SERIES

Convergent series

• **Def** : Let a_n be a sequence. The series S_n of a_n is defined as

$$\boxed{\forall n \in \mathbb{N}^* : S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n}$$

↳ We oftentimes write $S_n = \sum a_n$ as an abbreviation.

• **Def** : If $S_n = \sum a_n$ is convergent, then we write:

$$\boxed{\sum_{k=1}^{+\infty} a_k = \lim S_n}$$

Series whose limit can be evaluated

1) Geometric series

Since:

$$\boxed{S_n = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}, \quad \forall n \in \mathbb{N}^*}$$

it follows that:

$$\boxed{a \in (-1,1) \Rightarrow \lim S_n = \sum_{k=0}^{+\infty} a^k = \frac{1}{1-a}}$$

$$\boxed{a \in (-\infty,-1] \cup [1,+\infty) \Rightarrow \sum_{k=0}^{+\infty} a^k \text{ diverges}}$$

2) <u>Collapsing series</u>

Any series of the form

$$S_n = \sum_{k=1}^{n} (a_{k+1} - a_k) = a_{n+1} - a_1$$

is called a collapsing or telescopic series, and if it converges, its limit is given by

$$\boxed{\lim \sum_{k=1}^{n} (a_{k+1} - a_k) = (\lim a_n) - a_1}$$

Properties of convergent series

① <u>Linearity of convergent series</u>

<u>Thm</u> : If $\sum a_n$, $\sum b_n$ are convergent series, then

$$\boxed{\sum_{k=1}^{+\infty} (a_k + b_k) = \sum_{k=1}^{+\infty} a_k + \sum_{k=1}^{+\infty} b_k}$$

$$\boxed{\forall \lambda \in \mathbb{R} : \sum_{k=1}^{+\infty} (\lambda a_k) = \lambda \sum_{k=1}^{+\infty} a_k}$$

② <u>Necessary condition for series convergence</u>

$$\boxed{\sum_{k=1}^{+\infty} a_k \text{ converges} \Rightarrow \lim a_n = 0}$$

↳ The contrapositive statement gives a criterion for showing series divergence:

$$\boxed{\lim a_n \neq 0 \Rightarrow \sum_{k=1}^{+\infty} a_k \text{ diverges}}$$

<u>Proof</u>

Let $s_n = \sum_{k=1}^{n} a_k$ and assume that s_n convergent.

Then: $\exists l \in \mathbb{R} : \lim s_n = l$

and therefore:

$$\lim a_n = \lim \left[a_n + \sum_{k=1}^{n-1} a_k - \sum_{k=1}^{n-1} a_k \right] = \lim \left[\sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k \right]$$

$$= \lim (s_n - s_{n-1}) = \lim s_n - \lim s_{n-1} =$$

$$= l - l = 0 \qquad \square$$

EXAMPLES

Evaluate the following series, if they converge, or show that they diverge.

a) $\displaystyle\sum_{k=1}^{+\infty} \frac{1}{(k+2)(k+3)}$

Solution

We note that

$$\forall n \in \mathbb{N}^* : a_n = \frac{1}{(n+2)(n+3)} = \frac{A}{n+2} + \frac{B}{n+3}$$

with

$$A = \left.\frac{1}{x+3}\right|_{x=-2} = \frac{1}{(-2)+3} = \frac{1}{1} = 1$$

$$B = \left.\frac{1}{x+2}\right|_{x=-3} = \frac{1}{(-3)+2} = \frac{1}{(-1)} = -1$$

and therefore:

$$\sum_{k=1}^{+\infty} \frac{1}{(k+2)(k+3)} = \sum_{k=1}^{+\infty} \left[\frac{1}{k+2} - \frac{1}{k+3}\right] =$$

$$= \lim_{n \in \mathbb{N}} \sum_{k=1}^{n} \left[\frac{1}{k+2} - \frac{1}{k+3}\right] = \lim_{n \in \mathbb{N}} \left[\frac{1}{3} - \frac{1}{n+3}\right]$$

$$= \frac{1}{3} - \lim \frac{1}{n+3} = \frac{1}{3} - 0 = \frac{1}{3}$$

$\hookrightarrow$ This is an example of a telescopic series.

b) $\displaystyle\sum_{k=2}^{+\infty} \frac{3^k - 2^k}{7^k}$

<u>Solution</u>

We note that

$$\forall n \in \mathbb{N}^*: \quad S_n = \sum_{k=2}^{n} \frac{3^k - 2^k}{7^k} = \sum_{k=0}^{n} \frac{3^k - 2^k}{7^k} - \sum_{k=0}^{1} \frac{3^k - 2^k}{7^k} =$$

$$= \sum_{k=0}^{n} \left[\left(\frac{3}{7}\right)^k - \left(\frac{2}{7}\right)^k \right] - \frac{3-2}{7} - \frac{1-1}{1} =$$

$$= \frac{1-(3/7)^{n+1}}{1-(3/7)} - \frac{1-(2/7)^{n+1}}{1-(2/7)} - \frac{1}{7} \Rightarrow$$

$$\Rightarrow \sum_{k=2}^{+\infty} \frac{3^k - 2^k}{7^k} = \lim S_n = \lim \left[\frac{1-(3/7)^{n+1}}{1-(3/7)} - \frac{1-(2/7)^{n+1}}{1-(2/7)} - \frac{1}{7} \right] =$$

$$= \frac{1-\lim(3/7)^{n+1}}{1-(3/7)} - \frac{1-\lim(2/7)^{n+1}}{1-(2/7)} - \frac{1}{7} =$$

$$= \frac{1}{1-(3/7)} - \frac{1}{1-(2/7)} - \frac{1}{7} =$$

$$= \frac{7}{7-3} - \frac{7}{7-2} - \frac{1}{7} = \frac{7}{4} - \frac{7}{5} - \frac{1}{7} =$$

$$= \frac{7\cdot 5\cdot 7 - 7\cdot 4\cdot 7 - 4\cdot 5}{4\cdot 5\cdot 7} = \frac{245 - 196 - 20}{140}$$

$$= \frac{29}{140}$$

c) $\displaystyle\sum_{n=1}^{+\infty} \frac{n^3}{3n^3+2n^2+1}$

<u>Solution</u>

Since:

$$\lim_{x\to+\infty} \frac{x^3}{3x^3+2x^2+1} = \lim_{x\to+\infty} \frac{x^3}{3x^3} = \frac{1}{3} \Rightarrow$$

$$\Rightarrow \lim \frac{n^3}{3n^3+2n^2+1} = \frac{1}{3} \neq 0 \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{n^3}{3n^3+2n^2+1} \text{ diverges.}$$

Convergence of positive series

- A <u>positive series</u> $S_n = \sum_{k=1}^{n} a_k$ is any series that satisfies:

  $\forall k \in \mathbb{N}^* : a_k \geq 0$.

- The convergence of positive series can be determined by the following tests:
  1) Integral test
  2) Comparison test
  3) Limit comparison test
  4) Ratio test
  5) Root test

1 → Integral test

<u>Thm</u>: Let $f : [1, +\infty) \to \mathbb{R}$ be a function such that

$$\begin{cases} f \text{ continuous on } [1, +\infty) \\ f \searrow [1, +\infty) \wedge (\forall x \in [1, +\infty) : f(x) > 0) \\ \forall n \in \mathbb{N}^* : a_n = f(n) \end{cases}$$

Then:

a) $\sum_{k=1}^{+\infty} a_k$ converges $\Leftrightarrow \int_1^{+\infty} f(x)\,dx$ converges

b) $\sum_{k=1}^{+\infty} a_k$ diverges $\Leftrightarrow \int_1^{+\infty} f(x)\,dx$ diverges.

Proof

Since,

$$f \searrow [1,+\infty) \Rightarrow \forall k \in \mathbb{N}^* : \forall x \in [k,k+1] : f(k) \geq f(x) \geq f(k+1)$$
$$\Rightarrow \forall k \in \mathbb{N}^* : \left(\min_{x \in [k,k+1]} f(x) = f(k+1) \wedge \max_{x \in [k,k+1]} f(x) = f(k)\right)$$

it follows that the Riemann sums of f are given by

$$L_n(f|1,n) = \sum_{k=1}^{n-1} \left[\min_{x \in [k,k+1]} f(x)\right] ((k+1)-k) =$$
$$= \sum_{k=1}^{n-1} f(k+1) = \sum_{k=2}^{n} f(k) = \sum_{k=2}^{n} a_k$$

and

$$U_n(f|1,n) = \sum_{k=1}^{n-1} \left[\max_{x \in [k,k+1]} f(x)\right] ((k+1)-k) =$$
$$= \sum_{k=1}^{n-1} f(k) = \sum_{k=1}^{n-1} a_k.$$

Define $\forall n \in \mathbb{N}^* : I_n = \int_1^n f(t)\,dt$

Then: $\forall n \in \mathbb{N}^* : L_n(f|1,n) \leq I_n \leq U_n(f|1,n)$.

<u>($\Rightarrow$)</u>: Assume that $\sum_{k=1}^{+\infty} a_k$ converges.

