))$

to justify the above argument. Also note that using just one of the two conjugate roots is sufficient for obtaining both coefficients.

↳ Before dealing with the coefficients of the subleading terms, we use $A=1 \land D=-5 \land E=0$ to simplify Eq.(1)

$(1) \Leftrightarrow (x^2+2)^2 + (Bx+C)(x+3)(x^2+2) + (-5x)(x+3) = 6x^2-15x+22, \forall x \in \mathbb{R}$

To find $B, C \in \mathbb{R}$ we try:

For $x=0$; we have:

$(0+2)^2 + (0+C)(0+3)(0+2) + 0 = 0 - 0 + 22 \Leftrightarrow$

$\Leftrightarrow 4 + 6C = 22 \Leftrightarrow 6C = 22-4 = 18 \Leftrightarrow C = 3$

For $x=1$, we have:

$(1^2+2)^2 + (B+C)(1+3)(1^2+2) + (-5\cdot 1)(1+3) = 6\cdot 1^2 - 15\cdot 1 + 22 \Leftrightarrow$

$\Leftrightarrow 9 + 12(B+C) - 20 = 6 - 15 + 22 \Leftrightarrow 12(B+C) - 11 = 13 \Leftrightarrow$

$\Leftrightarrow \quad 12(B+C) = 24 \Leftrightarrow B+C = 2 \Leftrightarrow B+3 = 2 \Leftrightarrow B = -1.$

To summarize: $(A,B,C,D,E) = (1,-1,3,-5,0)$

and therefore:

$$f(x) = \frac{1}{x+3} + \frac{-x+3}{x^2+2} + \frac{-5x}{(x^2+2)^2}$$

Since

$$I_1 = \int \frac{dx}{x+3} = \ln|x+3| + C$$

$$I_2 = \int \frac{-x+3}{x^2+2}\,dx = \int \frac{-x\,dx}{x^2+2} + \int \frac{3\,dx}{x^2+2} =$$

$$= \frac{-1}{2}\int \frac{2x\,dx}{x^2+2} + 3\int \frac{dx}{x^2+2} =$$

$$= \frac{-1}{2}\int \frac{(x^2+2)'}{x^2+2}\,dx + 3\int \frac{dx}{x^2+(\sqrt{2})^2} =$$

$$= \frac{-1}{2}\ln|x^2+2| + \frac{3}{\sqrt{2}}\operatorname{Arctan}\left(\frac{x}{\sqrt{2}}\right)$$

and

$$I_3 = \int \frac{-5x\,dx}{(x^2+2)^2} = \frac{-5}{2}\int \frac{2x\,dx}{(x^2+2)^2} = \frac{-5}{2}\int \frac{(x^2+2)'\,dx}{(x^2+2)^2} =$$

$$= \frac{-5}{2}\int \left[\frac{-1}{x^2+2}\right]' dx = \frac{-5}{2}\,\frac{-1}{x^2+2} + c =$$

$$= \frac{5}{2(x^2+2)} + c$$

and it follows that

$$I = I_1 + I_2 + I_3 =$$

$$= \ln|x+3| - \frac{1}{2}\ln|x^2+2| + \frac{3}{\sqrt{2}}\operatorname{Arctan}\left(\frac{x}{\sqrt{2}}\right) + \frac{5}{2(x^2+2)} + c$$

↳ Note that if $E \neq 0$, the problem becomes much harder. However as long as the quadratic term goes up to power 2, this method is safe given the following method for calculating

$$I = \int \frac{dx}{(x^2+a^2)^2}$$

We use a clever rewrite and integration by parts:

$$I = \int \frac{dx}{(x^2+a^2)^2} = \frac{1}{a^2}\int \frac{x^2+a^2-x^2}{(x^2+a^2)^2}\,dx = \frac{1}{a^2}\int \frac{dx}{x^2+a^2} - \frac{1}{a^2}\int \frac{x^2\,dx}{(x^2+a^2)^2}$$

$$= \frac{1}{a^2}\,\frac{1}{a}\operatorname{Arctan}\left(\frac{x}{a}\right) + \frac{1}{a^2}\int \frac{x}{2}\,\frac{-2x}{(x^2+a^2)^2}\,dx =$$

$$= \frac{1}{a^3}\operatorname{Arctan}\left(\frac{x}{a}\right) + \frac{1}{a^2}\int \frac{x}{2}\left(\frac{1}{x^2+a^2}\right)' dx$$

$$= \frac{1}{a^3}\operatorname{Arctan}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)} - \frac{1}{a^2}\int \left(\frac{x}{2}\right)' \frac{1}{x^2+a^2}\,dx =$$

$$= \frac{1}{a^3}\,\mathrm{Arctan}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)} - \frac{1}{2a^2}\int \frac{dx}{x^2+a^2} =$$

$$= \frac{1}{a^3}\,\mathrm{Arctan}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)} - \frac{1}{2a^2}\,\frac{1}{a}\,\mathrm{Arctan}\left(\frac{x}{a}\right) + C$$

$$= \frac{1}{2a^3}\,\mathrm{Arctan}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)} + C$$

A second method to circumvent doing difficult integrals is to use the partial fraction decomposition:

$$f(x) = \frac{6x^2-15x+22}{(x+3)(x^2+2)^2} = \frac{A}{x+3} + \frac{B}{x+i\sqrt{2}} + \frac{C}{(x+i\sqrt{2})^2} + \frac{D}{x-i\sqrt{2}} + \frac{E}{(x-i\sqrt{2})}$$

however B, C, D, E may be complex numbers. Now, we may use Parāvartya-Sutrā method to find A, C, E and undetermined coefficients to find B, D. However the price you pay is that you need to use complex analysis theory to deal with the resulting integrals. Alternatively, you can use the partial fraction decomposition above and stay with real integrals at the expense of getting more challenging integrals.

EXERCISES

(20) Evaluate the following integrals.

a) $I = \int \frac{dx}{(x^2+4)^2}$ b) $I = \int_0^{\sqrt{3}} \frac{dx}{(x^2+3)^2}$

c) $I = \int \frac{(5x-2)}{(x^2+5)^2} dx$ d) $I = \int \frac{3x^4+2x^3+22x^2+10x+47}{(x+2)(x^2+3)^2} dx$

e) $I = \int \frac{8x^4-6x^3+90x^2-39x+243}{(x-2)(x^2+7)^2} dx$

f) $I = \int_0^1 \frac{dx}{x^4+1}$ g) $I = \int \frac{3x}{x^4+x^2+1} dx$

↳ Note the 4^{th}-order factorization

$$x^4+a^4 = x^4+2a^2x^2+a^4-2a^2x^2 =$$
$$= (x^2+a^2)^2 - 2a^2x^2 =$$
$$= [(x^2+a^2)-(a\sqrt{2})x][(x^2+a^2)+(a\sqrt{2})x]$$
$$= (x^2-(a\sqrt{2})x+a^2)(x^2+(a\sqrt{2})x+a^2)$$

This approach can also be used to factor a biquadratic polynomial of the form ax^4+bx^2+c.

h) $I = \int \frac{x^3 dx}{(x^2+3)^2}$ i) $I = \int \frac{dx}{x^6-64}$

↳ Note the 6^{th} order factorization combining difference of squares and difference of cubes:

$$x^6-a^6 = (x^3-a^3)(x^3+a^3) =$$
$$= (x-a)(x^2+ax+a^2)(x+a)(x^2-ax+a^2)$$

6 → Integrals $I = \int \frac{P(x)}{Q(x)}\,dx$ with $\deg P(x) \geq \deg Q(x)$

Before doing partial fraction decomposition, we use long polynomial division to divide the numerator with the denominator

EXAMPLE

$$I = \int \frac{x^4}{x^2+x-2}\,dx$$

Solution

We divide: $x^4/(x^2+x-2)$:

$$\begin{array}{r|l}
x^4+0x^3+0x^2+0x+0 & x^2+x-2 \\ \cline{2-2}
-x^4-1x^3+2x^2 & x^2-x+3 \\ \hline
-1x^3+2x^2+0x+0 & \\
+1x^3+x^2-2x+0 & \\ \hline
3x^2-2x+0 & \\
-3x^2-3x+6 & \\ \hline
-5x+6 &
\end{array}$$

thus:

$$f(x) = \frac{x^4}{x^2+x-2} = x^2-x+3+\frac{-5x+6}{x^2+x-2} =$$

$$= x^2-x+3+\frac{-5x+6}{(x+2)(x-1)} =$$

$$= x^2-x+3+\frac{A}{x+2}+\frac{B}{x-1}$$

with

$$A = \left.\frac{-5x+6}{x-1}\right|_{x=-2} = \frac{-5(-2)+6}{-2-1} = \frac{10+6}{-3} = \frac{-16}{3}$$

$$B = \left.\frac{-5x+6}{x+2}\right|_{x=1} = \frac{-5\cdot 1+6}{1+2} = \frac{1}{3}$$

and therefore

$$I = \int \frac{x^4}{x^2+x-2}\,dx = \int \left[x^2 - x + 3 + \frac{-16/3}{x+2} + \frac{1/3}{x-1} \right] dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + 3x - \frac{16\ln|x+2|}{3} + \frac{\ln|x-1|}{3} + C_1$$

EXERCISES

(21) Evaluate the following integrals:

a) $I = \int \frac{2x^3+3x^2-16}{x^2-4}\,dx$

b) $I = \int \frac{x^5-3x^4-2x^3-4x^2+2}{x^3+x^2+x+1}\,dx$

c) $I = \int \frac{x^6+2x^4-8x^2+3x+1}{x^2+4}\,dx$

d) $I = \int \frac{x^7+6x^5+7x^3+x^2-5x+3}{x^4+6x^2+9}\,dx$

e) $I = \int \frac{x^6+3x^5+8x^4+23x^3+9x^2+21x+3}{x^3+3x^2+7x+21}\,dx$

f) $I = \int \frac{x^6-3x^4+4x^3-3x^2+6x-7}{(x^2+1)^2-x^2}\,dx$

Integrals that reduce to rational integrals

1 → <u>Form</u> $\boxed{I = \int P(x) \operatorname{Arctan}(f(x))\,dx}$

We use integration by parts to eliminate the $\operatorname{Arctan}(f(x))$ factor.

EXAMPLE

$$I = \int (2x+1) \operatorname{Arctan}(x-1)\,dx$$

<u>Solution</u>

$$I = \int (2x+1) \operatorname{Arctan}(x-1)\,dx = \int (x^2+x)' \operatorname{Arctan}(x-1)\,dx =$$

$$= (x^2+x) \operatorname{Arctan}(x-1) - \int (x^2+x) [\operatorname{Arctan}(x-1)]'\,dx =$$

$$= (x^2+x) \operatorname{Arctan}(x-1) - \int (x^2+x) \frac{(x-1)'}{(x-1)^2+1}\,dx =$$

$$= (x^2+x) \operatorname{Arctan}(x-1) - \int \frac{x^2+x}{x^2-2x+1+1}\,dx$$

$$= (x^2+x) \operatorname{Arctan}(x-1) - \int \frac{x^2+x}{x^2-2x+2}\,dx$$

$$= (x^2+x) \operatorname{Arctan}(x-1) - I_1$$

Divide x^2+x with x^2-2x+2

$$\begin{array}{r|l} x^2+x+0 & x^2-2x+2 \\ \cline{2-2} \underline{-x^2+2x-2} & 1 \\ 3x-2 & \end{array}$$

therefore

$$I_1 = \int \frac{x^2+x}{x^2-2x+2}\,dx = \int \left[1 + \frac{3x-2}{x^2-2x+2} \right] dx =$$

$$= \int \left[1 + \frac{(3/2)(2x-2)+1}{x^2-2x+2} \right] dx =$$

$$= \int \left[1 + \frac{3}{2} \frac{(x^2-2x+2)'}{x^2-2x+2} + \frac{1}{x^2-2x+2} \right] dx =$$

$$= x + \frac{3}{2} \ln|x^2-2x+2| + \int \frac{dx}{(x-1)^2+1} =$$

$$= x + \frac{3}{2} \ln|x^2-2x+2| + \text{Arctan}(x-1) + c \Rightarrow$$

$$\Rightarrow I = (x^2+x)\,\text{Arctan}(x-1) - I_1 =$$

$$= (x^2+x)\,\text{Arctan}(x-1) - [x + (3/2)\ln|x^2-2x+2| + \text{Arctan}(x-1)] + c =$$

$$= (x^2+x-1)\,\text{Arctan}(x-1) - x - (3/2)\ln|x^2-2x+2| + c$$

② → Form $\boxed{I = \int P(x)\ln|f(x)|\,dx}$

We use integration by parts to eliminate the $\ln|f(x)|$ factor. The resulting integral most generally will be a rational integral.

