9/9/2019 Lecture 4 - Sets wrap-up

Cartesian product

- an ordered pair (a,b) is an ordered collection of two elements, with a as 1st element and b as 2nd element

Definition

$$(a,b) = (c,d) \Leftrightarrow a = c \wedge b = d$$

definition can be rebuilt depending on what elements are

compare with $\{a,b\} = \{c,d\} \Leftrightarrow (a=c \wedge b=d) \vee (a=d \wedge b=c)$

$(\quad)$ = order matters $\{\quad\}$ = order varies

Definition Let A, B be sets. We define

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

$$A^2 = A \times A$$

Examples

Let $A = \{1,2,3\}$ and $B = \{5,6\}$
Find $A \times B$ and B^2

Solution

$$A \times B = \{1,2,3\} \times \{5,6\}$$

$$= \{(1,5),(1,6),(2,5),(2,6),(3,5),(3,6)\}$$

9/9/2019 <u>Cartesian Product</u>

<u>Example</u> Let $B = \{5, 6\}$ Find B^2

<u>Solution</u>

$$B^2 = B \times B = \{5,6\} \times \{5,6\}$$

$$= \{(5,5), (5,6), (6,5), (6,6)\}$$

In general → n-tuples are an ordered collection of n elements

$$X = (x_1, x_2, x_3, \dots, x_n)$$

<u>Definition</u> given two n-tuples $x = (x_1, x_2, \dots, x_n)$
$y = (y_1, y_2, \dots, y_n)$

we say that

$$x = y \iff x_1 = y_1 \wedge x_2 = y_2 \wedge \dots \wedge x_n = y_n \iff$$

$$\iff \forall a \in [n] : x_a = y_a$$

<u>Definition</u> given sets $A_1, A_2, \dots, A_n$

$$\prod_{k=1}^{n} A_k = A_1 \times A_2 \times A_3 \times \dots \times A_n =$$

$$= \{(x_1, x_2, x_3, \dots, x_n) \mid \forall a \in [n] : x_a \in A_a\}$$

$$A^n = \underbrace{A \times A \times A \times \dots \times A}_{n \text{ times}} = \prod_{k=1}^{n} A$$

<u>HW</u> 5

9/9/2019 <u>Matrices - definitions</u>

A matrix A with size $n \times m$ is a collection of numbers arranged in n rows and m columns

<u>Notation</u>

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1m} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & \cdots & A_{nm} \end{bmatrix}$$

m-tuple of
n-tuple of
numbers

$A_{rc} = A_{\text{row column}}$ $\quad$ $A_{vh} = A_{\text{vertical horizontal}}$

A_{ab} = element of A on <u>row a</u> and <u>column b</u>
with $a \in [n]$ and $b \in [m]$

$M_{nm}(\mathbb{R})$: set of all matrices with size $n \times m$
of real numbers

$M_n(\mathbb{R}) = M_{nn}(\mathbb{R}) \leftarrow$ <u>square matrices</u> with size $n \times n$

<u>Definition</u>: Given $A, B \in M_{nm}(\mathbb{R})$ we say that

$$A = B \iff \forall a \in [n] : \forall b \in [m] : A_{ab} = B_{ab}$$

9/9/2019 Special matrices

Definition Given $A \in M_{nm}(\mathbb{R})$,

$$A = \mathbf{0} \iff \forall a \in [n] : \forall b \in [m] : A_{ab} = 0$$

Zero matrix alternate notation $\mathbf{0}^{(n \times m)}$

Definition The ~~square~~ $n \times n$ identity matrix $I \in M_n(\mathbb{R})$ (can only be square) is defined such that

$$\forall a, b \in [n] : I_{ab} = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases}$$

identity matrix notation: $I^{(n)}$

$$I^{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

identity matrix multiplied by any other matrix will return original matrix

Basic Operations Given $A, B \in M_{nm}(\mathbb{R})$ and $\lambda \in \mathbb{R}$ we define

- $A + B$
- λA
- $-A$
- $A - B$

9/9/2019 Basic Operations with Matrices

Definition Given $A, B \in M_{nm}(\mathbb{R})$ and $\lambda \in \mathbb{R}$
we define: $A+B$, λA, $-A$, $A-B$ such that

$$\forall a \in [n] : \forall b \in [m] : (A+B)_{ab} = A_{ab} + B_{ab}$$

$$\forall a \in [n] : \forall b \in [m] : (\lambda A)_{ab} = \lambda A_{ab}$$

$-A = (-1)A$, set $\lambda = -1$

$A - B = A + (-B)$

example Given $A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix}$

Calculate $2A - 3B$

Solution $2A - 3B = 2\begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix}$

multiply first

then add

$$= \begin{bmatrix} 2 & 6 & 4 \\ 6 & 2 & 8 \end{bmatrix} + \begin{bmatrix} -6 & -9 & -3 \\ 3 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -4 & -3 & 1 \\ 9 & 2 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 2+(-6) & 6+(-9) & 4+(-3) \\ 6+3 & 2+0 & 8+6 \end{bmatrix}$$