We define

$$\forall x \in [1,+\infty) : F(x) = \int_1^x f(t)\,dt$$

$$\rightarrow \forall x \in [1,+\infty) : F'(x) = \frac{d}{dx} \int_1^x f(t)\,dt = f(x) > 0$$

$\Rightarrow F \nearrow [1, +\infty)$. (1)

▸ We will show that I_n is increasing and upper bounded.

Let $n \in \mathbb{N}^*$ be given. Then:

$$I_n = \int_1^n f(t)\,dt = F(n) < F(n+1) = \int_1^{n+1} f(t)\,dt = I_{n+1} \Rightarrow I_n < I_{n+1}.$$

and therefore:

$(\forall n \in \mathbb{N}^*: I_n < I_{n+1}) \Rightarrow I_n$ increasing. (2)

Let $n \in \mathbb{N}^*$ be given. Then:

$$I_n \leqslant U_n(f|1,n) = \sum_{k=1}^{n-1} a_k \leqslant \sum_{k=1}^{+\infty} a_k$$

and therefore:

$$\left(\forall n \in \mathbb{N}^*: I_n \leqslant \sum_{k=1}^{+\infty} a_k\right) \Rightarrow I_n \text{ upper-bounded.} \quad (3)$$

From Eq. (2) and Eq. (3):

$$\begin{cases} I_n \text{ increasing} \\ I_n \text{ upper-bounded} \end{cases} \Rightarrow I_n \text{ convergent} \Rightarrow \exists l \in \mathbb{R}: \lim I_n = l.$$

We note that; from Eq. (1):

$$\forall n \in \mathbb{N}^*: \forall x \in [n, n+1]: I_n \leqslant \int_1^x f(t)\,dt \leqslant I_{n+1} \Rightarrow$$

$$\Rightarrow \lim I_n \leqslant \int_1^{+\infty} f(t)\,dt \leqslant \lim I_{n+1} \Rightarrow l \leqslant \int_1^{+\infty} f(t)\,dt \leqslant l$$

$$\Rightarrow \int_1^{+\infty} f(t)\,dt = l \Rightarrow \int_1^{+\infty} f(t)\,dt \text{ converges.}$$

($\Leftarrow$): Assume that $\int_1^{+\infty} f(x)dx$ converges and define $\forall n \in \mathbb{N}^*: S_n = \sum_{k=1}^{n} a_k$.

Since ($\forall n \in \mathbb{N}^*: a_n > 0$), it follows that:

$$\forall n \in \mathbb{N}^*: S_{n+1} = \sum_{k=1}^{n+1} a_k = a_{n+1} + \sum_{k=1}^{n} a_k > \sum_{k=1}^{n} a_k = S_n$$

$\Rightarrow (\forall n \in \mathbb{N}^*: S_{n+1} > S_n) \Rightarrow S_n$ strictly increasing. (4)

We also note that

$$\forall n \in \mathbb{N}^*: S_n = \sum_{k=1}^{n} a_k = a_1 + \sum_{k=2}^{n} a_k = a_1 + L_n(f|1,n)$$

$$\leqslant a_1 + I_n = a_1 + \int_1^n f(x)dx \leqslant a_1 + \int_1^{+\infty} f(x)dx$$

$\Rightarrow S_n$ upper bounded. (5)

From Eq. (4) and Eq. (5):

$$\begin{cases} S_n \text{ strictly increasing} \\ S_n \text{ upper bounded} \end{cases} \Rightarrow S_n \text{ converges} \Rightarrow \sum_{k=1}^{+\infty} a_k \text{ converges.} \quad \square$$

$\hookrightarrow$ <u>Application</u> : Using the integral test, we can show that:

$$\boxed{\begin{array}{ll} \displaystyle\sum_{k=1}^{+\infty} \frac{1}{k^p} & \text{converges} \iff p > 1 \\ \displaystyle\sum_{k=1}^{+\infty} \frac{1}{k^p} & \text{diverges} \iff p \leq 1 \end{array}}$$

- This result can be used in conjunction with forthcoming convergence tests (i.e. comparison test, limit comparison test) It also motivates the definition of the famous Riemann-Zeta function, defined as:

$$\boxed{\forall s \in [1, +\infty) : \zeta(s) = \sum_{k=1}^{+\infty} \frac{1}{k^s}}$$

- A remarkable result, originally shown by Euler, is that $\zeta(2) = \pi^2/6$.

<u>Proof</u>

We define $\forall x \in [1, +\infty) : f(x) = \dfrac{1}{x^p}$

It is easy to see that for $p < 0$:

$$\lim_{x \to +\infty} \frac{1}{x^p} = \lim_{x \to +\infty} x^{-p} = +\infty \Rightarrow \lim_{n \in \mathbb{N}^*} \frac{1}{n^p} \neq 0 \Rightarrow$$

$$\Rightarrow \sum_{k=1}^{+\infty} \frac{1}{k^p} \text{ diverges}$$

and for $p=0$:

$$\lim_{x\to+\infty} \frac{1}{x^p} = \lim_{x\to+\infty} \frac{1}{x^0} = 1 \Rightarrow \lim_{n\in\mathbb{N}^*} \frac{1}{n^p} = 1 \Rightarrow$$

$$\Rightarrow \sum_{k=1}^{+\infty} \frac{1}{k^p} \text{ diverges}$$

We may now assume, with no loss of generality, that $p>0$.
It follows that

$$\forall x \in [1,+\infty): f'(x) = \frac{d}{dx}\left(\frac{1}{x^p}\right) = (x^{-p})' = -px^{-p-1} =$$

$$= \frac{-p}{x^{p+1}} < 0$$

$$\Rightarrow f \searrow [1,+\infty) \qquad (1)$$

and also that

$$f \text{ continuous on } [1,+\infty) \qquad (2)$$

From Eq.(1) and Eq.(2), via the integral test, it follows that:

$$\sum_{k=1}^{+\infty} \frac{1}{k^p} \text{ converges} \Leftrightarrow \int_1^{+\infty} \frac{dx}{x^p} \text{ converges}$$

$$\Leftrightarrow p>1. \qquad \square$$

EXAMPLE

a) Show that $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{(n+1)\ln(n+1)}$ diverges.

Solution

We define: $\forall x \in [1,+\infty): f(x) = \dfrac{1}{(x+1)\ln(x+1)}$

and note that

f continuous on $[1,+\infty)$ (1)

and

$$\forall x \in [1,+\infty): f'(x) = \frac{-[(x+1)\ln(x+1)]'}{[(x+1)\ln(x+1)]^2} =$$

$$= \frac{-[(x+1)'\ln(x+1) + (x+1)[\ln(x+1)]']}{(x+1)^2[\ln(x+1)]^2}$$

$$= \frac{-\ln(x+1) - (x+1)(1/(x+1))}{(x+1)^2[\ln(x+1)]^2} =$$

$$= \frac{-\ln(x+1) - 1}{(x+1)^2[\ln(x+1)]^2}$$

Let $x \in [1,+\infty)$ be given. Then:

$x \in [1,+\infty) \Rightarrow x \geq 1 \Rightarrow x+1 \geq 2 > 1 \Rightarrow \ln(x+1) > \ln 1 \Rightarrow$

$\Rightarrow -\ln(x+1) < 0 \Rightarrow -\ln(x+1) - 1 < 0 \Rightarrow$

$\Rightarrow f'(x) < 0$

since $(x+1)^2[\ln(x+1)]^2 > 0$. It follows that:

$(\forall x \in [1,+\infty): f'(x) < 0) \Rightarrow f \searrow [1,+\infty)$ (2)

Finally, we note that:

$$\int_1^x f(t)\,dt = \int_1^x \frac{dt}{(t+1)\ln(t+1)} = \int_2^{x+1} \frac{dt}{t\ln t} = \int_2^{x+1} \frac{(\ln t)'}{\ln t}\,dt$$

$$= \Big[\ln|\ln t|\Big]_2^{x+1} = \ln|\ln(x+1)| - \ln|\ln(2)|, \ \forall x \in [1, +\infty).$$

and that:

$$\lim_{x\to+\infty}(x+1) = \lim_{x\to+\infty} x = +\infty \Rightarrow \lim_{x\to+\infty} \ln(x+1) = +\infty \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} |\ln(x+1)| = +\infty \Rightarrow \lim_{x\to+\infty} \ln|\ln(x+1)| = +\infty$$

$$\Rightarrow \lim_{x\to+\infty} \left[\ln|\ln(x+1)| - \ln|\ln 2|\right] = +\infty \Rightarrow$$

$$\Rightarrow \int_1^{+\infty} \frac{dx}{(x+1)\ln(x+1)} \text{ diverges} \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{1}{(n+1)\ln(n+1)} \text{ diverges.}$$

② → Comparison test

Thm: Let a_n, b_n be two sequences and assume that

$$\forall n \in \mathbb{N}^* : (n > n_0 \Rightarrow 0 \leq a_n \leq b_n)$$

Then,

$$\sum_{n=1}^{+\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges}$$

$$\sum_{n=1}^{+\infty} a_n \text{ diverges} \Rightarrow \sum_{n=1}^{+\infty} b_n \text{ diverges}$$

Proof

We define: $\forall n \in \mathbb{N}^* : \left(A_n = \sum_{k=1}^{n} a_n \wedge B_n = \sum_{k=1}^{n} b_k\right)$

Let us assume that B_n converges. Then:

B_n converges $\Rightarrow B_n$ bounded

$\Rightarrow B_n$ upper bounded

$\Rightarrow \exists B > 0 : \forall n \in \mathbb{N}^* : B_n \leq B$

Now we will show that A_n is bounded.