EXAMPLE

$$I = \int 2x \ln|x^2-1|\,dx$$

Solution

$$I = \int 2x \ln|x^2-1|\,dx = \int (x^2)' \ln|x^2-1|\,dx =$$

$$= x^2\ln|x^2-1| - \int x^2 [\ln|x^2-1|]'\,dx =$$

$$= x^2\ln|x^2-1| - \int x^2 \frac{(x^2-1)'}{x^2-1}\,dx =$$

$$= x^2\ln|x^2-1| - \int x^2 \frac{2x}{x^2-1}\,dx =$$

$$= x^2\ln|x^2-1| - \int \frac{2x^3}{x^2-1}\,dx \qquad (1)$$

We divide $2x^3/(x^2-1)$

$$\begin{array}{r|l} 2x^3+0x^2+0x+0 & x^2-1 \\ \hline -2x^3+0x^2+2x+0 & 2x \\ \hline 2x & \end{array}$$

and note that

$$f(x) = \frac{2x^3}{x^2-1} = 2x + \frac{2x}{x^2-1} = 2x + \frac{(x^2-1)'}{x^2-1}$$

and therefore from Eq. (1):

$$I = x^2 \ln|x^2-1| - \int \left[2x + \frac{(x^2-1)'}{x^2-1} \right] dx =$$

$$= x^2 \ln|x^2-1| - \left[x^2 + \ln|x^2-1| \right] + c =$$

$$= x^2 \ln|x^2-1| - x^2 - \ln|x^2+1| + c =$$

$$= (x^2-1) \ln|x^2-1| - x^2 + c$$

3 → Form:

$$I=\int F(x,\sqrt[n]{ax+b})\,dx$$

$$I=\int F\left(x,\sqrt[n]{\frac{ax+b}{cx+d}}\right)dx$$

Methodology:

- 1 Let $y=\sqrt[n]{ax+b}$ or $y=\sqrt[n]{(ax+b)/(cx+d)}$
- 2 Solve for x to obtain $x=g(y)$.
- 3 Calculate the $dx=g'(y)dy$
- 4 It follows that

$$I=\int F(g(y),y)\,g'(y)\,dy$$

which is, in general, a rational integral.

EXAMPLE

$$I=\int_{-1/2}^{-1/3}\frac{1}{x}\sqrt{\frac{1+x}{1-x}}\,dx$$

Solution

Let $y=\sqrt{\dfrac{1+x}{1-x}}\Leftrightarrow y^2=\dfrac{1+x}{1-x}\Leftrightarrow y^2(1-x)=1+x\Leftrightarrow$

$$\Leftrightarrow y^2-xy^2=1+x\Leftrightarrow x(1+y^2)=y^2-1\Leftrightarrow x=\frac{y^2-1}{y^2+1}\qquad(1)$$

For $x=-1/2\Rightarrow y=\sqrt{\dfrac{1-1/2}{1+1/2}}=\sqrt{\dfrac{1/2}{3/2}}=\sqrt{\dfrac{1}{3}}=\dfrac{\sqrt{3}}{3}$ (2)

For $x = -1/3 \Rightarrow y = \sqrt{\dfrac{1-1/3}{1+1/3}} = \sqrt{\dfrac{2/3}{4/3}} = \sqrt{\dfrac{1}{2}} = \dfrac{\sqrt{2}}{2}$ (3)

Also:

$$dx = \left(\frac{y^2-1}{y^2+1}\right)' dy = \frac{(y^2-1)'(y^2+1)-(y^2-1)(y^2+1)'}{(y^2+1)^2}\, dy =$$

$$= \frac{2y(y^2+1)-2y(y^2-1)}{(y^2+1)^2}\, dy = \frac{2y(y^2+1-y^2+1)}{(y^2+1)^2}\, dy$$

$$= \frac{4y}{(y^2+1)^2}\, dy \qquad (4)$$

From Eq.(1), Eq.(2), Eq.(3), Eq.(4):

$$I = \int_{\sqrt{3}/3}^{\sqrt{2}/2} \frac{y^2+1}{y^2-1}\cdot y \cdot \frac{4y}{(y^2+1)^2}\, dy =$$

$$= \int_{\sqrt{3}/3}^{\sqrt{2}/2} \frac{4y^2}{(y^2-1)(y^2+1)}\, dy = \int_{\sqrt{3}/3}^{\sqrt{2}/2} \frac{4y^2}{(y-1)(y+1)(y^2+1)}\, dy$$

$$= \int_{\sqrt{3}/3}^{\sqrt{2}/2} \left[\frac{A}{y-1} + \frac{B}{y+1} + \frac{Cy+D}{y^2+1}\right] dy \qquad (5)$$

with

$$A = \left.\frac{4y^2}{(y+1)(y^2+1)}\right|_{y=1} = \frac{4\cdot 1^2}{(1+1)(1^2+1)} = \frac{4}{2\cdot 2} = 1$$

$$B = \left.\frac{4y^2}{(y-1)(y^2+1)}\right|_{y=-1} = \frac{4(-1)^2}{((-1)-1)((-1)^2+1)} = \frac{4}{(-2)2} = -1$$

$$Ci+D = \left.\frac{4y^2}{(y-1)(y+1)}\right|_{y=i} = \frac{4i^2}{(i-1)(i+1)} = \frac{4(-1)}{i^2-1} =$$

$$= \frac{-4}{(-1)-1} = \frac{-4}{-2} = 2 \Leftrightarrow \begin{cases} C = 0 \\ D = 2 \end{cases}$$

and from Eq. (5)

$$I = \int_{\sqrt{3}/3}^{\sqrt{2}/2} \left[\frac{1}{y-1} - \frac{1}{y+1} + \frac{2}{y^2+1} \right] dy =$$

$$= \Big[\ln|y-1| - \ln|y+1| + 2\,\mathrm{Arctan}\,(y) \Big]_{\sqrt{3}/3}^{\sqrt{2}/2} =$$

$$= \left[\ln|\sqrt{2}/2-1| - \ln|\sqrt{2}/2+1| + 2\,\mathrm{Arctan}\,(\sqrt{2}/2) \right] -$$

$$- \left[\ln|\sqrt{3}/3-1| - \ln|\sqrt{3}/3+1| + 2\,\mathrm{Arctan}\,(\sqrt{3}/3) \right]$$

$$= \ln\left| \frac{(\sqrt{2}/2-1)(\sqrt{3}/3+1)}{(\sqrt{2}/2+1)(\sqrt{3}/3-1)} \right| + 2\left[\mathrm{Arctan}(\sqrt{2}/2) - \mathrm{Arctan}(\sqrt{3}/3) \right]$$

$$= \ln\left| \frac{(\sqrt{2}-2)(\sqrt{3}+3)}{(\sqrt{2}+2)(\sqrt{3}-3)} \right| + 2\left[\mathrm{Arctan}\,(\sqrt{2}/2) - \pi/6 \right]$$

$$= \ln\left| \frac{(\sqrt{2}-2)^2(\sqrt{3}+3)^2}{(\sqrt{2}+2)(\sqrt{3}-3)(\sqrt{2}-2)(\sqrt{3}+3)} \right| +$$

$$+ 2\left[\mathrm{Arctan}\,(\sqrt{2}/2) - \pi/6 \right]$$

$$= \ln\left| \frac{(2-4\sqrt{2}+4)(3+6\sqrt{3}+9)}{(2-4)(3-9)} \right| + 2\left[\mathrm{Arctan}\,(\sqrt{2}/2) - \pi/6 \right]$$

$$= \ln\left| \frac{(6-4\sqrt{2})(12+6\sqrt{3})}{(-2)(-6)} \right| + 2\left[\mathrm{Arctan}\,(\sqrt{2}/2) - \pi/6 \right]$$

$$= \ln\left| \frac{2(3-2\sqrt{2})\,6(2+\sqrt{3})}{(-2)(-6)} \right| + 2\left[\mathrm{Arctan}\,(\sqrt{2}/2) - \pi/6 \right]$$

$$= \ln\left| (3-2\sqrt{2})(2+\sqrt{3}) \right| + 2\left[\mathrm{Arctan}\,(\sqrt{2}/2) - \pi/6 \right]$$

$$= \ln\left| 6 - 4\sqrt{2} + 3\sqrt{3} - 2\sqrt{6} \right| + 2\left[\mathrm{Arctan}\,(\sqrt{2}/2) - \pi/6 \right]$$

EXERCISES

(22) Evaluate the following indefinite integrals:

a) $I = \int \text{Arctan}(x)\,dx$ b) $I = \int x\,\text{Arctan}(x)\,dx$

c) $I = \int x^2\,\text{Arctan}(x)\,dx$

(23) Use the indefinite integrals from the previous exercise to evaluate the integral

$$I = \int_0^{\sqrt{3}} (x+1)(x-2)\,\text{Arctan}(x)\,dx$$

(24) Evaluate the following integrals:

a) $I = \int \text{Arctan}(x^2)\,dx$ b) $I = \int x\,\text{Arctan}(x^2)\,dx$

c) $I = \int (3x-2)\ln x\,dx$ d) $I = \int 2x\ln(x^2-x+2)\,dx$

e) $I = \int x^2\ln(x^2+5x+6)\,dx$ f) $I = \int 2x\ln(x^4+1)\,dx$

g) $I = \int x\ln(x^6-1)\,dx$

(25) Evaluate the following integrals

a) $I = \int_2^3 x\sqrt{2x-3}\,dx$ b) $I = \int_0^4 \frac{x\,dx}{1+\sqrt{2x+1}}$

c) $I = \int \frac{2x-1}{\sqrt[3]{3x+2}}\,dx$ d) $I = \int \frac{x\,dx}{\sqrt[3]{x+1} + \sqrt{x+1}}$

d) $I = \int_{-1/2}^{1/2} \sqrt{\frac{1-x}{1+x}}\,dx$ e) $I = \int_0^1 \frac{x\,dx}{\sqrt[3]{2x+1}}$

f) $I = \int_0^1 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$ f) $I = \int_1^2 \frac{dx}{\sqrt{2x+1} - \sqrt[4]{2x+1}}$

g) $I = \int_{-1}^1 x\sqrt{\frac{2-x}{2+x}}\,dx$

↳ Hint: For (d) use $y = \sqrt[6]{x+1}$ such that
$\sqrt{x+1} = y^3$ and $\sqrt[3]{x+1} = y^2$.
Similar idea applies to (f)

Trigonometric Integrals

- Be sure to remove all trigonometric identities

① → Form:

$I = \int f(\sin x)\cos x\,dx$	$\leftarrow y = \sin x$
$I = \int f(\cos x)\sin x\,dx$	$\leftarrow y = \cos x$

EXAMPLES

a) $I = \int_0^{\pi/4} \frac{\sin x}{2+\cos x}\,dx$

Solution

Let $y = \cos x$. Then $dy = -\sin x\,dx \Rightarrow \sin x\,dx = -dy$.