Homework: #1-4 matrices

in basic lin alg file or all lecture notes

9/11/2019 Lecture 5 Matrix Properties

▶ Properties of matrix addition

$\forall A, B \in M_{nm}(\mathbb{R}) : A + B = B + A$ (commutative)

$\forall A, B, C \in M_{nm}(\mathbb{R}) : A + (B+C) = (A+B)+C$ (associa

$\forall A \in M_{nm}(\mathbb{R}) : A + \mathbb{0} = \mathbb{0} + A = A$ ($\mathbb{0}$ neutral ele of "+"

$\forall A \in M_{nm}(\mathbb{R}) : \exists B \in M_{nm}(\mathbb{R}) : A+B = B+A = \mathbb{0}$

with respect to "+" → (B is an inverse element of A)

▶ Properties of Scalar multiplication

$\forall \lambda \in \mathbb{R} : \forall A, B \in M_{nm}(\mathbb{R}) : \lambda(A+B) = \lambda A + \lambda B$

$\forall \lambda, \mu \in \mathbb{R} : \forall A \in M_{nm}(\mathbb{R}) : (\lambda+\mu)A = \lambda A + \mu A$

distributive

$\forall \lambda, \mu \in \mathbb{R} : \forall A \in M_{nm}(\mathbb{R}) : (\lambda\mu)A = \lambda(\mu A)$ ← pseudo-associative

$\forall A \in M_{nm}(\mathbb{R}) : 1A = A$ (1 is unit element with respect to scalar multiplication)

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Complete argument of a property

Proof of Associative property

Let $A, B, C \in M_{nm}(\mathbb{R})$ be given

Let $a \in [n]$ and $b \in [m]$ be given

THEN

$$
\begin{aligned}
[A+(B+C)]_{ab} &= A_{ab} + (B+C)_{ab} \\
&= A_{ab} + (B_{ab} + C_{ab}) \\
&= (A_{ab} + B_{ab}) + C_{ab} \\
&= (A+B)_{ab} + C_{ab} \\
&= [(A+B)+C]_{ab}
\end{aligned}
$$

It follows that

$$\forall a \in [n] : \forall b \in [m] : [A+(B+C)]_{ab} = [(A+B)+C]_{ab}$$

$$\Rightarrow A+(B+C) = (A+B)+C$$

We conclude that

$$\forall A, B, C \in M_{nm}(\mathbb{R}) : A+(B+C) = (A+B)+C$$

Homework: 1, 2, 3, 4, 5 # from lecture notes

9/11/2019 **Matrix Multiplication**

Definition Let $A \in M_{nk}(\mathbb{R})$ and $B \in M_{km}(\mathbb{R})$ be given.

Then we define $AB \in M_{nm}(\mathbb{R})$ such that

$$\forall a \in [n] : \forall b \in [m] : (AB)_{ab} = \sum_{c=1}^{k} A_{ac} B_{cb}$$

a) **Row matrix x Column matrix**

$A \in M_{1n}(\mathbb{R})$ and $B \in M_{n1}(\mathbb{R})$

$$[a_1, a_2, \dots, a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = [a_1 b_1 + a_2 b_2 + \dots + a_n b_n]$$

b) **Linear transformation; Square matrix x column matrix**

$A \in M_n(\mathbb{R})$ and $X \in M_{n1}$

For $n=2$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} A_{11}X_1 + A_{12}X_2 \\ A_{21}X_1 + A_{22}X_2 \end{bmatrix}$$

$$\text{vector} \xrightarrow{\text{(matrix)}} \text{vector}$$ (rotations, reflections, stretching)

c) **Square matrix x Square matrix**

$A, B \in M_n(\mathbb{R}) \Rightarrow AB \in M_n(\mathbb{R})$

For $n=2$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

given $X \in M_{n1}(\mathbb{R})$: $(AB)X = A(BX)$

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Properties of Matrix Multiplication

- we do not have a commutative property

$\boxed{\forall A, B \in M_n(\mathbb{R}) : AB = BA} \rightarrow$ FALSE

Potential course project

generalize a counterexample for all values of n where $AB \neq BA$