Let $n \in \mathbb{N}^*$ be given, with $n > n_0$. Then:

$$A_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n_0} a_k + \sum_{k=n_0+1}^{n} a_k \leq \sum_{k=1}^{n_0} a_k + \sum_{k=n_0+1}^{n} b_k \leq$$

$$\leq \sum_{k=1}^{n_0} a_k + \sum_{k=1}^{n} b_k = \sum_{k=1}^{n_0} a_k + B_n \leq \sum_{k=1}^{n_0} a_k + B = A_{n_0} + B$$

For $n \leq n_0$: $A_n = \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n_0} a_k \leq \sum_{k=1}^{n_0} a_k + B = A_{n_0} + B$

It follows that:

$(\forall n \in \mathbb{N}^*: A_n \leq A_n + B) \Rightarrow A_n$ upper bounded. (1)

Let $n \in \mathbb{N}^*$ be given. Then:

$$A_{n+1} = \sum_{k=1}^{n+1} a_k \geq \sum_{k=1}^{n} a_k = A_n$$

and it follows that

$(\forall n \in \mathbb{N}^*: A_{n+1} \geq A_n) \Rightarrow A_n$ increasing (2)

From Eq. (1) and Eq. (2):

$$\begin{cases} A_n \text{ upper bounded} \\ A_n \text{ increasing} \end{cases} \Rightarrow A_n \text{ converges.}$$

An immediate consequence is the contrapositive statement

A_n diverges $\Rightarrow B_n$ diverges. $\square$

EXAMPLES

a) Show that $\sum_{k=1}^{+\infty} \frac{1}{k!}$ converges.

Solution

We define $\forall n \in \mathbb{N}^* : a_n = 1/n! \geqslant 0$ (1)

and note that

$$\forall n \in \mathbb{N}^* - [2] : a_n = \frac{1}{n!} = \frac{1}{n(n-1)(n-2)!} \leqslant \frac{1}{n(n-1)}$$

$$\leqslant \frac{1}{(n-1)^2} \quad (2)$$

From Eq. (1) and Eq. (2)

$\forall n \in \mathbb{N}^* - [2] : 0 \leqslant a_n \leqslant 1/(n-1)^2$ (3)

From Eq. (3), via the comparison test, it follows that

$$\sum_{k=1}^{+\infty} \frac{1}{k^2} \text{ converges} \Rightarrow \sum_{k=2}^{+\infty} \frac{1}{(k-1)^2} \text{ converges} \Rightarrow$$

$$\Rightarrow \sum_{k=2}^{+\infty} \frac{1}{k!} \text{ converges}$$

$$\Rightarrow \sum_{k=1}^{+\infty} \frac{1}{k!} \text{ converges}$$

↳ Note that Eq. (2) is valid only for $n \geqslant 2$, which is needed to write $n! = n(n-1)(n-2)!$. It is therefore important to indicate that in the quantifier $\forall n \in \mathbb{N}^* - [2]$. Note that $\mathbb{N}^* - [2] = \{1, 2, 3, \ldots\} - \{1, 2\} = \{3, 4, 5, \ldots\}$

b) Show that $\sum_{n=1}^{+\infty} \frac{1}{2+n5^n}$ converges.

Solution

We define $\forall n \in \mathbb{N}^*: a_n = \frac{1}{2+n5^n}$

and note that: $\forall n \in \mathbb{N}^*: a_n \geq 0$ (1)

and

$\forall n \in \mathbb{N}^*: a_n = \frac{1}{2+n5^n} \leq \frac{1}{n5^n} \leq \frac{1}{5^n} = \left(\frac{1}{5}\right)^n$ (2)

From Eq.(1) and Eq.(2):

$\forall n \in \mathbb{N}^*: 0 \leq a_n \leq (1/5)^n$

and from the comparison test, it follows that

$\sum_{n=1}^{+\infty} (1/5)^n$ converges $\Rightarrow \sum_{n=1}^{+\infty} \frac{1}{2+n5^n}$ converges.

c) Show that $\sum_{n=1}^{+\infty} \frac{n^2}{4n^3-3}$ diverges.

Solution

We define $\forall n \in \mathbb{N}^*: a_n = \frac{n^2}{4n^3-3}$.

Let $n \in \mathbb{N}^*$ be given. Since

$$a_n = \frac{n^2}{4n^3-3} \geq \frac{n^2}{4n^3} = \frac{1}{4n} \geq 0$$

It follows that

$$\forall n \in \mathbb{N}^*: \left(0 \leq \frac{1}{4n} \leq a_n\right) \qquad (1)$$

From Eq.(1), via the comparison theorem, it follows that

$$\sum_{n=1}^{+\infty} \frac{1}{n} \text{ diverges} \Rightarrow \sum_{n=1}^{+\infty} \frac{1}{4n} \text{ diverges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{n^2}{4n^3-3} \text{ diverges.} \qquad \square$$

③ → Limit Comparison Test

Thm: Let a_n, b_n be two sequences and assume that

$$\begin{cases} \forall n \in \mathbb{N}^*: (a_n \geq 0 \wedge b_n > 0) \\ \lim (a_n/b_n) = l \in \mathbb{R} \end{cases}$$

Then:

$$l > 0 \Rightarrow \left(\sum_{n=1}^{+\infty} a_n \text{ converges} \Leftrightarrow \sum_{n=1}^{+\infty} b_n \text{ converges} \right)$$

$$l = 0 \Rightarrow \left(\sum_{n=1}^{+\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges} \right)$$

and the corresponding contrapositive statements are:

$$l > 0 \Rightarrow \left(\sum_{n=1}^{+\infty} a_n \text{ diverges} \Leftrightarrow \sum_{n=1}^{+\infty} b_n \text{ diverges} \right)$$

$$l = 0 \Rightarrow \left(\sum_{n=1}^{+\infty} a_n \text{ diverges} \Rightarrow \sum_{n=1}^{+\infty} b_n \text{ diverges} \right)$$

Proof

Case 1: Assume that $l > 0$. Then

$\lim (a_n/b_n) = l > 0 \Rightarrow$

$\Rightarrow \forall \varepsilon > 0: \exists n_0 \in \mathbb{N}^*: \forall n \in \mathbb{N}^* - [n_0]: |a_n/b_n - l| < \varepsilon$

Choose $\varepsilon = l/2$ and fix a $n_0 \in \mathbb{N}^*$ such that

$\forall n \in \mathbb{N}^* - [n_0]: |a_n/b_n - l| < l/2$

Let $n \in \mathbb{N}^* - [n_0]$ be given. Then:

$|a_n/b_n - l| < l/2 \Rightarrow -l/2 < a_n/b_n - l < l/2 \Rightarrow$

$\Rightarrow l - l/2 < a_n/b_n < l + l/2 \Rightarrow$

$$\Rightarrow l/2 < a_n/b_n < 3l/2 \Rightarrow \begin{cases} l/2 < a_n/b_n \\ a_n/b_n < 3l/2 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} b_n < (2/l)\, a_n \\ a_n < (3l/2)\, b_n \end{cases}$$

It follows that

$$\forall n \in \mathbb{N}^* - [n_0] : \begin{cases} b_n < (2/l)\, a_n \\ a_n < (3l/2)\, b_n \end{cases}$$

and therefore, from the comparison test:

$$\sum_{n=1}^{+\infty} a_n \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} (2/l)\, a_n \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} b_n \text{ converges}$$

and

$$\sum_{n=1}^{+\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} (3l/2)\, b_n \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges.}$$

We have thus shown that

$$\sum_{n=1}^{+\infty} a_n \text{ converges} \Leftrightarrow \sum_{n=1}^{+\infty} b_n \text{ converges.}$$

<u>Case 2</u>: Assume that $l = 0$. Then

$\lim(a_n/b_n) = 0 \Rightarrow$

$\Rightarrow \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n/b_n| < \varepsilon$

Choose any $\varepsilon > 0$ and fix $n_0 \in \mathbb{N}^*$ such that

$\forall n \in \mathbb{N}^* - [n_0] : |a_n/b_n| < \varepsilon.$

Let $n \in \mathbb{N}^* - [n_0]$ be given. Then, since $a_n \geq 0$ and $b_n > 0$,

it follows that

$|a_n/b_n| < \varepsilon \Rightarrow a_n/b_n < \varepsilon \Rightarrow a_n < \varepsilon b_n$

and therefore:

$\forall n \in \mathbb{N}^* - [n_0] : 0 \leq a_n < \varepsilon b_n$

From the comparison test, it follows that

$$\sum_{n=1}^{+\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} \varepsilon b_n \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges.} \qquad \blacksquare$$

EXAMPLE

a) Show that $\sum_{n=1}^{+\infty} \frac{3n-2}{n^3+2n^2+11}$ converges.