For $x = 0$: $y = \cos 0 = 1$

For $x = \pi/4$: $y = \cos(\pi/4) = \sqrt{2}/2$

It follows that

$$I = \int_1^{\sqrt{2}/2} \frac{-dy}{2+y} = \left[-\ln|y+2|\right]_1^{\sqrt{2}/2} =$$

$$= [-\ln|\sqrt{2}/2+2|] - [-\ln|1+2|] =$$

$$= -\ln(2+\sqrt{2}/2) + \ln 3$$

b) $I = \int_0^{\pi/2} \sin^5 x\, dx$

Solution

$$I = \int_0^{\pi/2} \sin^5 x\, dx = \int_0^{\pi/2} \sin^4 x \sin x\, dx = \int_0^{\pi/2} (\sin^2 x)^2 \sin x\, dx$$

$$= \int_0^{\pi/2} (1-\cos^2 x)^2 \sin x\, dx$$

Let $y = \cos x \Rightarrow dy = -\sin x\, dx \Rightarrow \sin x\, dx = -dy$

For $x=0$: $y = \cos 0 = 1$

For $x=\pi/2$: $y = \cos(\pi/2) = 0$

It follows that:

$$I = \int_1^0 (1-y^2)^2 (-1) dy = \int_0^1 (1-y^2)^2 dy =$$

$$= \int_0^1 (1-2y^2+y^4) dy = \left[y - \frac{2y^3}{3} + \frac{y^5}{5} \right]_0^1 =$$

$$= \left[1 - \frac{2 \cdot 1^3}{3} + \frac{1^5}{5} \right] - (0-0+0) =$$

$$= 1 - \frac{2}{3} + \frac{1}{5} = \frac{15 - 2 \cdot 5 + 1 \cdot 3}{15} = \frac{15-10+3}{15} =$$

$$= \frac{8}{15}$$

c) $I = \int \frac{dx}{\sin x}$

Solution

$$I = \int \frac{dx}{\sin x} = \int \frac{\sin x \, dx}{\sin^2 x} = \int \frac{\sin x \, dx}{1 - \cos^2 x} =$$

$$= \int \frac{\sin x \, dx}{(1-\cos x)(1+\cos x)}$$

Let $y = \cos x \Rightarrow dy = -\sin x \, dx \Rightarrow \sin x \, dx = -dy$

Then,

$$I = \int \frac{-dy}{(1-y)(1+y)} = \int \frac{dy}{(y-1)(y+1)} = \int \left[\frac{A}{y-1} + \frac{B}{y+1} \right] dy$$

with

$$A = \left. \frac{1}{y+1} \right|_{y=1} = \frac{1}{1+1} = \frac{1}{2}$$

$$B = \left. \frac{1}{y-1} \right|_{y=-1} = \frac{1}{-1-1} = \frac{-1}{2}$$

and therefore

$$I = \int \left[\frac{1/2}{y-1} + \frac{-1/2}{y+1} \right] dy =$$

$$= (1/2) \ln|y-1| - (1/2) \ln|y+1| + c$$

$$= (1/2) \left[\ln|y-1| - \ln|y+1| \right] + c$$

$$= \frac{1}{2} \ln\left(\frac{|y-1|}{|y+1|} \right) = \frac{1}{2} \ln\left| \frac{y-1}{y+1} \right| + c$$

$$= \frac{1}{2} \ln\left| \frac{\cos x - 1}{\cos x + 1} \right| + c$$

d) $I = \int_0^{\pi/4} \tan^5 x \, dx$

Solution

$$I = \int_0^{\pi/4} \tan^5 x \, dx = \int_0^{\pi/4} \frac{\sin^5 x}{\cos^5 x} dx = \int_0^{\pi/4} \frac{\sin^4 x}{\cos^5 x} \sin x \, dx =$$

$$= \int_0^{\pi/4} \frac{(1-\cos^2 x)^2}{\cos^5 x} \sin x \, dx$$

Let $y = \cos x \Rightarrow dy = -\sin x \, dx \Rightarrow \sin x \, dx = -dy$

For $x = 0$: $y = \cos 0 = 1$

For $x = \pi/4$: $y = \cos(\pi/4) = \sqrt{2}/2$

It follows that

$$I = \int_1^{\sqrt{2}/2} \frac{(1-y^2)^2}{y^5} (-1) dy = \int_{\sqrt{2}/2}^1 \frac{1-2y^2+y^4}{y^5} dy =$$

$$= \int_{\sqrt{2}/2}^1 [y^{-5} - 2y^{-3} + y^{-1}] dy = \left[\frac{y^{-4}}{-4} - 2\frac{y^{-2}}{-2} + \ln|y| \right]_{\sqrt{2}/2}^1$$

$$= \left[\frac{-1}{4y^4} + \frac{1}{y^2} + \ln|y| \right]_{\sqrt{2}/2}^1 =$$

$$= \left[\frac{-1}{4 \cdot 1^4} + \frac{1}{1^2} + \ln|1| \right] - \left[\frac{-1}{4(\sqrt{2}/2)^4} + \frac{1}{(\sqrt{2}/2)^2} + \ln(\sqrt{2}/2) \right]$$

$$= \left(1 - \frac{1}{4}\right) + \frac{1}{4(1/4)} - \frac{1}{1/2} - \ln 2^{-1/2} =$$

$$= 1 - \frac{1}{4} + 1 - 2 - (-1/2)\ln 2 = -\frac{1}{4} + \frac{1}{2}\ln 2$$

Note that all of the integrals above follow the general form

$$I = \int \sin^a x \cos^b x \, dx$$

with $a, b \in \mathbb{Z}$ and $\boxed{a \text{ odd} \vee b \text{ odd}}$

Using the identity $\sin^2 x + \cos^2 x = 1$ we see that such integrals easily reduce to form 1.

EXERCISES

(26) Evaluate the following integrals:

a) $I = \int_0^{\pi/6} \frac{\cos x}{\sqrt{1-\sin^2 x}}\, dx$

b) $I = \int_0^{\pi/4} \frac{\sin x}{1+\cos^2 x}\, dx$

c) $I = \int_0^{\pi/4} \sin^3 x\, dx$

d) $I = \int_0^{\pi/3} \cos^3 x\, dx$

e) $I = \int \sin^7 x\, dx$

f) $I = \int \cos^9 x\, dx$

g) $I = \int \sin^2 x \cos^3 x\, dx$

h) $I = \int_{\pi/4}^{\pi/3} \cos^3 x \sin^4 x\, dx$

i) $I = \int \frac{dx}{\cos^3 x}$

j) $I = \int_{\pi/6}^{\pi/3} \frac{dx}{\sin^5 x}$

k) $I = \int \tan^3 x\, dx$

l) $I = \int_0^{\pi/4} \tan^5 x\, dx$

m) $I = \int_0^{\pi/3} \cot^3(x)\, dx$

n) $I = \int_0^{\pi/8} \cot^5(2x)\, dx$

o) $I = \int_{\pi/6}^{\pi/2} \sin(2x)[\cos^2 x - \sin^2 x]\, dx$

p) $I = \int \cos(3x)[\sin x \cos(2x) + \sin(2x)\cos x]\, dx$

$\hookrightarrow$ Integrals (o) and (p) can be reduced to form I by using the appropriate trig. identities for $\sin(a \pm b)$ and $\cos(a \pm b)$.

2 $\rightarrow$ <u>Form</u> :

$I=\int \dfrac{f(\tan x)}{\cos^2 x}\,dx$	$\rightarrow$ Let $y=\tan x$
$I=\int \dfrac{f(\cot x)}{\sin^2 x}\,dx$	$\rightarrow$ Let $y=\cot x$
$I=\int f(\tan x)\,dx$	$\rightarrow$ Let $y=\tan x$

EXAMPLES

a) $I=\displaystyle\int_{\pi/3}^{\pi/4} \frac{\ln(\tan x)}{\cos^2 x}\,dx$

<u>Solution</u>

Let $y=\tan x=g(x) \Longrightarrow \begin{cases} dy = dx/\cos^2 x \\ g(\pi/3)=\tan(\pi/3)=\sqrt{3} \\ g(\pi/4)=\tan(\pi/4)=1 \end{cases} \Longrightarrow$

$$\Longrightarrow I=\int_{\sqrt{3}}^{1} \ln y\,dy=\int_{\sqrt{3}}^{1}(y)'\ln y\,dy=$$

$$=\left[y\ln y\right]_{\sqrt{3}}^{1}-\int_{\sqrt{3}}^{1} y(\ln y)'\,dy=$$

$$=\left[1\ln 1-\sqrt{3}\ln(\sqrt{3})\right]-\int_{\sqrt{3}}^{1} y(1/y)\,dy=$$

$$=\left[0-\sqrt{3}\ln(\sqrt{3})\right]-\int_{\sqrt{3}}^{1} dy=-\sqrt{3}\ln\sqrt{3}-\left[y\right]_{\sqrt{3}}^{1}=$$

$$=-\sqrt{3}\ln(\sqrt{3})-\left[1-\sqrt{3}\right]=\sqrt{3}-1-\sqrt{3}\ln(\sqrt{3})=$$

$$=-1+\sqrt{3}\left(1-\ln(\sqrt{3})\right)$$

b) $I = \int_0^{\pi/3} \tan^4 x \, dx$

Solution

Let $y = \tan x = g(x) \Rightarrow dy = (1+\tan^2 x)\,dx = (1+y^2)\,dx \Rightarrow$

$$\Rightarrow dx = \frac{dy}{1+y^2}$$

and note that

$g(0) = \tan 0 = 0$

$g(\pi/3) = \tan(\pi/3) = \sqrt{3}$

It follows that $I = \int_0^{\sqrt{3}} \frac{y^4}{1+y^2}\,dy$

We divide y^4 by $1+y^2$:

$$\begin{array}{r|l} y^4 + 0y^3 + 0y^2 + 0y + 0 & y^2 + 0y + 1 \\ \hline -y^4 + 0y^3 - y^2 & y^2 - 1 \\ \hline -y^2 + 0y + 0 & \\ +y^2 + 0y + 1 & \\ \hline 1 & \end{array}$$

and therefore:

$$I = \int_0^{\sqrt{3}} \frac{y^4}{1+y^2}\,dy = \int_0^{\sqrt{3}} \left[y^2 - 1 + \frac{1}{y^2+1} \right] dx =$$

$$= \left[\frac{y^3}{3} - y + \text{Arctan}(y) \right]_0^{\sqrt{3}} =$$

$$= \frac{(\sqrt{3})^3 - 0^3}{3} - [\sqrt{3} - 0] + [\text{Arctan}(\sqrt{3}) - \text{Arctan}(0)] =$$

$$= \frac{3\sqrt{3}}{3} - \sqrt{3} + \frac{\pi}{3} - 0 = \sqrt{3} - \sqrt{3} + \pi/3 = \pi/3.$$

EXERCISES

(27) Evaluate the following functions.

a) $I = \displaystyle\int_0^{\pi/4} \frac{\ln(1+\tan x)}{\cos^2 x}\,dx$ b) $I = \displaystyle\int_0^{\pi/6} \frac{\tan(2x)}{\cos^2 x}\,dx$

c) $I = \displaystyle\int \frac{\tan x + \tan 2x}{\cos^2(3x)\left[1-\tan x \tan(2x)\right]}\,dx$

↳ For (b) and (c) use the trigonometric identity for $\tan(a+b)$.

d) $I = \displaystyle\int_0^{\pi/4} \frac{\sin(\pi \tan x)}{\cos^2 x}\,dx$ e) $I = \displaystyle\int_0^{\pi/6} \tan^2 x\,dx$

f) $I = \displaystyle\int_0^{\pi/3} \tan^6 x\,dx$ g) $I = \displaystyle\int \frac{3\,dx}{\tan x + 2}$

h) $I = \displaystyle\int_0^{\pi/4} \frac{\tan x}{1+\tan x}\,dx$ i) $I = \displaystyle\int \frac{dx}{\tan(1+\tan x)}$

j) $I = \displaystyle\int_0^{\pi/3} \frac{dx}{1+\tan^3 x}$ k) $I = \displaystyle\int \frac{dx}{1+\tan^4 x}$

(28) Let $I_n = \displaystyle\int_0^{\pi/4} \tan^n(x)\,dx$, $\forall n \in \mathbb{N}^*$. Show that:

$$I_n = \frac{1}{n-1} - I_{n-2}, \quad \forall n \in \mathbb{N} - \{0, 1, 2\}$$

(29) Let $I_n = \int_0^{\pi/3} \frac{dx}{\cos^n x}$, $\forall n \in \mathbb{N}^*$.

a) Show that: $I_n = \frac{2^{n-2}\sqrt{3}}{n-1} + \frac{n-2}{n-1} I_{n-2}$, $\forall n \in \mathbb{N} - \{0,1,2\}$

b) Evaluate I_1 and use the recurrence relation to show that

$$\int_0^{\pi/3} \frac{dx}{\cos^3 x} = \sqrt{3} + \frac{1}{2} \ln\left(\frac{3}{2} + \sqrt{3}\right)$$

3 → Form :

$$I = \int \frac{\tan^a x}{\cos^b x}\, dx$$

$$I = \int \frac{\cot^a x}{\sin^b x}\, dx$$

We distinguish between the following cases :

Case 1 : Assume that $a \in \mathbb{R} \wedge b$ even integer

Use the identities

$$\frac{1}{\cos^2 x} = 1 + \tan^2 x \qquad \frac{1}{\sin^2 x} = 1 + \cot^2 x$$

Then reduce integral to form 2:

$$\int \frac{f(\tan x)}{\cos^2 x}\, dx \quad \text{or} \quad \int \frac{f(\cot x)}{\sin^2 x}\, dx$$

Case 2 : Assume that a odd integer $\wedge\ b \in \mathbb{R}$.

Use the identity $\sin^2 x + \cos^2 x = 1$

Then reduce integral to form 1:

$$\int f(\cos x) \sin x\, dx \quad \text{or} \quad \int f(\sin x) \cos x\, dx$$

Then integrals of form 1 or 2 can be evaluated as explained previously. The final result should be expressed using root notation whenever fractional powers are involved.

