Math 1450.01

Exam 2 Solution

Use the trigonometric numbers of known angles to evaluate $\cos(15^\circ)$.

Solution

$$\cos(15^\circ) = \sqrt{\frac{1+\cos(30^\circ)}{2}} = \sqrt{\frac{1+\sqrt{3}/2}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}}$$

$$= \frac{\sqrt{2+\sqrt{3}}}{2}$$

2nd method

$$\cos(15^\circ) = \cos(45^\circ - 30^\circ) = \cos(45^\circ)\cos(30^\circ) + \sin(45^\circ)\sin(30^\circ)$$

$$= \frac{\sqrt{2}}{2}\,\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\,\frac{1}{2} = \frac{\sqrt{2}(\sqrt{3}+1)}{4}$$

Note that while the two results may look different, they are in fact equal to each other.

Use known angles to evaluate $\tan(19\pi/12)$. Rationalize any denominators.

Solution

$$\tan(19\pi/12) = \tan(\pi + 7\pi/12) = \tan(7\pi/12) = \tan(3\pi/12 + 4\pi/12) =$$

$$= \tan(\pi/4 + \pi/3) = \frac{\tan(\pi/4) + \tan(\pi/3)}{1 - \tan(\pi/4)\tan(\pi/3)} =$$

$$= \frac{1+\sqrt{3}}{1 - 1\cdot\sqrt{3}} = \frac{1+\sqrt{3}}{1-\sqrt{3}} = \frac{(1+\sqrt{3})^2}{(1-\sqrt{3})(1+\sqrt{3})} =$$

$$= \frac{1^2 + 2\sqrt{3} + 3}{1-3} = \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3}$$

2nd method

$$\tan(19\pi/12) = \tan(\pi + 7\pi/12) = \tan(7\pi/12) = \tan(\pi/2 + \pi/12) =$$

$$= \tan(\pi/2 - (-\pi/12)) = \cot(-\pi/12) = -\cot(\pi/12) =$$

$$= -\sqrt{\frac{1+\cos(\pi/6)}{1-\cos(\pi/6)}} = -\sqrt{\frac{1+\sqrt{3}/2}{1-\sqrt{3}/2}} =$$

$$= -\sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}} = -\sqrt{\frac{(2+\sqrt{3})^2}{(2-\sqrt{3})(2+\sqrt{3})}} =$$

$$= -\sqrt{\frac{2^2 + 2\cdot 2\sqrt{3} + (\sqrt{3})^2}{2^2 - (\sqrt{3})^2}} = -\sqrt{\frac{4+4\sqrt{3}+3}{4-3}} =$$

$$= -\sqrt{7+4\sqrt{3}}$$

Show that

$\sin(a+b)\sin(a-b) = \sin^2 a - \sin^2 b$

Solution

$\sin(a+b)\sin(a-b) =$

$= [\sin a \cos b + \sin b \cos a][\sin a \cos b - \sin b \cos a] =$

$= \sin^2 a \cos^2 b - \sin^2 b \cos^2 a =$

$= \sin^2 a (1 - \sin^2 b) - \sin^2 b (1 - \sin^2 a)$

$= \sin^2 a - \sin^2 a \sin^2 b - \sin^2 b + \sin^2 a \sin^2 b$

$= \sin^2 a - \sin^2 b$

Show that

$$\tan(\pi/4 - a) = \frac{\cos(2a)}{1+\sin(2a)}$$

Solution

$$\tan(\pi/4 - a) = \frac{\tan(\pi/4) - \tan a}{1+\tan(\pi/4)\tan a} = \frac{1-\tan a}{1+\tan a} \qquad (1)$$

$$\frac{\cos(2a)}{1+\sin(2a)} = \frac{\frac{1-\tan^2 a}{1+\tan^2 a}}{1+\frac{2\tan a}{1+\tan^2 a}} = \frac{\frac{1-\tan^2 a}{1+\tan^2 a}}{\frac{1+\tan^2 a+2\tan a}{1+\tan^2 a}} =$$

$$= \frac{1-\tan^2 a}{1+2\tan a+\tan^2 a} = \frac{(1-\tan a)(1+\tan a)}{(1+\tan a)^2} =$$

$$= \frac{1-\tan a}{1+\tan a} \qquad (2)$$

From Eq. (1) and Eq. (2):

$$\tan(\pi/4 - a) = \frac{\cos(2a)}{1+\sin(2a)}$$

Show that $\dfrac{\cos(3a)-\cos(5a)}{\sin(5a)-\sin(3a)} = \tan(4a)$

Solution

$$\frac{\cos(3a)-\cos(5a)}{\sin(5a)-\sin(3a)} = \frac{2\sin\left(\frac{5a-3a}{2}\right)\sin\left(\frac{5a+3a}{2}\right)}{2\sin\left(\frac{5a-3a}{2}\right)\cos\left(\frac{5a+3a}{2}\right)} =$$

$$= \frac{\sin((5a+3a)/2)}{\cos((5a+3a)/2)} = \frac{\sin(8a/2)}{\cos(8a/2)} =$$

$$= \frac{\sin(4a)}{\cos(4a)} = \tan(4a)$$