Solution

We define:

$$\forall n \in \mathbb{N}^*: \left(a_n = \frac{3n-2}{n^3+2n^2+11} \wedge b_n = \frac{1}{n^2} \right)$$

It follows that

$$\forall n \in \mathbb{N}^*: \frac{a_n}{b_n} = \frac{3n-2}{n^3+2n^2+11} \left(\frac{1}{n^2}\right)^{-1} = \frac{n^2(3n-2)}{n^3+2n^2+11} \quad (1)$$

and since

$$\lim_{x \to +\infty} \frac{x^2(3x-2)}{x^3+2x^2+11} = \lim_{x \to +\infty} \frac{3x^3}{x^3} = 3 \Rightarrow$$

$$\Rightarrow \lim \frac{a_n}{b_n} = \lim \frac{n^2(3n-2)}{n^3+2n^2-11} = 3 > 0 \quad (2)$$

it follows from the limit comparison test that

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} \frac{3n-2}{n^3+2n^2+11} \text{ converges.}$$

↳ Note that for large n we can reasonably expect a_n to scale as $3n/n^3 = 3/n^2$, which is why we are comparing it with $b_n = 1/n^2$.

b) Show that $\sum_{n=1}^{+\infty} \frac{n}{2^n(n+1)}$ converges.

<u>Solution</u>

We define:

$$\forall n \in \mathbb{N}^* : \left(a_n = \frac{n}{2^n(n+1)} \wedge b_n = \frac{1}{2^n} \right)$$

It follows that

$$\forall n \in \mathbb{N}^* : \frac{a_n}{b_n} = \frac{\frac{n}{2^n(n+1)}}{\frac{1}{2^n}} = \frac{2^n n}{2^n(n+1)} = \frac{n}{n+1}$$

and since:

$$\lim_{x \to +\infty} \frac{x}{x+1} = \lim_{x \to +\infty} \frac{x}{x} = 1 \Rightarrow \lim \frac{a_n}{b_n} = 1$$

and therefore, from the limit comparison theorem, it follows that

$$\sum_{n=1}^{+\infty} \frac{1}{2^n} \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} \frac{n}{2^n(n+1)} \text{ converges.}$$

c) Show that $\sum_{n=1}^{+\infty} \frac{\ln n}{n^2}$ converges

<u>Solution</u>

We define:

$$\forall n \in \mathbb{N}^* : \left(a_n = \frac{\ln n}{n^2} \wedge b_n = \frac{1}{n^s} \right)$$

and note that

$$\forall n \in \mathbb{N}^*: \frac{a_n}{b_n} = \frac{\frac{\ln n}{n^2}}{\frac{1}{n^s}} = \frac{n^s \ln n}{n^2} = \frac{\ln n}{n^{2-s}}$$

Let us assume that $2-s>0$. Then:

$$\lim_{x\to+\infty} \frac{\ln x}{x^{2-s}} \underset{\infty/\infty}{=} \lim_{x\to+\infty} \frac{(\ln x)'}{(x^{2-s})'} = \lim_{x\to+\infty} \frac{x^{-1}}{(2-s)x^{1-s}} =$$

$$= \lim_{x\to+\infty} \frac{1}{(2-s)x^{2-s}} = \frac{1}{2-s} \cdot 0 = 0$$

$$\Rightarrow \lim \frac{a_n}{b_n} = 0 \quad , \forall s \in (-\infty, 2) \qquad (1)$$

For $s=3/2$, via the limit comparison test,

$$\sum_{n=1}^{+\infty} \frac{1}{n^{3/2}} \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} \frac{\ln n}{n^2} \text{ converges.} \quad \square$$

↳ Since it is not obvious what to compare with, we try the general case $b_n = 1/n^s$. Note that the De L'Hospital step requires $2-s>0$, otherwise the limit $x\to+\infty$ will not be a ∞/∞ limit. So the proof of Eq.(1) is valid only when $s<2$. Note that any choice $1<s<2$ will work as well as $s=3/2$. For $s\leq 1$ there is a problem. We cannot say

$$\sum_{n=1}^{+\infty} \frac{1}{n^s} \text{ diverges} \Rightarrow \sum_{n=1}^{+\infty} \frac{\ln n}{n^2} \text{ diverges}$$

because the implication is going in the wrong direction. Since $\lim (a_n/b_n) = 0$, the implication does not work in both directions. From the limit comparison test, we can only conclude:

$$\sum_{n=1}^{+\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges}$$

$$\sum_{n=1}^{+\infty} a_n \text{ diverges} \Rightarrow \sum_{n=1}^{+\infty} b_n \text{ diverges.}$$

It is very important to apply the theorem correctly to avoid any errors in logic.

(4) → Ratio test

Let a_n be a sequence. Then:

$$\begin{cases} \forall n \in \mathbb{N}^*: a_n > 0 \\ \lim (a_{n+1}/a_n) = \rho < 1 \end{cases} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges}$$

$$\begin{cases} \forall n \in \mathbb{N}^*: a_n > 0 \\ \lim (a_{n+1}/a_n) = \rho > 1 \end{cases} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ diverges}$$

Proof

▸ Part 1: Assume that $\rho < 1$. Since:

$(\forall n \in \mathbb{N}^*: a_n > 0) \Rightarrow (\forall n \in \mathbb{N}^*: a_{n+1}/a_n > 0)$

$\Rightarrow \rho = \lim (a_{n+1}/a_n) \geq 0$

thus, we know that $0 \leq \rho < 1$. Since

$\lim (a_{n+1}/a_n) = \rho \Rightarrow \forall \varepsilon > 0: \exists n_0 \in \mathbb{N}^*: \forall n \in \mathbb{N}^* - [n_0]: |a_{n+1}/a_n - \rho| < \varepsilon$.

Choose an $r \in \mathbb{R}$ such that $r \in (\rho, 1)$ and choose $\varepsilon = r - \rho > 0$.

We also fix $n_0 \in \mathbb{N}^*$ such that

$\forall n \in \mathbb{N}^* - [n_0]: |a_{n+1}/a_n - \rho| < r - \rho$

Let $n \in \mathbb{N}^* - [n_0]$ be given. Then:

$|a_{n+1}/a_n - \rho| < r - \rho \Rightarrow -(r-\rho) < (a_{n+1}/a_n) - \rho < r - \rho \Rightarrow$

$\Rightarrow (a_{n+1}/a_n) - \rho < r - \rho \Rightarrow a_{n+1}/a_n < r \Rightarrow a_{n+1} < r a_n$

The last step uses the assumption $a_n > 0$. It follows that

$\forall n \in \mathbb{N}^* - [n_0]: a_{n+1} < r a_n$

▸ We claim that $\forall n \in \mathbb{N}^* - [n_0]: a_n < r^{n-n_0} a_{n_0}$ (1)

For $n = n_0 + 1$: $a_n = a_{n_0+1} < r a_{n_0} = r^{(n_0+1)-n_0} a_{n_0} = r^{n-n_0} a_{n_0}$

$\Rightarrow a_n < r^{n-n_0} a_{n_0}$

For $n = n_0 + k$ with $k \in \mathbb{N}^*$ we assume that

$$a_n < r^{n-n_0} a_{n_0} \Leftrightarrow a_{n_0+k} < r^{(n_0+k)-n_0} a_{n_0} \Leftrightarrow$$

$$\Leftrightarrow a_{n_0+k} < r^k a_{n_0}$$

For $n = n_0 + k + 1$ we show that

$$a_n = a_{n_0+k+1} < r a_{n_0+k} < r(r^k a_{n_0}) = r^{k+1} a_{n_0}$$

$$= r^{(n_0+k+1)-n_0} a_{n_0} = r^{n-n_0} a_{n_0} \Rightarrow a_n < r^{n-n_0} a_{n_0}$$

The claim is thus shown by induction.

From Eq. (1) and our assumptions we get

$$\forall n \in \mathbb{N}^* - [n_0] : 0 \leq a_n < r^{n-n_0} a_{n_0}$$

and, via the comparison test, it follows that

$$\sum_{n=n_0+1}^{+\infty} r^{n-n_0} \text{ converges} \Rightarrow \sum_{n=n_0+1}^{+\infty} r^{n-n_0} a_{n_0} \text{ converges}$$

$$\Rightarrow \sum_{n=n_0+1}^{+\infty} a_n \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges.}$$

▸ <u>Part 2</u> : Assume that $\rho > 1$. Then:

$$\lim (a_{n+1}/a_n) = \rho > 1 \Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : a_{n+1}/a_n > 1$$

Fix $n_0 \in \mathbb{N}^*$ such that $\forall n \in \mathbb{N}^* - [n_0] : a_{n+1}/a_n > 1$

Then $\forall n \in \mathbb{N}^* - [n_0] : a_{n+1} > a_n$

We claim that $\forall n \in \mathbb{N}^* - [n_0] : a_n \geq a_{n_0+1}$

For $n = n_0 + 1$: $a_n = a_{n_0+1} \Rightarrow a_n \geq a_{n_0+1}$

For $n = k > n_0 + 1$, we assume that $a_k \geq a_{n_0+1}$

For $n = k+1$: $a_n = a_{k+1} > a_k \geq a_{n_0+1} \Rightarrow a_n \geq a_{n_0+1}$.