EXAMPLE

a) $I = \int \tan^3 x \sqrt{\cos x} \, dx$

Solution

$$I = \int \tan^3 x \sqrt{\cos x} \, dx = \int \frac{\sin^3 x}{\cos^3 x} \cos^{1/2} x \, dx =$$

$$= \int \frac{\sin^2 x \cos^{1/2} x}{\cos^3 x} \sin x \, dx =$$

$$= \int \frac{(1-\cos^2 x)\cos^{1/2} x}{\cos^3 x} \sin x \, dx$$

Let $y = \cos x \Rightarrow dy = -\sin x \, dx \Rightarrow \sin x \, dx = -dy$ and therefore,

$$I = \int \frac{(1-y^2) y^{1/2}}{y^3} (-1) dy = \int (y^{-3} - y^{-1}) y^{1/2} (-1) \, dy =$$

$$= \int (y^{-1} - y^{-3}) y^{1/2} dy = \int (y^{-1/2} - y^{-5/2}) \, dy =$$

$$= \frac{y^{1/2}}{1/2} - \frac{y^{-3/2}}{-3/2} + C = 2\sqrt{y} + \frac{2}{3} \frac{1}{y\sqrt{y}} + C$$

$$= 2\sqrt{\cos x} + \frac{2}{3} \frac{1}{\cos x \sqrt{\cos x}} + C$$

b) $I = \int \dfrac{dx}{\cos^4 x \tan x \sqrt{\tan x}}$

<u>Solution</u>

$$I = \int \frac{dx}{\cos^4 x \tan x \sqrt{\tan x}} = \int \frac{\tan^{-3/2} x}{\cos^2 x} \frac{1}{\cos^2 x} dx =$$

$$= \int \frac{\tan^{-3/2} x (1 + \tan^2 x)}{\cos^2 x} dx$$

Let $y = \tan x \Rightarrow dy = \dfrac{dx}{\cos^2 x}$ and therefore:

$$I = \int y^{-3/2}(1+y^2)\,dy = \int (y^{-3/2} + y^{1/2})\,dy =$$

$$= \frac{y^{-1/2}}{-1/2} + \frac{y^{3/2}}{3/2} + C = \frac{-2}{\sqrt{y}} + \frac{2}{3} y\sqrt{y} + C =$$

$$= \frac{-2}{\sqrt{\tan x}} + \frac{2 \tan x \sqrt{\tan x}}{3} + C$$

EXERCISES

(30) Evaluate the following integrals:

a) $I = \int \frac{dx}{\tan^3 x \cos^4 x}$

b) $I = \int_0^{\pi/6} \frac{\sqrt{\tan x}}{\cos^4 x} dx$

c) $I = \int \frac{\tan^3 x}{\cos^6 x} dx$

d) $I = \int_0^{\pi/3} \frac{\tan^5 x}{\sqrt{\cos x}} dx$

e) $I = \int \frac{\tan^3 x}{\cos x \sqrt{\cos x}} dx$

f) $I = \int_0^{\pi/4} \frac{\tan x \sqrt{\tan x}}{\cos^6 x} dx$

(4) → Form: Products of trig. functions

Products of trigonometric functions can be eliminated using the following identities:

$$\boxed{\begin{aligned} \sin a \cdot \cos b &= (1/2)[\sin(a-b)+\sin(a+b)] \\ \cos a \cos b &= (1/2)[\cos(a-b)+\cos(a+b)] \\ \sin a \sin b &= (1/2)[\cos(a-b)-\cos(a+b)] \end{aligned}}$$

EXAMPLE

a) $I = \int_0^{\pi} \sin(3x)\cos(2x)dx =$

$= \int_0^{\pi} (1/2)[\sin(3x-2x)+\sin(3x+2x)]dx$

$= \int_0^{\pi} (1/2)[\sin x + \sin 5x]\, dx =$

$= \frac{1}{2}\left[-\cos x - \frac{\cos 5x}{5}\right]_0^{\pi} =$

$= \frac{1}{2}\left[-(\cos\pi - \cos 0) - \frac{\cos 5\pi - \cos 0}{5}\right]$

$= \frac{1}{2}\left[-((-1)-1) - \frac{(-1)-1}{5}\right] =$

$= \frac{1}{2}\left[1+1+\frac{2}{5}\right] = \frac{1}{2}\,\frac{12}{5} = \frac{6}{5}$

b) $I = \int_0^{\pi/2} x^2 \sin(2x)\cos(3x)\,dx$

Solution

$$I = \int_0^{\pi/2} x^2 \sin(2x)\cos(3x)\,dx = \int_0^{\pi/2} x^2 (1/2)\left[\sin(2x+3x) + \sin(2x-3x)\right] dx$$

$$= \frac{1}{2}\int_0^{\pi/2} x^2\left[\sin(5x) + \sin(-x)\right] dx =$$

$$= \frac{1}{2}\int_0^{\pi/2} x^2 \sin(5x)\,dx - \frac{1}{2}\int_0^{\pi/2} x^2 \sin x\,dx$$

Define $\forall a \in \mathbb{R}^* : F(a) = \int_0^{\pi/2} x^2 \sin(ax)\,dx$

and note that

$$I = \frac{F(5) - F(1)}{2} \qquad (1)$$

$\mapsto$ Note that it is more convenient to evaluate the general integral $F(a)$ and the use Eq. (1) than to do the two integrals separately.

$$F(a) = \int_0^{\pi/2} x^2 \sin(ax)\,dx = \int_0^{\pi/2} x^2 \left[\frac{-1}{a}\cos(ax)\right]' dx =$$

$$= \left[x^2 \cdot \frac{(-1)}{a}\cos(ax)\right]_0^{\pi/2} - \int_0^{\pi/2} (x^2)' \left[\frac{-1}{a}\cos(ax)\right] dx =$$

$$= -\left(\frac{\pi}{2}\right)^2 \frac{1}{a}\cos\left(\frac{a\pi}{2}\right) - \int_0^{\pi/2} 2x\left(\frac{-1}{a}\right)\cos(ax)\,dx =$$

$$= \frac{-\pi^2}{4a}\cos\left(\frac{a\pi}{2}\right) + \frac{2}{a}\int_0^{\pi/2} x\left[\frac{\sin(ax)}{a}\right]' dx =$$

$$= \frac{-\pi^2\cos(a\pi/2)}{4a} + \frac{2}{a}\left[x\,\frac{\sin(ax)}{a}\right]_0^{\pi/2} - \frac{2}{a}\int_0^{\pi/2}(x)'\,\frac{\sin(ax)}{a}\,dx$$

$$= \frac{-\pi^2\cos(a\pi/2)}{4a} + \frac{2}{a}\,\frac{\pi}{2}\,\frac{\sin(a\pi/2)}{a} - \frac{2}{a^2}\int_0^{\pi/2}\sin(ax)\,dx$$

$$= \frac{-\pi^2\cos(a\pi/2)}{4a} + \frac{\pi\sin(a\pi/2)}{a^2} - \frac{2}{a^2}\left[\frac{-\cos(ax)}{a}\right]_0^{\pi/2}$$

$$= \frac{-\pi^2\cos(a\pi/2)}{4a} + \frac{\pi\sin(a\pi/2)}{a^2} + \frac{2}{a^3}\left[\cos(a\pi/2) - 1\right]$$

and therefore

$$F(1) = \frac{-\pi^2\cos(\pi/2)}{4} + \frac{\pi\sin(\pi/2)}{1^2} + \frac{2}{1^3}\left[\cos(\pi/2) - 1\right]$$

$$= 0 + \pi + 2[0-1] = \pi - 2$$

and

$$F(5) = \frac{-\pi^2\cos(5\pi/2)}{4\cdot 5} + \frac{\pi\sin(5\pi/2)}{5^2} + \frac{2}{5^3}\left[\cos(5\pi/2) - 1\right]$$

$$= \frac{-\pi^2\cos(2\pi+\pi/2)}{20} + \frac{\pi\sin(2\pi+\pi/2)}{25} + \frac{2}{125}\left[\cos(2\pi+\pi/2) - 1\right]$$

$$= \frac{-\pi^2\cos(\pi/2)}{20} + \frac{\pi\sin(\pi/2)}{25} + \frac{2}{125}\left[\cos(\pi/2) - 1\right]$$

$$= 0 + \frac{\pi}{25} + \frac{2}{125}[0-1] = \frac{\pi}{25} - \frac{2}{125}$$

It follows that

$$I = (1/2)[F(5) - F(1)] = \frac{1}{2}\left[\left(\frac{\pi}{25} - \frac{2}{125}\right) - (\pi - 2)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{25} - \frac{2}{125} - \pi + 2\right] = \frac{1}{2}\,\frac{5\pi - 2 - 125\pi + 250}{125}$$

$$= \frac{1}{2} \frac{248 - 120\pi}{125} = \frac{1}{2} \cdot \frac{2(124 - 60\pi)}{125} =$$

$$= \frac{124 - 60\pi}{125}$$

↳ Note that more complicated products can be broken down in a similar manner one at a time.

For example:

$$\sin(5x)\cos(3x)\cos(7x) = \sin(5x)(1/2)[\cos(3x-7x) + \cos(3x+7x)]$$

$$= (1/2)\sin(5x)[\cos(-4x) + \cos(10x)] =$$

$$= (1/2)[\sin(5x)\cos(4x) + \sin(5x)\cos(10x)]$$

$$= (1/2)(1/2)[\sin(5x-4x) + \sin(5x+4x) + \sin(5x-10x) + \sin(5x+10x)]$$

$$= (1/4)[\sin x + \sin(9x) + \sin(-5x) + \sin(15x)]$$

$$= (1/4)(\sin x + \sin(9x) - \sin(5x) + \sin(15x))$$

EXERCISES

(31) Evaluate the following integrals:

a) $I = \int_0^{\pi/6} \sin(2x)\cos(3x)\,dx$ b) $I = \int_0^{\pi/3} \cos(5x)\cos(2x)\,dx$

c) $I = \int_0^{\pi/12} \sin(2x)\cdot\sin(4x)\,dx$ d) $I = \int_0^{3\pi/4} \sin x\cos(2x)\cos(3x)\,dx$

e) $I = \int_0^{\pi/6} \sin x\sin(2x)\sin(3x)\,dx$ f) $I = \int_0^{\pi/4} x^2\cos(3x)\sin(5x)\,dx$

g) $I = \int_0^{\pi/2} x^3\cos(2x)\cos(4x)\,dx$ h) $I = \int_0^{\pi} e^{-x}\sin x\sin(2x)\,dx$

i) $I = \int_0^{2\pi/3} e^{-x}\cos(2x)\sin(5x)\,dx$

j) $I = \int_0^{\pi/3} \sin\left(5x - \frac{\pi}{4}\right)\cos\left(x + \frac{\pi}{4}\right)dx$

k) $I = \int_0^{\pi} \sin\left(x - \frac{\pi}{3}\right)\cos\left(2x - \frac{\pi}{4}\right)dx$

5 → Form: Squares of trigonometric functions

Squares of trigonometric functions can be eliminated by the half-angle formulas:

$\sin^2 x = \dfrac{1-\cos 2x}{2}$	$\tan^2 x = \dfrac{1-\cos 2x}{1+\cos 2x}$
$\cos^2 x = \dfrac{1+\cos 2x}{2}$	$\cot^2 x = \dfrac{1+\cos 2x}{1-\cos 2x}$

EXAMPLES

a) $$I = \int_0^{\pi/2} \sin^2 x \cos^2 x \, dx = \frac{1}{4} \int_0^{\pi/2} (2\sin x \cos x)^2 \, dx =$$

$$= \frac{1}{4} \int_0^{\pi/2} \sin^2 (2x) \, dx = \frac{1}{4} \int_0^{\pi/2} \frac{1-\cos 4x}{2} \, dx$$

$$= \frac{1}{8} \int_0^{\pi/2} dx - \frac{1}{8} \int_0^{\pi/2} \cos 4x \, dx =$$

$$= \frac{1}{8} \frac{\pi}{2} - \frac{1}{8} \left[\frac{\sin 4x}{4} \right]_0^{\pi/2} =$$

$$= \frac{1}{8} \left[\frac{\pi}{2} - \frac{\sin(2\pi) - \sin 0}{4} \right] =$$

$$= \frac{1}{8} \left[\frac{\pi}{2} - \frac{0-0}{4} \right] = \frac{\pi}{16}$$

b) $I = \int_0^{\pi/8} \tan^2 x \sin(2x)\,dx = \int_0^{\pi/8} \frac{1-\cos(2x)}{1+\cos(2x)} \sin(2x)\,dx$

