10/28/2019 # Lecture 18: Eigenvalue/vector Wrap Up

Take Home exam will be given Weds
will cover up to eigenvalues/eigenvec

Theoretic Project ☆ Cayley-Hamilton theorem

☆ Cayley-Hamilton theorem

Theorem Let $A \in M_n(\mathbb{R})$ with
$$\det(A - \lambda I) = (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1 \lambda + c_0$$
Then:
$$(-1)^n A^n + c_{n-1} A^{n-1} + \dots + c_1 A + c_0 I = 0$$

instead of A^0, use I instead

Example a) $A = \begin{bmatrix} 5 & 4 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix}$ Find A^{-1} using Cayley-Hamilton theorem

Solution $\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 4 & 0 \\ 1 & 2-\lambda & 0 \\ 1 & 2 & 2-\lambda \end{vmatrix} =$

$= (+1)(2-\lambda)\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\left[(5-\lambda)(2-\lambda) - (4)(1)\right] =$

$= (2-\lambda)\left[10 - 5\lambda - 2\lambda + \lambda^2 - 4\right] = (2-\lambda)\left[\lambda^2 - 7\lambda + 6\right] =$

$= \underline{2\lambda^2} - \underline{14\lambda} + 12 - \lambda^3 + \underline{7\lambda^2} - \underline{6\lambda} = -\lambda^3 + (2+7)\lambda^2 + (-14-6)\lambda +$

$= -\lambda^3 + 9\lambda^2 - 20\lambda + 12 = \det(A - \lambda I)$

10/28/2019 Cayley-Hamilton theorem.

Solution a) $A = \begin{bmatrix} 5 & 4 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix}$ Find A^{-1} using Cayley-Hamilton theorem

we know $\det(A-\lambda I) = -\lambda^3+9\lambda^2-20\lambda+12 \Rightarrow$

$\Rightarrow -A^3+9A^2-20A+12I = \mathbf{0} \Rightarrow$

$\Rightarrow -A^3+9A^2-20A = -12I \Rightarrow A(-A^2+9A-20I) = -12I$

$\Rightarrow A\left[\frac{-1}{12}(-A^2+9A-20I)\right] = I \Rightarrow$

$\Rightarrow A^{-1} = \frac{-1}{12}(-A^2+9A-20I) \Rightarrow \boxed{A^{-1} = \frac{1}{12}(A^2-9A+20I)}$

Example $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ show that $\forall n \in \mathbb{N} - \{0,1\}: A^n = nA-(n-1)I$

<inductive>
Solution for $n=2$ $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2-(0)\cdot(1) =$

$= (1-\lambda)^2 = \lambda^2-\lambda-\lambda+1 \Rightarrow \lambda^2-2\lambda+1 \Rightarrow$

$\Rightarrow A^2-2A+I = \mathbf{0} \Rightarrow A^2 = 2A-I$ - Base step Holds

Assume for $n=K$: $A^K = KA-(K-1)I$ - Hypothesis $p \Rightarrow q$
Prove for $n=K+1$: $A^{K+1} = (K+1)A-((K+1)-1)I$ - conclusion

if q is true must assume p is true

10/28/2019 Example continued, Inductive proof

Example $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ show that $\forall n \in \mathbb{N} - \{0,1\} : A^n = nA - (n-1)I$

Assume $A^K = KA - (K-1)I$ Prove $A^{K+1} = (K+1)A - ((K+1)-1)$

Note that $A^{K+1} = A^K A$, then using inductive hypothesis

$A^K = KA - (K-1)I$, then $A^K A = [KA - (K-1)I]A \Rightarrow$

$\Rightarrow KA^2 - (K-1)A \Rightarrow$ earlier we showed $A^2 = 2A - I$, so $\rightarrow$

$\Rightarrow K(2A - I) - (K-1)A \Rightarrow 2KA - KI - KA + A \Rightarrow (2K - K + I)A -$

$\Rightarrow (K+I)A - [(K+1)-1]I \Rightarrow$ we match our conclusion

note $K+1-1 = K$

It follows by induction that $\forall n \in \mathbb{N} - \{0,1\}$: $A^n = nA - (n-1)$

Homework: 16, 17

Note { If zero eigenvalue, det = 0, then no inverse }

det tells you if matrix has/doesn't have inve

Use of theorem is easiest way to prove

Exam 2 ↑ covers everything up to this point

10/30/2019 Lecture 19 - Vector Spaces

$\boxed{\text{Exam given - due Monday}}$

Cramer Rule, eigenvalues
Cayley-Hamilton theorem, theoretical

▶ Definition - (Vector Spaces Operations) ← Recall set of Cartesian product

Let A, B, C be sets such that $A \times B \neq \emptyset$ and $C \neq \emptyset$: an operation is any mapping $f: A \times B \to C$ such that every element $(a,b) \in A \times B$ is mapped to a unique element $afb \in C$ [afb = a operate b]