It follows, by induction, that

$(\forall n \in \mathbb{N}^* - [n_0]: a_n \geq a_{n_0+1}) \Rightarrow$

$\Rightarrow \lim a_n \geq a_{n_0+1} \vee \lim a_n$ does not exist

$\Rightarrow \lim a_n \neq 0 \Rightarrow \sum_{n=1}^{+\infty} a_n$ diverges. $\square$

The ratio test is particularly useful in series involving the <u>factorial</u> or the <u>double factorial</u>, defined as follows:

1) <u>Factorial</u> : It is defined recursively by:

$$\boxed{\begin{cases} 0! = 1 \\ n! = n(n-1)! \ , \ \forall n \in \mathbb{N}^* \end{cases}}$$

or equivalently by

$$\boxed{n! = \prod_{a=1}^{n} a = n(n-1) \cdot \dots \cdot 3 \cdot 2 \cdot 1}$$

2) <u>Double factorial</u>: It is defined recursively by

$$\boxed{\begin{cases} 0!! = 1 \ \wedge \ 1!! = 1 \\ (2n)!! = 2n(2n-2)!! \ , \ \forall n \in \mathbb{N}^* \\ (2n+1)!! = (2n+1)(2n-1)!! \ , \ \forall n \in \mathbb{N}^* \end{cases}}$$

or equivalently

$$\boxed{\begin{aligned} (2n)!! &= \prod_{a=1}^{n} 2a = 2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n) \ , \ \forall n \in \mathbb{N}^* \\ (2n+1)!! &= \prod_{a=0}^{n} (2a+1) = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n+1) \ , \ \forall n \in \mathbb{N} \end{aligned}}$$

EXAMPLES

a) Show that $\sum_{n=1}^{+\infty} \frac{(n!)^2}{(2n)!}$ converges

Solution

We define

$$a_n = \frac{(n!)^2}{(2n)!} > 0, \ \forall n \in \mathbb{N}^* \qquad (1)$$

and note that

$$\frac{a_{n+1}}{a_n} = \frac{\left[\frac{((n+1)!)^2}{(2n+2)!}\right]}{\left[\frac{(n!)^2}{(2n)!}\right]} = \frac{(2n)!\,((n+1)!)^2}{(n!)^2\,(2n+2)!} =$$

$$= \frac{(2n)!\,[(n+1)n!]^2}{(n!)^2\,(2n+2)(2n+1)(2n)!} = \frac{(n+1)^2}{(2n+2)(2n+1)}, \ \forall n \in \mathbb{N}^*$$

Since

$$\lim_{x \to +\infty} \frac{(x+1)^2}{(2x+2)(2x+1)} = \lim_{x \to +\infty} \frac{x^2}{(2x)^2} = \lim_{x \to +\infty} \frac{x^2}{4x^2}$$

$$= \frac{1}{4} \Rightarrow$$

$$\Rightarrow \lim_{n \in \mathbb{N}^*} \frac{a_{n+1}}{a_n} = \lim_{n \in \mathbb{N}^*} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1 \qquad (2)$$

From Eq.(1) and Eq.(2):

$$\sum_{n=1}^{+\infty} \frac{(n!)^2}{(2n)!} \text{ converges}$$

b) Show that $\sum_{n=1}^{+\infty} \frac{5^n}{(2n+1)!!}$ converges.

<u>Solution</u>

We define

$$a_n = \frac{5^n}{(2n+1)!!} > 0 \ , \ \forall n \in \mathbb{N}^* \qquad (1)$$

and note that

$$\frac{a_{n+1}}{a_n} = \frac{\left[\frac{5^{n+1}}{(2n+3)!!}\right]}{\left[\frac{5^n}{(2n+1)!!}\right]} = \frac{5^{n+1}(2n+1)!!}{5^n\,(2n+3)!!} =$$

$$= \frac{5(2n+1)!!}{(2n+3)(2n+1)!!} = \frac{5}{2n+3} \ , \ \forall n \in \mathbb{N}^*$$

Since

$$\lim_{x \to +\infty} \frac{5}{2x+3} = \lim_{x \to +\infty} \frac{5}{2x} = 0 \Rightarrow$$

$$\Rightarrow \lim_{n \in \mathbb{N}^*} \frac{a_{n+1}}{a_n} = \lim_{n \in \mathbb{N}^*} \frac{5}{2n+3} = 0 < 1 \qquad (2)$$

From Eq.(1) and Eq.(2):

$$\sum_{n=1}^{+\infty} \frac{5^n}{(2n+1)!!} \quad \text{converges}$$

c) Show that $\sum_{n=1}^{+\infty} \frac{(3n+1)!}{2^n (n!)^2 (2n)!!}$ diverges.

<u>Solution</u>

We define $\forall n \in \mathbb{N}^* : a_n = \frac{(3n+1)!}{2^n (n!)^2 (2n)!!} > 0$ (1)

and note that

$$\frac{a_{n+1}}{a_n} = \frac{\left[\frac{(3n+4)!}{2^{n+1}((n+1)!)^2 (2n+2)!!}\right]}{\left[\frac{(3n+1)!}{2^n (n!)^2 (2n)!!}\right]} =$$

$$= \frac{2^n (n!)^2 (2n)!! (3n+4)!}{2^{n+1}((n+1)!)^2 (2n+2)!! (3n+1)!}$$

$$= \frac{2^n (n!)^2 (2n)!! (3n+4)(3n+3)(3n+2)(3n+1)!}{2 \cdot 2^n (n+1)^2 (n!)^2 (2n+2)(2n)!! (3n+1)!}$$

$$= \frac{(3n+4)(3n+3)(3n+2)}{2(n+1)^2 (2n+2)}, \quad \forall n \in \mathbb{N}^*$$

It follows that

$$\lim_{x \to +\infty} \frac{(3x+4)(3x+3)(3x+2)}{2(x+1)^2 (2x+2)} = \lim_{x \to +\infty} \frac{(3x)^3}{2x^2 (2x)} =$$

$$= \lim_{x \to +\infty} \frac{27x^3}{4x^3} = \frac{27}{4} \Rightarrow \lim_{n \in \mathbb{N}^*} \frac{a_{n+1}}{a_n} = \frac{27}{4} > 1 \quad (2)$$

From Eq. (1) and Eq. (2) via the ratio test:

$$\sum_{n=1}^{+\infty} \frac{(3n+1)!}{2^n (n!)^2 (2n)!!} \text{ diverges}$$

5 → Root test

Thm: Let a_n be a sequence. Then

$$\begin{cases} \forall n \in \mathbb{N}^* : a_n > 0 \\ \lim (a_n)^{1/n} = \rho < 1 \end{cases} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges}$$

$$\begin{cases} \forall n \in \mathbb{N}^* : a_n > 0 \\ \lim (a_n)^{1/n} = \rho > 1 \end{cases} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ diverges}$$

Proof

Case 1 : Assume that $\rho < 1$. Since

$\lim (a_n)^{1/n} = \rho \Rightarrow \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n^{1/n} - \rho| < \varepsilon$

Choose an $r \in [\rho, 1)$ and choose $\varepsilon = r - \rho > 0$. Fix an $n_0 \in \mathbb{N}^*$ such that

$$\forall n \in \mathbb{N}^* - [n_0] : |a_n^{1/n} - \rho| < r - \rho.$$

Let $n \in \mathbb{N}^* - [n_0]$ be given. We note that $a_n > 0$ and also

$$|a_n^{1/n} - \rho| < r - \rho \Rightarrow -(r-\rho) < a_n^{1/n} - \rho < r - \rho \Rightarrow$$
$$\Rightarrow a_n^{1/n} - \rho < r - \rho \Rightarrow a_n^{1/n} < r$$
$$\Rightarrow a_n < r^n.$$

It follows that $\forall n \in \mathbb{N}^* - [n_0] : 0 < a_n < r^n$.

and from the comparison test, we have:

$$0 < r < 1 \Rightarrow \sum_{n=n_0+1}^{+\infty} r^n \text{ converges} \Rightarrow \sum_{n=n_0+1}^{+\infty} a_n \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges.}$$

Case 2 : Assume that $\rho > 1$. Choose an $r \in [1, \rho]$. Then

$\lim (a_n)^{1/n} = \rho > r \Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : (a_n)^{1/n} > r$

Fix $n_0 \in \mathbb{N}^*$ such that $\forall n \in \mathbb{N}^* - [n_0] : (a_n)^{1/n} > r$

Then $\forall n \in \mathbb{N}^* - [n_0] : a_n > r^n > 0$

and from the comparison test, it follows that

$r > 1 \Rightarrow \sum_{n=n_0+1}^{+\infty} r^n$ diverges $\Rightarrow \sum_{n=n_0+1}^{+\infty} a_n$ diverges

$\Rightarrow \sum_{n=1}^{+\infty} a_n$ diverges. $\square$

EXAMPLES

a) Show that $\sum_{n=1}^{+\infty} \left(\frac{3n^2+2n+1}{n^2+5n+2}\right)^n$ diverges.