Let $y = g(x) = \cos(2x) \Rightarrow \begin{cases} dy = -2\sin(2x)dx \\ g(0) = \cos(0) = 1 \\ g(\pi/8) = \cos(2\cdot(\pi/8)) = \cos(\pi/4) = \frac{\sqrt{2}}{2} \end{cases} \Rightarrow$

$$\Rightarrow I = \int_1^{\sqrt{2}/2} \frac{1-y}{1+y} \cdot (-1/2)\,dy = -\frac{1}{2}\int_1^{\sqrt{2}/2} \frac{1-y}{1+y}\,dy$$

$$= -\frac{1}{2}\int_1^{\sqrt{2}/2} \frac{2-1-y}{1+y}\,dy = \frac{-1}{2}\int_1^{\sqrt{2}/2} \left[\frac{2}{1+y} - 1\right] dy =$$

$$= \frac{-1}{2}\Big[2\ln|1+y| - y\Big]_1^{\sqrt{2}/2} =$$

$$= \frac{-1}{2}\left[2\ln|1+\sqrt{2}/2| - \frac{\sqrt{2}}{2} - (2\ln|1+1| - 1)\right] =$$

$$= \frac{-1}{2}\left[2\ln(1+\sqrt{2}/2) - 2\ln 2 + 1 - \frac{\sqrt{2}}{2}\right]$$

c) $I = \int_0^\pi \sqrt{1+\cos x}\, dx$

<u>Solution</u>

$$I = \int_0^\pi \sqrt{1+\cos x}\, dx = \int_0^\pi \sqrt{2\cos^2(x/2)}\, dx =$$

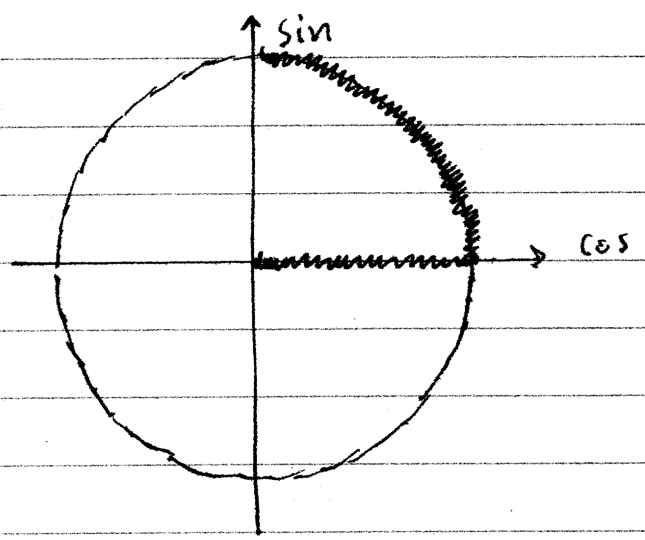
$$= \int_0^\pi \sqrt{2}\, |\cos(x/2)|\, dx$$

Let $x \in [0,\pi]$ be given. Then:

$x \in [0,\pi] \Rightarrow 0 \leq x \leq \pi \Rightarrow 0 \leq x/2 \leq \pi/2 \overset{(*)}{\Rightarrow} \cos(x/2) \geq 0 \Rightarrow$

$\Rightarrow |\cos(x/2)| = \cos(x/2)$

and therefore $\forall x \in [0,\pi] : |\cos(x/2)| = \cos(x/2)$



We use the trig circle to justify the (*) step and note that:

$\sqrt{x^2} = |x|$

$(\sqrt{x})^2 = x$

It follows that

$$I = \int_0^\pi \sqrt{2}\cos(x/2)\,dx = \sqrt{2}\int_0^\pi \cos(x/2)\,dx = \sqrt{2}\left[\frac{\sin(x/2)}{1/2}\right]_0^\pi =$$

$$= 2\sqrt{2}\left[\sin(x/2)\right]_0^\pi = 2\sqrt{2}\left[\sin(\pi/2) - \sin 0\right] = 2\sqrt{2}(1-0)$$

$$= 2\sqrt{2}.$$

EXERCISES

(32) Evaluate the following integrals.

a) $I = \int_0^{\pi/12} \cos^2 x \, dx$ b) $I = \int_0^{\pi/8} \sin^4 x \, dx$

c) $I = \int_0^{2\pi/3} \cos^6 x \, dx$ d) $I = \int_0^{\pi/4} \sin^2 x \tan^2 x \sin(2x) \, dx$

e) $I = \int_0^{\pi/8} \frac{\cos^2 x \sin(2x)}{\sin^2 x} \, dx$ f) $I = \int_0^{\pi/4} \sin^2 x \cos^4 x \, dx$

(33) Evaluate the following integrals.

a) $I = \int_0^{\pi/6} \sqrt{1 - \cos x} \, dx$ b) $I = \int_0^{\pi/2} \sqrt{1 + \sin(2x)} \, dx$

c) $I = \int_{\pi/6}^{\pi/2} \sqrt{1 - \sin(3x)} \, dx$

$\hookrightarrow$ Note that for (b) and (c) we can use the cofunction identity:

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$

to convert the signs into cosines. If necessary, to eliminate the absolute value, we may have to split the integral into subintervals.

(34) Consider the integrals

$$I_1 = \int_0^{\pi/2} \sin^4 x \cos^2 x \, dx$$

$$I_2 = \int_0^{\pi/2} \sin^2 x \cos^4 x \, dx$$

a) Evaluate $I_1 + I_2$ and $I_1 - I_2$.

b) Use the results of (a) to solve for I_1 and I_2.

↳ In the above argument use:

$$\sin^2 x + \cos^2 x = 1$$

$$\cos^2 x - \sin^2 x = \cos(2x)$$

to simplify the calculation.

(35) Evaluate the following integrals

a) $I = \displaystyle\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^4 x}$ b) $I = \displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos^2 x \, dx}{\sin^4 x}$

c) $I = \displaystyle\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^4 x \cos^2 x}$

↳ Introduce $\sin^2 x + \cos^2 x$ in the numerator to evaluate (a), using form $f(\tan x)/\cos^2 x$ or $f(\cot x)/\sin^2 x$. Then use (a) to do (b) and (c).

↳ In general integrals of the form $\int \sin^a x \cos^a x \, dx$ with a even $\wedge$ b even can be very challenging. Problems 32, 34, 35 illustrate some methods for handling such integrals.

6 → Form: $\boxed{I=\int f(\sin x, \cos x, \tan x, \cot x)\,dx}$

<u>Method of desperation</u>: We use this method if all trigonometric functions use the same angle x and we don't know what to do.

• 1 Use the identities

$$\boxed{\begin{array}{ll} \sin x = \dfrac{2\tan(x/2)}{1+\tan^2(x/2)} & \tan x = \dfrac{2\tan(x/2)}{1-\tan^2(x/2)} \\ \cos x = \dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)} & \cot x = \dfrac{1-\tan^2(x/2)}{2\tan(x/2)} \end{array}}$$

to rewrite the integral as

$$\boxed{I=\int F(\tan(x/2))\,dx}$$

• 2 Let $y=\tan(x/2)$. It follows that

$$dy=(1/2)(1+\tan^2(x/2))\,dx=(1/2)(1+y^2)\,dx \Rightarrow$$

$$\Rightarrow dx=\frac{2}{1+y^2}\,dy \Rightarrow \boxed{I=\int F(y)\frac{2}{1+y^2}\,dy}=\dots$$

thus the integral reduces to a rational integral.

↳ The above method is known as <u>tangent substitution</u>.

EXAMPLES

a) $I = \int \frac{d\vartheta}{3-\cos\vartheta}$

Let $y = \tan(\vartheta/2) \Rightarrow dy = (1/2)(1+\tan^2(\vartheta/2))\, d\vartheta =$

$= (1/2)(1+y^2)\, d\vartheta \Rightarrow$

$\Rightarrow d\vartheta = \frac{2dy}{1+y^2}$

Note that $\cos\vartheta = \frac{1-\tan^2(\vartheta/2)}{1+\tan^2(\vartheta/2)} = \frac{1-y^2}{1+y^2}$. It follows that

$$I = \int \frac{\frac{2}{1+y^2}}{3-\frac{1-y^2}{1+y^2}}\, dy = \int \frac{2}{3(1+y^2)-(1-y^2)}\, dy =$$

$$= \int \frac{2}{3+3y^2-1+y^2}\, dy = \int \frac{2}{4y^2+2}\, dy =$$

$$= \int \frac{dy}{2y^2+1}.$$ Let $z = \sqrt{2}\,y \Rightarrow dz = \sqrt{2}\,dy \Rightarrow$

$\Rightarrow dy = (1/\sqrt{2})dz \Rightarrow$

$$\Rightarrow I = \int \frac{(1/\sqrt{2})dz}{z^2+1} = \frac{1}{\sqrt{2}} \mathrm{Arctan}(z) + c =$$

$$= \frac{1}{\sqrt{2}} \mathrm{Arctan}(\sqrt{2}\,y) + c =$$

$$= \frac{1}{\sqrt{2}} \mathrm{Arctan}\left(\sqrt{2}\tan(\vartheta/2)\right) + c$$

8) Evaluate the following integral

$$I = \int_0^{\pi/4} \frac{dx}{(3\sin x + \cos x)^2}$$

Solution

We note that

$$(3\sin x + \cos x)^2 = 9\sin^2 x + 6\sin x\cos x + \cos^2 x =$$
$$= (\sin^2 x + \cos^2 x) + 8\sin^2 x + 3(2\sin x\cos x)$$
$$= 1 + 4(2\sin^2 x) + 3\sin(2x)$$
$$= 1 + 4(1 - \cos(2x)) + 3\sin(2x)$$
$$= 1 + 4 - 4\cos(2x) + 3\sin(2x)$$
$$= 5 - 4\cos(2x) + 3\sin(2x) \equiv f(x)$$

and therefore

$$I = \int_0^{\pi/4} \frac{dx}{5 - 4\cos(2x) + 3\sin(2x)}$$

Let $y = \tan x$. It follows that

$$dy = (\tan x)' \, dx = (1 + \tan^2 x)\, dx = (1 + y^2)\, dx \Rightarrow$$
$$\Rightarrow dx = \frac{dy}{1 + y^2}$$

and

$$\cos(2x) = \frac{1 - \tan^2 x}{1 + \tan^2 x} = \frac{1 - y^2}{1 + y^2}$$
$$\sin(2x) = \frac{2\tan x}{1 + \tan^2 x} = \frac{2y}{1 + y^2}$$

therefore,

$$f(x) = 5 - 4\cos(2x) + 3\sin(2x)$$

$$= 5 - 4\left(\frac{1-y^2}{1+y^2}\right) + 3\left(\frac{2y}{1+y^2}\right)$$

$$= \frac{5(1+y^2) - 4(1-y^2) + 3(2y)}{1+y^2}$$

$$= \frac{5+5y^2-4+4y^2+6y}{1+y^2} = \frac{(5+4)y^2+6y+(5-4)}{1+y^2}$$

$$= \frac{9y^2+6y+1}{y^2+1} = \frac{(3y+1)^2}{y^2+1}$$

For $x=0 \Rightarrow y=\tan 0 = 0$

For $x=\pi/4 \Rightarrow y=\tan(\pi/4)=1$

It follows that

$$I = \int_0^1 \frac{y^2+1}{(3y+1)^2}\frac{dy}{y^2+1} = \int_0^1 \frac{dy}{(3y+1)^2} =$$

$$= \frac{1}{9}\int_0^1 \frac{dy}{(y+1/3)^2} = \frac{1}{9}\int_{1/3}^{4/3} \frac{dy}{y^2} =$$

$$= \frac{1}{9}\left[\frac{y^{-1}}{-1}\right]_{1/3}^{4/3} = \frac{-1}{9}\left[\frac{1}{y}\right]_{1/3}^{4/3} =$$

$$= \frac{-1}{9}\left[\frac{1}{4/3} - \frac{1}{1/3}\right] = \frac{-1}{9}\left[\frac{3}{4} - 3\right]$$

$$= \frac{-3}{9}\left[\frac{1}{4} - 1\right] = \frac{-1}{3}\left(\frac{-3}{4}\right) = \frac{1}{4}$$

EXERCISES

(36) Evaluate the following integrals using tangent substitution

a) $I = \int_0^{\pi/4} \frac{dx}{\sin x + \cos x}$

b) $I = \int_0^{\pi/3} \frac{dx}{\cos x}$

c) $I = \int_0^{\pi/6} \frac{dx}{1 + \sin x}$

d) $I = \int_0^{\pi/2} \frac{dx}{1 + \sin x + \cos x}$

e) $I = \int_0^{\pi/6} \frac{dx}{\sin x + 3\cos x}$

f) $I = \int_0^{\pi/4} \frac{\sin x - \cos x}{\sin x + \cos x}\, dx$

g) $I = \int \frac{dx}{2\cos x - 3\tan x}$

(37) Evaluate the following integrals using tangent substitution.

a) $I = \int_0^{\pi/4} \frac{dx}{(\sin x + \cos x)^2}$

b) $I = \int \frac{dx}{\sin^2 x - 4\cos^2 x}$

c) $I = \int_0^{\pi/6} \frac{dx}{5 + 2\cos^2 x}$

↳ The following identities can be used to simplify the integrand and to reduce powers before applying the tangent substitution.