▶ notation: operations are denoted with symbols (not letters) such as $+, \cdot, *, \times, \circ, \odot, \ldots$

example $*: (a,b) \in A \times B \to a*b \in C$

▶ proposition: Let $*$ be an operation $*: A \times B \to C$
Then:

$$\forall a, b \in A : \forall c \in B : (a = b \Rightarrow a*c = b*c)$$

and

$$\forall a, b \in B : \forall c \in A : (a = b \Rightarrow c*a = c*b)$$

▶ Definition: an operation $*: A \times A \to A$ is an internal operation

▶ Definition (closure): Let $*$ be an internal operation on A
Let $A_1 \subseteq A$
Then

$$* \text{ closed on } A_1 \Leftrightarrow \forall a, b \in A_1 : a*b \in A$$

10/30/2019 **Properties of (Vector Space) Operations**

Properties of Operations

Definition: Let $*$ be an internal operation on A

$*$ commutative $\Leftrightarrow \forall a, b \in A: a*b = b*a$

$*$ associative $\Leftrightarrow \forall a, b, c \in A: (a*b)*c = a*(b*c)$

note all operations in class so far have been associative but operations can be made to be not associative

$e \in A$ unit element of $* \Leftrightarrow \forall a \in A: e*a = a*e = a$

Definition: Let $*$ be an internal operation and let $e \in A$ unit element of $*$
Let $a, a' \in A$. Then

a, a' symmetric elements with respect to $*$ and $e \in A$ $\Leftrightarrow a*a' = a'*a = e$

These definitions are descriptors, not requirements
How many $e \in A$ unit element of $*$ can exist? one

Theorem Let $*$ be internal operation on A

(a) e_1, e_2 unit elements of $(A, *) \Rightarrow e_1 = e_2$

(b) $$\left.\begin{array}{l} a, a_1 \text{ symmetric on } (A,*) \\ a, a_2 \text{ symmetric on } (A,*) \\ (A,*) \text{ has unit element} \\ * \text{ associative} \end{array}\right\} \Rightarrow a_1 = a_2$$

10|30|2019 <u>Theorem Proofs</u>

(b) $\left.\begin{array}{l} a,\ a_1 \text{ symmetric on } (A,*) \\ a,\ a_2 \text{ symmetric on } (A,*) \\ (A,*) \text{ has unit element} \\ * \text{ associative} \end{array}\right\} \Rightarrow a_1 = a_2$

(b) <u>Proof</u> Assume that $*$ associative, and a, a_1 symmetric, and a, a_2 symmetric, and let $e \in A$ be unit element of $*$

Then
$$\begin{aligned} a_1 &= a_1 * e && [e \text{ unit element}] \\ &= a_1 * (a * a_2) && [a_1 \text{ and } a_2 \text{ symmetric}] \\ &= (a_1 * a) * a_2 && [\text{associative}] \\ &= e * a_2 && [a_1 \text{ and } a_1 \text{ symmetric}] \\ &= a_2 && [e \text{ unit element}] \end{aligned}$$

We conclude the claim is correct given assumptions

(a) e_1, e_2 unit elements of $(A,*)$ $\Rightarrow e_1 = e_2$

(a) <u>Proof</u> Assume that e_1, e_2 unit elements of $(A,*)$

Then:
$$\begin{aligned} e_1 * e_2 &= e_2 && [e_1 \text{ unit element}] \\ e_1 * e_2 &= e_1 && [e_2 \text{ unit element}] \end{aligned}$$

Therefore $e_1 = e_2$

10/30/2019 Vector Spaces

Example ⓐ Let $A=\mathbb{R}-\{\lambda\}$ with $\lambda\in\mathbb{R}$

define: $\forall x,y\in A: x*y = xy-\lambda(x+y)+\lambda(\lambda+1)$

Show that ① $*$ closed on A ② $*$ commutative

③ $*$ has a unit element on A

Solution

① Let $x,y\in A$ be given — contradiction what if $x*y=\lambda$?

Then $x,y\in\mathbb{R}-\{\lambda\} \Rightarrow x\neq\lambda \wedge y\neq\lambda$

To show a contradiction, assume that $x*y\notin A$ (1)

Then $x*y\notin\mathbb{R}-\{\lambda\} \Rightarrow x*y=\lambda \Rightarrow \underline{x*y-\lambda=0}$ substitute definition

Since $x*y-\lambda = [xy-\lambda(x+y)+\lambda(\lambda+1)]-\lambda$

$= xy-\lambda x-\lambda y+\lambda^2+\lambda-\lambda$

$= xy-\lambda x-\lambda y+\lambda^2$

$= x(y-\lambda)-\lambda(y-\lambda) = (x-\lambda)(y-\lambda)$ (2)

From Eq(1) and Eq(2) $(x-\lambda)(y-\lambda)=0 \Rightarrow x-\lambda=0 \vee y-\lambda=0$

$\Rightarrow x=\lambda \vee y=\lambda$ ← can't be true - contradiction

because $x\neq\lambda\wedge y\neq\lambda$ - we conclude that $\underline{x*y\in A}$

therefore $\forall x,y\in A: x*y\in A \Rightarrow *$ closed on A

② Let $x,y\in A$ be given, then

$x*y = xy-\lambda(x+y)+\lambda(\lambda+1) = yx-\lambda(y+x)+\lambda(\lambda+1) = y*x$

Therefore $\forall x,y\in A: x*y=y*x \Rightarrow *$ commutative

$x*e$ or $e*x$

③ with commutative property only need to show one case,

Let $x\in A$ be given. Solve with respect to e:

$x*e=x \Leftrightarrow x*e-x=0 \Leftrightarrow xe-\lambda(x+e)+\lambda(\lambda+1)-x=0 \Leftrightarrow$

$\Leftrightarrow \underline{xe}-\lambda x-\underline{\lambda e}+\lambda^2+\lambda-x \Leftrightarrow (x-\lambda)e = \lambda x-\lambda^2-\lambda+x$

Can also do factorization by grouping <u>TO BE continued</u>

Homework: 1,2,3