Solution

Define $\forall n \in \mathbb{N}^*: a_n = \left(\frac{3n^2+2n+1}{n^2+5n+2}\right)^n$

and we note that $\forall n \in \mathbb{N}^*: a_n > 0$ (1)

and

$$\forall n \in \mathbb{N}^*: a_n^{1/n} = \left[\left(\frac{3n^2+2n+1}{n^2+5n+2}\right)^n\right]^{1/n} = \frac{3n^2+2n+1}{n^2+5n+2}$$

Since:

$$\lim_{x \to +\infty} \frac{3x^2+2x+1}{x^2+5x+2} = \lim_{x \to +\infty} \frac{3x^2}{x^2} = 3 \Rightarrow$$

$$\Rightarrow \lim_{n \in \mathbb{N}^*} a_n^{1/n} = \lim_{n \in \mathbb{N}^*} \frac{3n^2+2n+1}{n^2+5n+2} = 3 > 1 \qquad (2)$$

From Eq.(1) and Eq.(2), via the root test, it follows that

$$\sum_{n=1}^{+\infty} \left(\frac{3n^2+2n+1}{n^2+5n+2}\right)^n \text{ diverges.}$$

b) Show that $\sum_{n=2}^{+\infty} \left(\frac{1}{\ln n}\right)^n$ converges

Solution

Define $\forall n \in \mathbb{N} - \{0,1\}: \; a_n = \left(\frac{1}{\ln n}\right)^n$

and we note that $\forall n \in \mathbb{N} - \{0,1\}: \; a_n > 0$ (1)

and

$$\forall n \in \mathbb{N} - \{0,1\}: \; a_n^{1/n} = \left[\left(\frac{1}{\ln n}\right)^n\right]^{1/n} = \frac{1}{\ln n}$$

Since

$$\lim_{x \to +\infty} \ln x = +\infty \Rightarrow \lim_{x \to +\infty} \frac{1}{\ln x} = 0 \Rightarrow$$

$$\Rightarrow \lim_{n \in \mathbb{N} - \{0,1\}} a_n^{1/n} = \lim_{n \in \mathbb{N} - \{0,1\}} \frac{1}{\ln n} = 0 \quad (2)$$

From Eq. (1) and Eq. (2): $\sum_{n=2}^{+\infty} \left(\frac{1}{\ln n}\right)^n$ converges.

c) Show that $\sum_{n=0}^{+\infty} n^3 (1/2)^n$ converges

<u>Solution</u>

Define $\forall n \in \mathbb{N}^*: \; a_n = n^3 (1/2)^n > 0$. It follows that

$$\lim_{n \in \mathbb{N}^*} \sqrt[n]{a_n} = \lim_{n \in \mathbb{N}^*} \sqrt[n]{n^3 (1/2)^n} = \lim_{n \in \mathbb{N}^*} \left[(1/2)\sqrt[n]{n^3}\right]$$

$$= (1/2) \lim_{n \in \mathbb{N}^*} (\sqrt[n]{n})^3 = (1/2) \cdot 1 < 1 \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{+\infty} n^3 (1/2)^n \text{ converges} \Rightarrow \sum_{n=0}^{+\infty} n^3 (1/2)^n \text{ converges.}$$

Other convergence tests

- Let a_n be a sequence. An alternating series, is a series of the form

$$S_n = \sum_{k=1}^{+\infty} (-1)^k a_k$$

with $\forall k \in \mathbb{N}^*: a_k \geq 0$

① → Leibnitz Rule (Alternating series test)

Thm: Let a_n be a sequence. Then

$$\left.\begin{array}{l} a_n \text{ decreasing} \\ \forall n \in \mathbb{N}^*: a_n \geq 0 \\ \lim a_n = 0 \end{array}\right\} \Rightarrow \sum_{n=1}^{+\infty} (-1)^n a_n \text{ converges}$$

Proof

We define $\forall n \in \mathbb{N}^*: S_n = \sum_{k=1}^{n} (-1)^k a_k$

▸ We claim that S_{2n} is decreasing and lower-bounded.

Let $n \in \mathbb{N}^*$ be given. We note that

$$S_{2n+2} = \sum_{k=1}^{2n+2} (-1)^k a_k = \sum_{k=1}^{2n} (-1)^k a_k + (-1)^{2n+1} a_{2n+1} + (-1)^{2n+2} a_{2n+2}$$

$$= S_{2n} + (a_{2n+2} - a_{2n+1}) \leq S_{2n}$$

because a_n decreasing $\Rightarrow a_{2n+2} \leq a_{2n+1} \Rightarrow$

$$\Rightarrow a_{2n+2} - a_{2n+1} \leq 0$$

We also note that

$$s_{2n} = \sum_{k=1}^{2n} (-1)^k a_k =$$

$$= (-1)^1 a_1 + \sum_{k=2}^{2n-1} (-1)^k a_k + (-1)^{2n} a_{2n} =$$

$$= -a_1 + \sum_{k=1}^{n-1} \left((-1)^{2k} a_{2k} + (-1)^{2k+1} a_{2k+1} \right) + a_{2n}$$

$$= -a_1 + \sum_{k=1}^{n-1} (a_{2k} - a_{2k+1}) + a_{2n}$$

Since $(\forall k \in \mathbb{N}^* : a_k \geq 0) \Rightarrow a_{2n} \geq 0$

and since

a_n decreasing $\Rightarrow (\forall k \in \mathbb{N}^* : a_{2k} \geq a_{2k+1})$

$\Rightarrow (\forall k \in \mathbb{N}^* : a_{2k} - a_{2k+1} \geq 0)$

$\Rightarrow \sum_{k=1}^{n-1} (a_{2k} - a_{2k+1}) \geq 0$

and it follows that $s_{2n} \geq -a_1$

From the above it follows that

$$\begin{cases} \forall n \in \mathbb{N}^* : s_{2n+2} \leq s_{2n} \\ \forall n \in \mathbb{N}^* : s_{2n} \geq -a_1 \end{cases} \Rightarrow \begin{cases} s_{2n} \text{ decreasing} \\ s_{2n} \text{ lower-bounded} \end{cases} \Rightarrow$$

$\Rightarrow s_{2n}$ converges $\Rightarrow \exists l \in \mathbb{R} : \lim_{n \in \mathbb{N}^*} s_{2n} = l.$ (1)

We also note that

$$\lim s_{2n+1} = \lim (s_{2n} + (-1)^{2n+1} a_{2n+1}) = \lim (s_{2n} - a_{2n+1})$$

$$= \lim s_{2n} - \lim a_{2n+1} = l - 0 = l \qquad (2)$$

From Eq. (1) and Eq. (2):

$\lim s_n = l \Rightarrow \sum_{n=1}^{+\infty} (-1)^n a_n$ converges. $\square$

Note that in order to apply the alternating series test, it is important to show that the sequence a_n is decreasing. Oftentimes, this can be established via Calculus 1 methods.

EXAMPLES

a) Show that the sequence $\sum_{n=1}^{+\infty} (-1)^n \frac{n^2}{n^3+1}$ converges

Solution

We define:

$$\left(\forall n \in \mathbb{N}^* : a_n = \frac{n^2}{n^3+1}\right) \wedge \left(\forall x \in [1,+\infty) : f(x) = \frac{x^2}{x^3+1}\right)$$

and we note that:

$$\forall n \in \mathbb{N}^* : a_n \geqslant 0 \qquad (1)$$

and since

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2}{x^3+1} = \lim_{x\to+\infty} \frac{x^2}{x^3} = \lim_{x\to+\infty} \frac{1}{x} = 0$$

$$\Rightarrow \lim a_n = 0. \qquad (2)$$

We also note that:

$$\forall x \in [1,+\infty) : f'(x) = \frac{(x^2)'(x^3+1) - x^2(x^3+1)'}{(x^3+1)^2} =$$

$$= \frac{2x(x^3+1) - x^2(3x^2)}{(x^3+1)^2} = \frac{x\left[2(x^3+1) - 3x^3\right]}{(x^3+1)^2}$$

$$= \frac{x(2x^3+2-3x^3)}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}$$

and since:

$\forall x \in [2,+\infty) : (x>0 \wedge 2-x^3<0 \wedge (x^3+1)^2>0)$

$\Rightarrow (\forall x \in [2,+\infty) : f'(x)<0) \Rightarrow f$ strictly decreasing on $[2,+\infty)$

$\Rightarrow (\forall n \in \mathbb{N}^* - \{1\} : a_{n+1} < a_n) \Rightarrow$

$\Rightarrow a_n$ decreasing on $\mathbb{N}^* - \{1\}$ (3)

From Eq.(1) and Eq.(2) and Eq.(3); via the alternating series test, we have:

$$\sum_{n=2}^{+\infty} (-1)^n a_n \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} (-1)^n \frac{n^2}{n^3+1} \text{ converges.}$$

b) Show that $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ converges

<u>Solution</u>

We define $\forall n \in \mathbb{N}^* : a_n = \frac{1}{n}$

and note that

$\forall n \in \mathbb{N}^* : a_n > 0$ (1)

Claim: We will show that a_n decreasing.