$2\sin^2 x = 1 - \cos(2x)$ $\qquad \sin^2 x + \cos^2 x = 1$

$2\cos^2 x = 1 + \cos(2x)$

$2\sin x \cos x = \sin(2x)$

▼ Rationalizing substitutions

Integrals that follow the general form

$$I = \int F(x, \sqrt{ax^2+bx+c}\,)\,dx$$

can be evaluated either by trigonometric substitution or an algebraic substitution (or both), as follows:

① → $\boxed{I = \int F(x, \sqrt{a^2-(bx+c)^2}\,)\,dx}$

For integrals where the coefficient of x^2 in the radical is negative, the only method available is to complete squares and let

$\boxed{bx+c = a\sin\vartheta \quad \text{with } \vartheta \in [-\pi/2, \pi/2]}$

It follows that

$bx+c = a\sin\vartheta \Leftrightarrow bx = a\sin\vartheta - c \Leftrightarrow x = (1/b)(a\sin\vartheta - c)$

and therefore

$dx = ((1/b)(a\sin\vartheta - c))'\,d\vartheta = (a/b)\cos\vartheta\,d\vartheta$

and

$a^2-(bx+c)^2 = a^2 - a^2\sin^2\vartheta = a^2(1-\sin^2\vartheta) = a^2\cos^2\vartheta$

$\Rightarrow \sqrt{a^2-(bx+c)^2} = \sqrt{a^2\cos^2\vartheta} = |a||\cos\vartheta| = |a|\cos\vartheta$

because $\vartheta \in [-\pi/2, \pi/2] \Rightarrow \cos\vartheta \geq 0$. We conclude that

$$I = \int F((1/b)(a\sin\vartheta - c), |a|\cos\vartheta)\,(a/b)\cos\vartheta\,d\vartheta$$

reducing the integral to a trigonometric integral.

EXAMPLES

a) Evaluate $I = \int_{-1}^{1} \frac{x^2+x+1}{\sqrt{4-x^2}}\, dx$

Solution

Let $x = 2\sin\vartheta$ with $\vartheta \in [-\pi/2, \pi/2]$. Then:

$dx = (2\sin\vartheta)'\, d\vartheta = 2\cos\vartheta\, d\vartheta$

and

$4-x^2 = 4-(2\sin\vartheta)^2 = 4-4\sin^2\vartheta = 4(1-\sin^2\vartheta) = 4\cos^2\vartheta \Rightarrow$

$\Rightarrow \sqrt{4-x^2} = \sqrt{4\cos^2\vartheta} = 2|\cos\vartheta| = 2\cos\vartheta$

For $x=-1 \Leftrightarrow 2\sin\vartheta = -1 \Leftrightarrow \sin\vartheta = -1/2 = -\sin(\pi/6) = \sin(-\pi/6)$

$\Leftrightarrow \vartheta = -\pi/6$

For $x = 1 \Leftrightarrow 2\sin\vartheta = 1 \Leftrightarrow \sin\vartheta = 1/2 = \sin(\pi/6) \Leftrightarrow$

$\Leftrightarrow \vartheta = \pi/6$

It follows that:

$$I = \int_{-\pi/6}^{\pi/6} \frac{(2\sin\vartheta)^2 + 2\sin\vartheta + 1}{2\cos\vartheta}\, 2\cos\vartheta\, d\vartheta =$$

$$= \int_{-\pi/6}^{\pi/6} [4\sin^2\vartheta + 2\sin\vartheta + 1]\, d\vartheta =$$

$$= \int_{-\pi/6}^{\pi/6} [2(1-\cos(2\vartheta)) + 2\sin\vartheta + 1]\, d\vartheta =$$

$$= \int_{-\pi/6}^{\pi/6} (3 - 2\cos(2\vartheta) + 2\sin\vartheta)\, d\vartheta$$

$$= \left[3\vartheta - \frac{2\sin(2\vartheta)}{2} - 2\cos\vartheta \right]_{-\pi/6}^{\pi/6}$$

$$= \Big[3\theta - \sin(2\theta) - 2\cos\theta\Big]_{-\pi/6}^{\pi/6} =$$

$$= \big[3(\pi/6) - \sin(\pi/3) - 2\cos(\pi/6)\big] -$$

$$-\big[3(-\pi/6) - \sin(-\pi/3) - 2\cos(-\pi/6)\big] =$$

$$= \pi/2 - \sqrt{3}/2 - \underline{2\sqrt{3}/2} + \pi/2 - \sqrt{3}/2 + \underline{2\sqrt{3}/2}$$

$$= \pi - \sqrt{3}.$$

B) Evaluate $I = \int_0^2 \frac{x}{\sqrt{4+4x-x^2}}\,dx$

Solution

We note that

$$4+4x-x^2 = 4-(x^2-4x) = 4-(x^2-4x+4)+4 =$$
$$= 8-(x-2)^2$$

and therefore let $x-2 = \sqrt{8}\sin\vartheta$ with $\vartheta \in [-\pi/2, \pi/2]$.

Then: $x = 2+\sqrt{8}\sin\vartheta$

and $dx = (2+\sqrt{8}\sin\vartheta)'\,d\vartheta = \sqrt{8}\cos\vartheta\,d\vartheta$

and

$$4+4x-x^2 = 8-(x-2)^2 = 8-(\sqrt{8}\sin\vartheta)^2 =$$
$$= 8-8\sin^2\vartheta = 8(1-\sin^2\vartheta) = 8\cos^2\vartheta \Rightarrow$$
$$\Rightarrow \sqrt{4+4x-x^2} = \sqrt{8\cos^2\vartheta} = \sqrt{8}\,|\cos\vartheta| = \sqrt{8}\cos\vartheta$$

For $x=0 \Leftrightarrow 2+\sqrt{8}\sin\vartheta = 0 \Leftrightarrow 2+2\sqrt{2}\sin\vartheta = 0$

$$\Leftrightarrow 2\sqrt{2}\sin\vartheta = -2 \Leftrightarrow$$
$$\Leftrightarrow \sin\vartheta = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2} = -\sin(\pi/4)$$
$$= \sin(-\pi/4) \Leftrightarrow \vartheta = -\pi/4$$

For $x=2 \Leftrightarrow 2+\sqrt{8}\sin\vartheta = 2 \Leftrightarrow \sqrt{8}\sin\vartheta = 0 \Leftrightarrow \sin\vartheta = 0$

$$\Leftrightarrow \vartheta = 0$$

It follows that

$$I = \int_{-\pi/4}^{0} \frac{2+\sqrt{8}\sin\vartheta}{\sqrt{8}\cos\vartheta}\sqrt{8}\cos\vartheta\,d\vartheta =$$
$$= \int_{-\pi/4}^{0} (2+2\sqrt{2}\sin\vartheta)\,d\vartheta$$

$$= \left[2\theta + 2\sqrt{2}\,(-\cos\theta) \right]_{-\pi/4}^{0} =$$

$$= \left[2\cdot 0 - 2\sqrt{2}\cos 0 \right] - \left[2(-\pi/4) - 2\sqrt{2}\cos(-\pi/4) \right]$$

$$= -2\sqrt{2} + \pi/2 + 2\sqrt{2}\cos(\pi/4)$$

$$= -2\sqrt{2} + \pi/2 + 2\sqrt{2}\,(\sqrt{2}/2)$$

$$= -2\sqrt{2} + \pi/2 + 2$$

EXERCISE

(38) Evaluate the following integrals

a) $I = \int_0^{\sqrt{2}} \sqrt{2-x^2}\, dx$

b) $I = \int \frac{x^3 dx}{\sqrt{1-x^2}}$

c) $I = \int \frac{(x-1)(x+3)}{\sqrt{9-2x^2}}\, dx$

d) $I = \int_0^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}$

e) $I = \int_0^{1/2} \frac{x^2 dx}{1+\sqrt{1-x^2}}$

f) $I = \int \frac{dx}{x+3\sqrt{9-x^2}}$

g) $I = \int \sqrt{1+x-x^2}\, dx$

h) $I = \int x^2\sqrt{9+6x-x^2}\, dx$

i) $I = \int \frac{dx}{\sqrt{6+5x-x^2}}$

j) $I = \int \frac{x(x+3)\, dx}{\sqrt{25+10x-x^2}}$

(2)

$$\boxed{I = \int F\left(x, \sqrt{ax^2+bx+c}\right) dx \quad \text{with } a>0}$$

Algebraic rationalizing substitution:

Let $\boxed{y - x\sqrt{a} = \sqrt{ax^2+bx+c}}$ (1)

•₁ We solve for x:

$$\text{Eq.}(1) \Leftrightarrow (y - x\sqrt{a})^2 = \left(\sqrt{ax^2+bx+c}\right)^2 \Leftrightarrow$$

$$\Leftrightarrow y^2 - 2xy\sqrt{a} + ax^2 = ax^2 + bx + c \Leftrightarrow$$

$$\Leftrightarrow y^2 - 2xy\sqrt{a} = bx + c \Leftrightarrow (2y\sqrt{a}+b)x = y^2 - c$$

$$\Leftrightarrow x = \frac{y^2 - c}{2y\sqrt{a}+b}$$

•₂ We calculate dx in terms of y:

$$dx = \left[\frac{y^2-c}{2y\sqrt{a}+b}\right]' dy =$$

$$= \frac{(y^2-c)'(2y\sqrt{a}+b) - (y^2-c)(2y\sqrt{a}+b)'}{(2y\sqrt{a}+b)^2} dy$$

$$= \frac{2y(2y\sqrt{a}+b) - (y^2-c)2\sqrt{a}}{(2y\sqrt{a}+b)^2} dy$$

$$= \frac{4y^2\sqrt{a} + 2by - 2y^2\sqrt{a} + 2c\sqrt{a}}{(2y\sqrt{a}+b)^2} dy$$

$$= \frac{2y^2\sqrt{a} + 2by + 2c\sqrt{a}}{(2y\sqrt{a}+b)^2} dy$$

$$= \frac{2(y^2\sqrt{a} + by + c\sqrt{a})}{(2y\sqrt{a}+b)^2} dy$$

• 3 We calculate $\sqrt{ax^2+bx+c}$ in terms of y

$$\sqrt{ax^2+bx+c} = y - x\sqrt{a} = y - \frac{y^2-c}{2y\sqrt{a}+b}\sqrt{a} =$$

$$= \frac{y(2y\sqrt{a}+b)-(y^2-c)\sqrt{a}}{2y\sqrt{a}+b} =$$

$$= \frac{2y^2\sqrt{a}+by-y^2\sqrt{a}+c\sqrt{a}}{2y\sqrt{a}+b} =$$

$$= \frac{y^2\sqrt{a}+by+c\sqrt{a}}{2y\sqrt{a}+b}$$

• 4 It follows that:

$$I = \int F\left(\frac{y^2-c}{2y\sqrt{a}+b}, \frac{y^2\sqrt{a}+by+c\sqrt{a}}{2y\sqrt{a}+b}\right)\frac{2(y^2\sqrt{a}+by+c\sqrt{a})}{(2y\sqrt{a}+b)^2}\,dy$$

EXAMPLES

a) $I = \int_0^2 \frac{dx}{\sqrt{x^2+3x+5}}$

Solution

Let $y - x = \sqrt{x^2+3x+5} \Leftrightarrow (y-x)^2 = x^2+3x+5 \Leftrightarrow$

$\Leftrightarrow y^2 - 2xy + x^2 = x^2+3x+5 \Leftrightarrow y^2 - 2xy = 3x+5$

$\Leftrightarrow 3x + 2xy = y^2 - 5 \Leftrightarrow x(3+2y) = y^2-5 \Leftrightarrow x = \frac{y^2-5}{2y+3}$

Then:

$$dx = \left(\frac{y^2-5}{2y+3}\right)' dy = \frac{(y^2-5)'(2y+3) - (y^2-5)(2y+3)'}{(2y+3)^2} dy =$$

$$= \frac{2y(2y+3) - 2(y^2-5)}{(2y+3)^2} dy = \frac{4y^2+6y-2y^2+10}{(2y+3)^2} dy$$

$$= \frac{2y^2+6y+10}{(2y+3)^2} dy = \frac{2(y^2+3y+5)}{(2y+3)^2} dy$$

and

$$\sqrt{x^2+3x+5} = y - x = y - \frac{y^2-5}{2y+3} = \frac{y(2y+3)-(y^2-5)}{2y+3}$$

$$= \frac{2y^2+3y-y^2+5}{2y+3} = \frac{y^2+3y+5}{2y+3}$$

Since $y = x + \sqrt{x^2+3x+5}$,

$x = 0 \Rightarrow y = 0 + \sqrt{0+0+5} = \sqrt{5}$

$x = 2 \Rightarrow y = 2 + \sqrt{2^2+3\cdot 2+5} = 2 + \sqrt{4+6+5} = 2+\sqrt{15}$