Let $n \in \mathbb{N}^*$ be given. Then:

$$a_{n+1} - a_n = \frac{1}{n+1} - \frac{1}{n} = \frac{n-(n+1)}{n(n+1)} = \frac{n-n-1}{n(n+1)}$$

$$= \frac{-1}{n(n+1)} < 0 \Rightarrow a_{n+1} < a_n$$

It follows that

$$(\forall n \in \mathbb{N}^* : a_{n+1} < a_n) \Rightarrow a_n \text{ decreasing} \qquad (2)$$

which proves the claim.

Finally we have

$$\lim_{x \to +\infty} \frac{1}{x} = 0 \Rightarrow \lim a_n = \lim \frac{1}{n} = 0 \qquad (3)$$

From Eq.(1), Eq.(2), Eq.(3), via the alternating series test, it follows that

$$\sum_{n=1}^{+\infty} (-1)^n \frac{1}{n} \text{ converges.}$$

c) Show that $\displaystyle\sum_{n=2}^{+\infty} (-1)^n \frac{n^2+1}{n^2-1}$ diverges.

<u>Solution</u>

We define:

$$\forall n \in \mathbb{N}^* \setminus \{1\} : a_n = \frac{n^2+1}{n^2-1} (-1)^n$$

and therefore

$$\forall n \in \mathbb{N}^* : a_{2n} = (-1)^{2n} \frac{(2n)^2+1}{(2n)^2-1} = \frac{4n^2+1}{4n^2-1}$$

$$\forall n \in \mathbb{N}^* : a_{2n+1} = (-1)^{2n+1} \frac{(2n+1)^2+1}{(2n+1)^2-1} = \frac{-(4n^2+4n+1+1)}{4n^2+4n+1-1}$$

$$= \frac{-(4n^2+4n+2)}{4n^2+4n}$$

Since $\displaystyle\lim_{x \to +\infty} \frac{4x^2+1}{4x^2-1} = \lim_{x \to +\infty} \frac{4x^2}{4x^2} = 1 \Rightarrow \lim a_{2n} = 1 \qquad (1)$

and

$$\lim_{x\to+\infty} \frac{-(4x^2+4x+2)}{4x^2+4x} = \lim_{x\to+\infty} \frac{-4x^2}{4x^2} = -1 \Rightarrow \lim a_{2n+1} = -1 \quad (2)$$

From Eq.(1) and Eq.(2):

a_n diverges $\Rightarrow \lim a_n \neq 0 \Rightarrow \sum_{n=2}^{+\infty} (-1)^n \frac{n^2+1}{n^2-1}$ diverges. $\square$

$\hookrightarrow$ One method for showing that an alternating series $\sum_{n=1}^{+\infty} (-1)^n a_n$ diverges is to show that $\lim (-1)^n a_n \neq 0$.

2 → Absolute convergence test

Thm: Let a_n be a sequence. Then:

$$\sum_{n=1}^{+\infty} |a_n| \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges.}$$

Proof

Let $n \in \mathbb{N}^*$ be given. Then:

$$-|a_n| \leq a_n \leq |a_n| \Rightarrow \begin{cases} -|a_n| \leq a_n \\ a_n \leq |a_n| \end{cases} \Rightarrow \begin{cases} a_n + |a_n| \geq 0 \\ a_n + |a_n| \leq 2|a_n| \end{cases}$$

$$\Rightarrow 0 \leq a_n + |a_n| \leq 2|a_n|$$

and it follows that

$\forall n \in \mathbb{N}^* : 0 \leq a_n + |a_n| \leq 2|a_n|$

From the comparison test, it follows that

$$\sum_{n=1}^{+\infty} |a_n| \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} 2|a_n| \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} (a_n + |a_n|) \text{ converges}$$

and since:

$$\begin{cases} \sum_{n=1}^{+\infty} (a_n + |a_n|) \text{ converges} \\ \sum_{n=1}^{+\infty} |a_n| \text{ converges} \end{cases} \Rightarrow \sum_{n=1}^{+\infty} \left[(a_n + |a_n|) - |a_n| \right] \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges} \qquad \square$$

→ Application to alternating series: If we show that $\sum_{n=1}^{+\infty} a_n$ converges, the absolute convergence test can be used to show $\sum_{n=1}^{+\infty} (-1)^n a_n$ converges.

EXAMPLES

a) Show that $\sum_{n=1}^{+\infty} (-1)^n \frac{n^2+3n+1}{2n^4+5n-1}$ converges.

Solution

We define $\forall n \in \mathbb{N}^*: \left(a_n = \frac{n^2+3n+1}{2n^4+5n-1} \wedge b_n = \frac{1}{n^2} \right)$

and note that

$$\forall n \in \mathbb{N}^*: \frac{a_n}{b_n} = \frac{\left(\frac{n^2+3n+1}{2n^4+5n-1}\right)}{\frac{1}{n^2}} = \frac{n^2(n^2+3n+1)}{2n^4+5n-1} =$$

$$= \frac{n^4+3n^3+n^2}{2n^4+5n-1}$$

Since $\lim_{x \to +\infty} \frac{x^4+3x^3+x^2}{2x^4+5x-1} = \lim_{x \to +\infty} \frac{x^4}{2x^4} = \frac{1}{2} \Rightarrow \lim_{n \in \mathbb{N}^*} \frac{a_n}{b_n} = \frac{1}{2}$

it follows from the limit comparison test that:

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} \frac{n^2+3n+1}{2n^4+5n-1} \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} \left| (-1)^n \frac{n^2+3n+1}{2n^4+5n-1} \right| \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} (-1)^n \frac{n^2+3n+1}{2n^4+5n-1} \text{ converges.}$$

B) Show that $\sum_{n=1}^{+\infty} (-1)^n \frac{n^3}{3^n}$ converges.

<u>Solution</u>

Define $\forall n \in \mathbb{N}^*: a_n = \frac{n^3}{3^n}$

and note that

$$\forall n \in \mathbb{N}^*: \frac{a_{n+1}}{a_n} = \frac{\left(\frac{(n+1)^3}{3^{n+1}}\right)}{\left(\frac{n^3}{3^n}\right)} = \frac{3^n (n+1)^3}{3^{n+1} n^3} = \frac{3^n (n+1)^3}{3^n 3 n^3} =$$

$$= \frac{n^3+3n^2+3n+1}{3n^3}$$

Since

$$\lim_{x \to +\infty} \frac{x^3+3x^2+3x+1}{3x^3} = \lim_{x \to +\infty} \frac{x^3}{3x^3} = \frac{1}{3} \Rightarrow$$

$$\Rightarrow \lim \frac{a_{n+1}}{a_n} = \frac{1}{3} \Rightarrow \sum_{n=1}^{+\infty} \frac{n^3}{3^n} \text{ converges} \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{+\infty} \left| (-1)^n \frac{n^3}{3^n} \right| \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} (-1)^n \frac{n^3}{3^n} \text{ converges.} \quad \square$$

The absolute convergence test can be combined with the comparison test to show that series of the form $\sum_{n=1}^{+\infty} a_n b_n$, with a_n bounded and $\sum_{n=1}^{+\infty} |b_n|$ convergent, also converge. Such arguments use the properties of absolute values:

$\forall a,b \in \mathbb{R}: |a+b| \leq |a|+|b|$ $\quad$ $\forall a,b \in \mathbb{R}: |ab| = |a||b|$

$\forall a,b \in \mathbb{R}: |a-b| \leq |a|+|b|$ $\quad$ $\forall a,b \in \mathbb{R}: \left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

and the following inequalities:

$\forall x \in \mathbb{R}: |\sin x| \leq 1$

$\forall x \in \mathbb{R}: |\cos x| \leq 1$

$\forall x \in [-1,1]: |\operatorname{Arcsin}(x)| \leq \pi/2$

$\forall x \in \mathbb{R}: |\operatorname{Arctan} x| < \pi/2$

c) Show that $\displaystyle\sum_{n=1}^{+\infty} \frac{n[\sin(n!)+\cos(n!)]}{n^3+1}$ converges.