From the above, it follows that:

$$I = \int_{\sqrt{5}}^{2+\sqrt{15}} \frac{\left[\dfrac{2(y^2+3y+5)}{(2y+3)^2}\right]}{\left[\dfrac{y^2+3y+5}{2y+3}\right]}\,dy = \int_{\sqrt{5}}^{2+\sqrt{15}} \frac{2dy}{2y+3}$$

$$= \int_{\sqrt{5}}^{2+\sqrt{15}} \frac{(2y+3)'dy}{2y+3} = \Big[\ln|2y+3|\Big]_{\sqrt{5}}^{2+\sqrt{15}} =$$

$$= \ln|2(2+\sqrt{15})+3| - \ln|2\sqrt{5}+3|$$

$$= \ln|4+2\sqrt{15}+3| - \ln|3+2\sqrt{5}|$$

$$= \ln(7+2\sqrt{15}) - \ln(3+2\sqrt{5})$$

b) $I = \int_2^4 \frac{\sqrt{x^2-4}}{x}\, dx$

Solution

Let $y - x = \sqrt{x^2-4} \Leftrightarrow (y-x)^2 = x^2-4 \Leftrightarrow y^2-2xy+x^2 = x^2-4$
$\Leftrightarrow y^2-2xy = -4 \Leftrightarrow 2xy = y^2+4 \Leftrightarrow x = \frac{y^2+4}{2y}$

We note that

$$dx = \left[\frac{y^2+4}{2y}\right]' dy = \frac{(y^2+4)'(2y)-(y^2+4)(2y)'}{(2y)^2}\, dy =$$

$$= \frac{(2y)(2y)-2(y^2+4)}{4y^2}\, dy = \frac{4y^2-2y^2-8}{4y^2}\, dy =$$

$$= \frac{2y^2-8}{4y^2}\, dy = \frac{y^2-4}{2y^2}\, dy$$

and

$$\sqrt{x^2-4} = y - x = y - \frac{y^2+4}{2y} = \frac{2y^2-(y^2+4)}{2y} =$$

$$= \frac{2y^2-y^2-4}{2y} = \frac{y^2-4}{2y}$$

and since $y = x + \sqrt{x^2-4}$

$x = 2 \Rightarrow y = 2 + \sqrt{2^2-4} = 2 + \sqrt{4-4} = 2$

$x = 4 \Rightarrow y = 4 + \sqrt{4^2-4} = 4 + \sqrt{16-4} = 4 + \sqrt{12}$
$= 4 + 2\sqrt{3}$

From the above, it follows that:

$$I = \int_2^{4+2\sqrt{3}} \frac{\left[\frac{y^2-4}{2y}\right]}{\left[\frac{y^2+4}{2y}\right]} \frac{y^2-4}{2y^2}\, dy$$

$$= \int_2^{4+2\sqrt{3}} \frac{(y^2-4)^2}{2y^2(y^2+4)}\, dy$$

It is sufficient to do the partial fraction decomposition of

$$g(z) = \frac{(z-4)^2}{2z(z+4)} = \frac{z^2-8z+16}{2z^2+8z}$$

Divide $z^2-8z+16$ with $2z^2+8z$

$$\begin{array}{l|l} z^2-8z+16 & 2z^2+8z \\ \hline -z^2-4z & (1/2) \\ \quad -12z+16 & \end{array}$$

thus:

$$g(z) = \frac{1}{2} + \frac{-12z+16}{2z(z+4)} = \frac{1}{2} + \frac{-6z+8}{z(z+4)} =$$

$$= \frac{1}{2} + \frac{A}{z} + \frac{B}{z+4}$$

with

$$A = \left.\frac{-6z+8}{z+4}\right|_{z=0} = \frac{-0+8}{0+4} = 2$$

$$B = \left.\frac{-6z+8}{z}\right|_{z=-4} = \frac{-6(-4)+8}{-4} = \frac{24+8}{-4} =$$

$$= -6-2 = -8$$

and therefore,

$$\frac{(z-4)^2}{2z(z+4)} = \frac{1}{2} + \frac{2}{z} - \frac{8}{z+4} \Rightarrow$$

$$\Rightarrow \frac{(y^2-4)^2}{2y^2(y^2+4)} = \frac{1}{2} + \frac{2}{y^2} - \frac{8}{y^2+4} \Rightarrow$$

$$\Rightarrow I = \int_2^{4+2\sqrt{3}} \frac{dy}{2} + \int_2^{4+2\sqrt{3}} \frac{2dy}{y^2} - \int_2^{4+2\sqrt{3}} \frac{8dy}{y^2+4}$$

$$= I_1 + I_2 + I_3$$

Since,

$$I_1 = \int_2^{4+2\sqrt{3}} \frac{dy}{2} = \frac{1}{2}\Big[y\Big]_2^{4+2\sqrt{3}} = \frac{(4+2\sqrt{3})-2}{2}$$

$$= \frac{2+2\sqrt{3}}{2} = 1+\sqrt{3}$$

and

$$I_2 = \int_2^{4+2\sqrt{3}} \frac{2dy}{y^2} = \left[\frac{2y^{-1}}{-1}\right]_2^{4+2\sqrt{3}} = \left[\frac{-2}{y}\right]_2^{4+2\sqrt{3}} =$$

$$= \frac{-2}{4+2\sqrt{3}} - \frac{-2}{2} = 1 - \frac{1}{2+\sqrt{3}} =$$

$$= 1 - \frac{2-\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})} = 1 - \frac{2-\sqrt{3}}{4-3} =$$

$$= 1-(2-\sqrt{3}) = 1-2+\sqrt{3} = \sqrt{3}-1$$

and

$$I_3 = \int_2^{4+2\sqrt{3}} \frac{-8dy}{y^2+4} = -8\int_2^{4+2\sqrt{3}} \frac{dy}{y^2+2^2} =$$

$$= -8\left[\frac{1}{2}\operatorname{Arctan}\left(\frac{y}{2}\right)\right]_2^{4+2\sqrt{3}} =$$

$$= -8\left[\frac{1}{2}\operatorname{Arctan}\left(\frac{4+2\sqrt{3}}{2}\right) - \frac{1}{2}\operatorname{Arctan}\left(\frac{2}{2}\right)\right]$$

$$= -4\left[\operatorname{Arctan}(2+\sqrt{3}) - \operatorname{Arctan}(1)\right]$$

$$= -4\left[\operatorname{Arctan}(2+\sqrt{3}) - \pi/4\right] =$$

$$= \pi - 4\operatorname{Arctan}(2+\sqrt{3})$$

It follows that

$$I = I_1 + I_2 + I_3 =$$

$$= (1+\sqrt{3}) + (\sqrt{3}-1) + (\pi - 4\operatorname{Arctan}(2+\sqrt{3}))$$

$$= 2\sqrt{3} + \pi - 4\operatorname{Arctan}(2+\sqrt{3}).$$

EXERCISE

(39) Evaluate the following integrals

a) $I = \int \frac{dx}{\sqrt{x^2+3x+2}}$

b) $I = \int_0^1 \frac{dx}{\sqrt{x^2+5x+7}}$

c) $I = \int \frac{x^2\,dx}{\sqrt{x^2+x+1}}$

d) $I = \int_0^1 \frac{x^3\,dx}{\sqrt{x^2+x+3}}$

(3) $\longrightarrow$ $\boxed{I = \int F\left(x, \sqrt{(ax+b)^2 - c^2}\right) dx}$

Trigonometric substitution:

Let $\boxed{ax+b = \dfrac{c}{\cos\vartheta} \quad \text{with} \quad \vartheta \in [0, \pi/2) \cup (\pi/2, \pi]}$

• 1 Solve for x:

$$ax+b = \frac{c}{\cos\vartheta} \Leftrightarrow ax = \frac{c}{\cos\vartheta} - b = \frac{c - b\cos\vartheta}{\cos\vartheta} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{c - b\cos\vartheta}{a\cos\vartheta}$$

• 2 Calculate dx

$$dx = \left[\frac{c - b\cos\vartheta}{a\cos\vartheta}\right]' d\vartheta = \left[\frac{c}{a\cos\vartheta} - \frac{b}{a}\right]' d\vartheta =$$

$$= \frac{-c(\cos\vartheta)'}{a\cos^2\vartheta} d\vartheta = \frac{c\sin\vartheta}{a\cos^2\vartheta} d\vartheta$$

• 3 Calculate $\sqrt{(ax+b)^2 - c^2}$

$$(ax+b)^2 - c^2 = \frac{c^2}{\cos^2\vartheta} - c^2 = \frac{c^2 - c^2\cos^2\vartheta}{\cos^2\vartheta} = \frac{c^2(1-\cos^2\vartheta)}{\cos^2\vartheta} =$$

$$= \frac{c^2\sin^2\vartheta}{\cos^2\vartheta} = c^2\tan^2\vartheta \Rightarrow$$

$$\Rightarrow \sqrt{(ax+b)^2 - c^2} = \sqrt{c^2\tan^2\vartheta} = |c||\tan\vartheta| =$$

$$= \begin{cases} |c|\tan\vartheta, & \text{if } \vartheta \in [0, \pi/2) \\ -|c|\tan\vartheta, & \text{if } \vartheta \in (\pi/2, \pi] \end{cases}$$

• 4 It follows that

$$I = \int F\left(\frac{c - b\cos\vartheta}{\cos\vartheta}, |c||\tan\vartheta|\right) \frac{c\sin\vartheta}{\cos^2\vartheta} d\vartheta$$

which is a trigonometric integral reducible to a rational integral.

EXAMPLE

a) $I = \int_2^4 \frac{\sqrt{x^2-4}}{x}\, dx$

Solution

Let $x = \frac{2}{\cos\vartheta}$ with $\vartheta \in [0, \pi/2) \cup (\pi/2, \pi]$. Then:

$$dx = \left(\frac{2}{\cos\vartheta}\right)' d\vartheta = \frac{-2(\cos\vartheta)'}{\cos^2\vartheta}\, d\vartheta = \frac{-2(-\sin\vartheta)}{\cos^2\vartheta}\, d\vartheta$$

$$= \frac{2\sin\vartheta}{\cos^2\vartheta}\, d\vartheta = \frac{2\tan\vartheta}{\cos\vartheta}\, d\vartheta$$

and

$$x^2-4 = \left(\frac{2}{\cos\vartheta}\right)^2 - 4 = \frac{4}{\cos^2\vartheta} - 4 = \frac{4(1-\cos^2\vartheta)}{\cos^2\vartheta} =$$

$$= \frac{4\sin^2\vartheta}{\cos^2\vartheta} = 4\tan^2\vartheta \Rightarrow \sqrt{x^2-4} = 2|\tan\vartheta|$$

For $x = 2 \Leftrightarrow \frac{2}{\cos\vartheta} = 2 \Leftrightarrow \cos\vartheta = 1 \Leftrightarrow \vartheta = 0$

$x = 4 \Leftrightarrow \frac{2}{\cos\vartheta} = 4 \Leftrightarrow \cos\vartheta = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right) \Leftrightarrow \vartheta = \frac{\pi}{3}$

It follows that:

$$I = \int_0^{\pi/3} \frac{2|\tan\vartheta|}{\left(\frac{2}{\cos\vartheta}\right)} \frac{2\tan\vartheta}{\cos\vartheta}\, d\vartheta =$$

$$= \int_0^{\pi/3} 2|\tan\vartheta|\tan\vartheta \, d\vartheta = \int_0^{\pi/3} 2\tan^2\vartheta \, d\vartheta$$

Let $y = \tan\vartheta \Rightarrow dy = (\tan\vartheta)' d\vartheta = (1+\tan^2\vartheta)\, d\vartheta = (1+y^2)\, d\vartheta$

$$\Rightarrow d\vartheta = \frac{dy}{1+y^2}$$

For $\vartheta = 0 \Rightarrow y = \tan 0 = 0$

For $\vartheta = \pi/3 \Rightarrow y = \tan(\pi/3) = \sqrt{3}$

and it follows that

$$I = \int_0^{\sqrt{3}} 2y^2 \frac{dy}{1+y^2} = 2\int_0^{\sqrt{3}} \frac{y^2}{1+y^2}\, dy = 2\int_0^{\sqrt{3}} \frac{(1+y^2)-1}{1+y^2}\, dy$$

$$= 2\int_0^{\sqrt{3}} \left[1 - \frac{1}{1+y^2}\right] dy = 2\left[y - \mathrm{Arctan}(y)\right]_0^{\sqrt{3}}$$

$$= 2\left[\sqrt{3} - \mathrm{Arctan}(\sqrt{3})\right] - 2\left[0 - \mathrm{Arctan}(0)\right]$$

$$= 2\left[\sqrt{3} - \pi/3\right] - 2(0-0)$$

$$= 2\sqrt{3} - \frac{2\pi}{3}$$

→ We have calculated this integral using both the algebraic and trigonometric methods and found