<u>Solution</u>

We define $\forall n \in \mathbb{N}^*: a_n = \dfrac{n[\sin(n!)+\cos(n!)]}{n^3+1}$

Let $n \in \mathbb{N}^*$ be given and since

$$|a_n| = \left|\frac{n[\sin(n!)+\cos(n!)]}{n^3+1}\right| = \frac{|n[\sin(n!)+\cos(n!)]|}{|n^3+1|}$$

$$= \frac{n}{n^3+1} \cdot |\sin(n!)+\cos(n!)|$$

$$\leq \frac{n(|\sin(n!)|+|\cos(n!)|)}{n^3+1} \leq \frac{n(1+1)}{n^3+1} =$$

$$= \frac{2n}{n^3+1} \leq \frac{2n}{n^3} = \frac{2}{n^2}$$

it follows that $\forall n \in \mathbb{N}^*: 0 \leq |a_n| \leq \dfrac{2}{n^2}$

and therefore, via the comparison test and the absolute convergence test:

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} \frac{2}{n^2} \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} |a_n| \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges.} \qquad \square$$

d) Show that $\displaystyle\sum_{n=1}^{+\infty} \frac{n^3 \sin(\pi n/4) - n^2 \cos(\pi n/3)}{5^n}$ converges

<u>Solution</u>

We define $\forall n \in \mathbb{N}^*$: $a_n = \dfrac{n^3 \sin(\pi n/4) - n^2 \cos(\pi n/3)}{5^n}$

Let $n \in \mathbb{N}^*$ be given. Then

$$|a_n| = \left| \frac{n^3 \sin(\pi n/4) - n^2 \cos(\pi n/3)}{5^n} \right| =$$

$$= \frac{|n^3 \sin(\pi n/4) - n^2 \cos(\pi n/3)|}{5^n}$$

$$\leq \frac{|n^3 \sin(\pi n/4)| + |n^2 \cos(\pi n/3)|}{5^n} =$$

$$= \frac{n^3 |\sin(\pi n/4)| + n^2 |\cos(\pi n/3)|}{5^n}$$

$$\leq \frac{n^3 + n^2}{5^n} \leq \frac{n^3 + n^3}{5^n} = \frac{2n^3}{5^n}$$

and it follows that

$$\forall n \in \mathbb{N}^* : 0 \leq |a_n| \leq \frac{2n^3}{5^n} \qquad (1)$$

Define $\forall n \in \mathbb{N}^* : b_n = \dfrac{2n^3}{5^n}$

and note that

$$\forall n \in \mathbb{N}^* : \frac{b_{n+1}}{b_n} = \frac{\left(\dfrac{2(n+1)^3}{5^{n+1}}\right)}{\left(\dfrac{2n^3}{5^n}\right)} = \frac{2 \cdot 5^n (n+1)^3}{5^{n+1}(2n^3)} =$$

$$= \frac{(n+1)^3}{5n^3}$$

Since:

$$\lim_{x \to +\infty} \frac{(x+1)^3}{5x^3} = \lim_{x \to +\infty} \frac{x^3+3x^2+3x+1}{5x^3} =$$

$$= \lim_{x \to +\infty} \frac{x^3}{5x^3} = \frac{1}{5} \Rightarrow$$

$$\Rightarrow \lim \frac{b_{n+1}}{b_n} = \frac{1}{5} < 1 \Rightarrow \sum_{n=1}^{+\infty} \frac{2n^3}{5^n} \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{+\infty} |a_n| \text{ converges} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ converges.}$$

Conditional convergence

Note that the contrapositive of the absolute convergence test reads:

$$\sum_{n=1}^{+\infty} a_n \text{ diverges} \Rightarrow \sum_{n=1}^{+\infty} |a_n| \text{ diverges}$$

However the converse of this statement, which is given by

$$\sum_{n=1}^{+\infty} |a_n| \text{ diverges} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ diverges}$$

is NOT always true and cannot be used in the solution of problems. Nevertheless, the <u>absolute ratio test</u> and <u>absolute root test</u> allow the formulation of similar correct arguments. We also note that the above observation motivates the following definition of conditional convergence:

Def: Let a_n be a sequence. Then:

$$\sum_{n=1}^{+\infty} a_n \text{ conditionally converges} \Leftrightarrow \begin{cases} \sum_{n=1}^{+\infty} a_n \text{ converges} \\ \sum_{n=1}^{+\infty} |a_n| \text{ diverges.} \end{cases}$$

An example of a conditionally convergent series is:

$$S_n = \sum_{k=1}^{n} (-1)^k \frac{1}{k}$$

The proof of the absolute ratio test and the absolute root test is based on the following lemma:

<u>Lemma</u> : Let a_n be a sequence. Then:

$$(\forall n \in \mathbb{N}^* - [n_0] : |a_n| \geq b > 0) \Rightarrow \lim a_n \neq 0$$

<u>Proof</u>

It is sufficient to show that

$\exists \varepsilon > 0 : \forall n_1 \in \mathbb{N}^* : \exists n \in \mathbb{N}^* - [n_1] : |a_n| \geq \varepsilon.$

We choose $\varepsilon = b > 0$. Let $n_1 \in \mathbb{N}^*$ be given and choose $n = \max\{n_1 + 1, n_0 + 1\}$. Then:

$$n = \max\{n_1 + 1, n_0 + 1\} \Rightarrow n \geq n_0 + 1 \Rightarrow n \in \mathbb{N}^* - [n_0]$$
$$\Rightarrow |a_n| \geq b = \varepsilon \Rightarrow |a_n| \geq \varepsilon$$

It follows that

$(\exists \varepsilon > 0 : \forall n_1 \in \mathbb{N}^* : \exists n \in \mathbb{N}^* - [n_1] : |a_n| \geq \varepsilon) \Rightarrow \lim a_n \neq 0.$

3 → <u>Absolute ratio test</u>

<u>Thm</u> : Let a_n be a sequence. Then:

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r > 1 \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ diverges}$$

<u>Proof</u>

Since:

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r > 1 \Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : \left| \frac{a_{n+1}}{a_n} \right| > 1.$$

Fix $n_0 \in \mathbb{N}^*$ such that $\forall n \in \mathbb{N}^* - [n_0] : |a_{n+1}/a_n| > 1.$

Let $n \in \mathbb{N}^* - [n_0]$ be given. Then:

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow \frac{|a_{n+1}|}{|a_n|} > 1 \Rightarrow |a_{n+1}| > |a_n| > 0$$

It follows that:

$$(\forall n \in \mathbb{N}^* - [n_0] : |a_{n+1}| > |a_n|) \Rightarrow$$
$$\Rightarrow (\forall n \in \mathbb{N}^* - [n_0] : |a_n| \geq |a_{n_0+1}| > 0)$$
$$\Rightarrow \lim a_n \neq 0 \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ diverges.} \quad \square$$

4 → <u>Absolute root test</u>

<u>Thm</u> : Let a_n be a sequence. Then:

$$\lim |a_n|^{1/n} > r > 1 \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ diverges}$$

<u>Proof</u>

Since $\lim |a_n|^{1/n} > r > 1 \Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n|^{1/n} > r > 1$

Fix $n_0 \in \mathbb{N}^*$ such that $\forall n \in \mathbb{N}^* - [n_0] : |a_n|^{1/n} > r > 1$

and let $n \in \mathbb{N}^* - [n_0]$ be given. Then:

$$|a_n|^{1/n} > r \Rightarrow |a_n| = (|a_n|^{1/n})^n > r^n > r > 0$$

It follows that

$$(\forall n \in \mathbb{N}^* - [n_0] : |a_n| > r > 0) \Rightarrow \lim a_n \neq 0 \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ diverges.} \quad \square$$

EXAMPLES

a) Show that $\sum_{n=1}^{+\infty} (-1)^n \left[\frac{3n^2+n}{n^2+2}\right]^n$ diverges

Solution

We define $\forall n \in \mathbb{N}^*: a_n = (-1)^n \left[\frac{3n^2+n}{n^2+2}\right]^n$.

Then:

$$\forall n \in \mathbb{N}^*: |a_n|^{1/n} = \left| (-1)^n \left(\frac{3n^2+n}{n^2+2}\right)^n \right|^{1/n} =$$

$$= \left[\left(\frac{3n^2+n}{n^2+2}\right)^n\right]^{1/n} = \frac{3n^2+n}{n^2+2} \qquad (1)$$

and we note that:

$$\lim_{x \to +\infty} \frac{3x^2+x}{x^2+2} = \lim_{x \to +\infty} \frac{3x^2}{x^2} = 3 \Rightarrow$$

$$\Rightarrow \lim |a_n|^{1/n} = 3 \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ diverges.}$$

b) Show that $\sum_{n=1}^{+\infty} (-1)^n \frac{(n+1)^n}{n!}$ diverges.

Solution

We define $\forall n \in \mathbb{N}^*: a_n = (-1)^n \frac{(n+1)^n}{n!}$

and we note that

$$\forall n \in \mathbb{N}^*: \left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{\frac{(-1)^{n+1}(n+2)^{n+1}}{(n+1)!}}{\frac{(-1)^n (n+1)^n}{n!}} \right| = \frac{\frac{(n+2)^{n+1}}{(n+1)!}}{\frac{(n+1)^n}{n!}} =$$

$$= \frac{n!\,(n+2)^{n+1}}{(n+1)!\,(n+1)^n} = \frac{n!\,(n+2)^{n+1}}{(n+1)n!\,(n+1)^n} = \frac{(n+2)^{n+1}}{(n+1)^{n+1}} =$$

$$= \left(\frac{n+2}{n+1}\right)^{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

It follows that

$$\lim\left|\frac{a_{n+1}}{a_n}\right| = \lim\left(1+\frac{1}{n+1}\right)^{n+1} = \lim\left(1+\frac{1}{n}\right)^{n} = e > 1$$

$$\Rightarrow \sum_{n=1}^{+\infty} (-1)^n \frac{(n+1)^n}{n!} \text{ diverges.}$$