$I = 2\sqrt{3} + \pi - 4\,\mathrm{Arctan}(2+\sqrt{3})$

$I = 2\sqrt{3} - 2\pi/3$

These are in fact equal to each other and it follows that:

$$\pi - 4\,\mathrm{Arctan}(2+\sqrt{3}) = -2\pi/3 \Leftrightarrow 4\,\mathrm{Arctan}(2+\sqrt{3}) = \pi + 2\pi/3$$

$$\Leftrightarrow 4\,\mathrm{Arctan}(2+\sqrt{3}) = 5\pi/3 \Leftrightarrow \mathrm{Arctan}(2+\sqrt{3}) = 5\pi/12.$$

EXERCISE

(40) Evaluate the following integrals

a) $I = \int \sqrt{x^2-5}\, dx$

b) $I = \int x^2\sqrt{x^2-1}\, dx$

c) $I = \int_0^{1/2} \frac{x^3\, dx}{\sqrt{x^2-1}}$

d) $I = \int_0^2 \sqrt{x^2+6x+8}\, dx$

(4) $\rightarrow$ $$\boxed{I = \int F\left(x, \sqrt{(ax+b)^2 + c^2}\right) dx}$$

Trigonometric substitution

Let $\boxed{ax + b = c\tan\vartheta \text{ with } \vartheta \in (-\pi/2, \pi/2)}$

- 1 Solve for x:

$$ax + b = c\tan\vartheta \Leftrightarrow ax = c\tan\vartheta - b \Leftrightarrow x = (1/a)(c\tan\vartheta - b)$$

- 2 Calculate dx:

$$dx = (1/a)(c\tan\vartheta - b)'\, d\vartheta = \frac{c}{a\cos^2\vartheta}\, d\vartheta$$

- 3 Calculate $\sqrt{(ax+b)^2 + c^2}$

$$(ax+b)^2 + c^2 = [c\tan\vartheta]^2 + c^2 = c^2(1 + \tan^2\vartheta) = \frac{c^2}{\cos^2\vartheta} \Rightarrow$$

$$\Rightarrow \sqrt{(ax+b)^2 + c^2} = \sqrt{\frac{c^2}{\cos^2\vartheta}} = \left|\frac{c}{\cos\vartheta}\right| = \frac{|c|}{|\cos\vartheta|} = \frac{|c|}{\cos\vartheta}$$

noting that $\vartheta \in (-\pi/2, \pi/2) \Rightarrow \cos\vartheta \geq 0$.

- 4 It follows that:

$$I = \int F\left(\frac{c\tan\vartheta - b}{a}, \frac{|c|}{\cos\vartheta}\right) \frac{c}{a\cos^2\vartheta}\, d\vartheta$$

which is a trigonometric integral that can be reduced to a rational integral.

EXAMPLE

Evaluate $I = \int_{-3}^{-2} x\sqrt{x^2+4x+5}\,dx$

Solution

We note that

$x^2+4x+5 = (x^2+4x+4)+1 = (x+2)^2+1$

so we define $x+2 = \tan\vartheta$ with $\vartheta \in (-\pi/2, \pi/2)$.

It follows that

$x = \tan\vartheta - 2$

$dx = (\tan\vartheta - 2)'\,d\vartheta = \dfrac{d\vartheta}{\cos^2\vartheta}$

and

$$x^2+4x+5 = (x+2)^2+1 = \tan^2\vartheta + 1 = \frac{1}{\cos^2\vartheta} \Rightarrow$$

$$\Rightarrow \sqrt{x^2+4x+5} = \sqrt{\frac{1}{\cos^2\vartheta}} = \frac{1}{|\cos\vartheta|} = \frac{1}{\cos\vartheta}$$

because $\vartheta \in (-\pi/2, \pi/2) \Rightarrow \cos\vartheta > 0 \Rightarrow |\cos\vartheta| = \cos\vartheta$

For $x = -3 \Leftrightarrow \tan\vartheta - 2 = -3 \Leftrightarrow \tan\vartheta = 2-3 = -1$

$\Leftrightarrow \vartheta = \text{Arctan}(-1) = -\text{Arctan}(1) = -\pi/4$

For $x = -2 \Leftrightarrow \tan\vartheta - 2 = -2 \Leftrightarrow \tan\vartheta = 0 \Leftrightarrow \vartheta = 0$

It follows that

$$I = \int_{-\pi/4}^{0} (\tan\vartheta - 2)\frac{1}{\cos\vartheta}\frac{d\vartheta}{\cos^2\vartheta} =$$

$$= \int_{-\pi/4}^{0} \left[\frac{\sin\vartheta}{\cos\vartheta} - 2\right]\frac{d\vartheta}{\cos^3\vartheta} =$$

$$= \int_{-\pi/4}^{0} \frac{\sin\theta}{\cos^4\theta}\, d\theta - 2\int_{-\pi/4}^{0} \frac{d\theta}{\cos^3\theta} = I_1 - 2I_2$$

To evaluate I_1, let $y = \cos\theta$. Then

$dy = (\cos\theta)'\, d\theta = -\sin\theta\, d\theta \Rightarrow \sin\theta\, d\theta = -dy.$

and

$\theta = -\pi/4 \Rightarrow y = \cos(-\pi/4) = \cos(\pi/4) = \sqrt{2}/2$

$\theta = 0 \Rightarrow y = \cos(0) = 1$

therefore

$$I_1 = \int_{-\pi/4}^{0} \frac{\sin\theta}{\cos^4\theta}\, d\theta = \int_{\sqrt{2}/2}^{1} \frac{-dy}{y^4} = \int_{\sqrt{2}/2}^{1} (-y^{-4})\, dy =$$

$$= \left[-\frac{y^{-3}}{-3} \right]_{\sqrt{2}/2}^{1} = \left[\frac{1}{3y^3} \right]_{\sqrt{2}/2}^{1} =$$

$$= \frac{1}{3} - \frac{1}{3(\sqrt{2}/2)^3} = \frac{1}{3} - \frac{1}{3(2\sqrt{2}/8)} =$$

$$= \frac{1}{3}\left[1 - \frac{8}{2\sqrt{2}} \right] = \frac{1}{3}\left[1 - \frac{4}{\sqrt{2}} \right] =$$

$$= \frac{1}{3}\left[1 - \frac{4\sqrt{2}}{2} \right] = \frac{1 - 2\sqrt{2}}{3}$$

To evaluate I_2, we write

$$I_2 = \int_{-\pi/4}^{0} \frac{d\theta}{\cos^3\theta} = \int_{-\pi/4}^{0} \frac{\cos\theta\, d\theta}{\cos^4\theta} = \int_{-\pi/4}^{0} \frac{\cos\theta\, d\theta}{(1-\sin^2\theta)^2}$$

Let $y = \sin\theta$. Then $dy = (\sin\theta)'\, d\theta = \cos\theta\, d\theta$

For $\theta = -\pi/4 \Rightarrow y = \sin(-\pi/4) = -\sin(\pi/4) = -\sqrt{2}/2$

For $\theta = 0 \Rightarrow y = \sin 0 = 0$

It follows that

$$I_2 = \int_{-\sqrt{2}/2}^{0} \frac{dy}{(1-y^2)^2} = \int_{-\sqrt{2}/2}^{0} \frac{dy}{(1-y)^2(1+y)^2} =$$

$$= \int_{-\sqrt{2}/2}^{0} \frac{dy}{(y-1)^2(y+1)^2} =$$

$$= \int_{-\sqrt{2}/2}^{0} \left[\frac{A}{(y-1)^2} + \frac{B}{y-1} + \frac{C}{(y+1)^2} + \frac{D}{y+1} \right] dy$$

with

$$A = \left.\frac{1}{(y+1)^2}\right|_{y=1} = \frac{1}{(1+1)^2} = \frac{1}{4}$$

$$B = \left.\frac{d}{dy}\left[\frac{1}{(y+1)^2}\right]\right|_{y=1} = \left.\frac{d}{dy}(y+1)^{-2}\right|_{y=1} =$$

$$= \left.-2(y+1)^{-3}\right|_{y=1} = \left.\frac{-2}{(y+1)^3}\right|_{y=1} = \frac{-2}{(1+1)^3}$$

$$= \frac{-2}{8} = \frac{-1}{4}$$

$$C = \left.\frac{1}{(y-1)^2}\right|_{y=-1} = \frac{1}{(-1-1)^2} = \frac{1}{(-2)^2} = \frac{1}{4}$$

$$D = \left.\frac{d}{dy}\frac{1}{(y-1)^2}\right|_{y=-1} = \left.\frac{d}{dy}(y-1)^{-2}\right|_{y=-1} =$$

$$= \left.-2(y-1)^{-3}\right|_{y=-1} = \left.\frac{-2}{(y-1)^3}\right|_{y=-1} = \frac{-2}{(-1-1)^3}$$

$$= \frac{-2}{-8} = \frac{1}{4}$$

and therefore,

$$I_2 = \int_{-\sqrt{2}/2}^{0} \left[\frac{1/4}{(y-1)^2} + \frac{-1/4}{y-1} + \frac{1/4}{(y+1)^2} + \frac{1/4}{y+1} \right] dy =$$

$$= \frac{1}{4}\left[\frac{-1}{y-1} - \ln|y-1| + \frac{-1}{y+1} + \ln|y+1| \right]_{-\sqrt{2}/2}^{0} =$$

$$= \frac{1}{4}\left[\frac{-(y+1)-(y-1)}{(y-1)(y+1)} + \ln\left|\frac{y+1}{y-1}\right| \right]_{-\sqrt{2}/2}^{0}$$

$$= \frac{1}{4}\left[\frac{-2y}{y^2-1} + \ln\left|\frac{y+1}{y-1}\right| \right]_{-\sqrt{2}/2}^{0} =$$

$$= \frac{1}{4}\left[0 + \ln\left|\frac{0+1}{0-1}\right| \right] -$$

$$- \frac{1}{4}\left[\frac{-2(-\sqrt{2}/2)}{(-\sqrt{2}/2)^2-1} + \ln\left|\frac{-\sqrt{2}/2+1}{-\sqrt{2}/2-1}\right| \right]$$

$$= (1/4)(0+\ln 1) - \frac{1}{4}\left[\frac{\sqrt{2}}{1/2-1} + \ln\left|\frac{\sqrt{2}-2}{\sqrt{2}+2}\right| \right]$$

$$= \frac{-1}{4}\left[\frac{\sqrt{2}}{-1/2} + \ln\left|\frac{(\sqrt{2}-2)^2}{(\sqrt{2}+2)(\sqrt{2}-2)}\right| \right]$$

$$= \frac{-1}{4}\left[-2\sqrt{2} + \ln\left|\frac{2-4\sqrt{2}+4}{2-4}\right| \right]$$

$$= \frac{-1}{4}\left[-2\sqrt{2} + \ln\left|\frac{6-4\sqrt{2}}{-2}\right| \right] =$$

$$= \frac{+\sqrt{2}}{2} - \frac{\ln(3-2\sqrt{2})}{4}$$

We conclude that

$$I = I_1 - 2I_2 = \frac{1-2\sqrt{2}}{3} - 2\left[\frac{\sqrt{2}}{2} - \frac{\ln(3-2\sqrt{2})}{4} \right]$$

$$= \frac{1-2\sqrt{2}}{3} - \sqrt{2} + \frac{\ln(3-2\sqrt{2})}{2} = \frac{1-5\sqrt{2}}{3} + \frac{\ln(3-2\sqrt{2})}{2}$$

EXERCISE

(41) Evaluate the following integrals

a) $I = \int_1^2 \frac{\sqrt{x^2+9}}{x}\, dx$

b) $I = \int_0^2 \sqrt{x^2+4}\, dx$

c) $I = \int_0^1 x^3\sqrt{x^2+1}\, dx$

d) $I = \int \frac{x^2 dx}{\sqrt{x^2+7}}$

e) $I = \int \frac{dx}{x^2\sqrt{x^2+1}}$

f) $I = \int_0^1 x^3\sqrt{x^2+x+2}\, dx$