The Nastrom-Gage energy spectrum of the atmosphere

proposed theoretical explanations and the double cascade theory.

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Publications

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The Big Questions

- Can the energy spectrum of atmospheric turbulence (the Nastrom-Gage spectrum) be reconciled with the predictions of 2D and QG turbulence theory?
- What are these predictions anyway and why?
- What is the simplest model that has the capability to reproduce the Nastrom-Gage spectrum?

Why study 2D and QG turbulence

- QG theory is based on the assumption of rapid rotation (small Rossby number) and thinness (10 km vertical vs 20000 km horizontal). The underlying physics involves a velocity equation and a temperature equation, and 1st-order perturbation expansion around the Rossby number. The vertical velocity is constrained.
- On 2D turbulence the vertical dimension is eliminated.
- Both 2D and QG theory conserve energy and potential enstrophy.
- However, 2D and QG are not isomorphic in terms of physics. (see Tung and Orlando (2003))

Overview

- Review predictions for 2D turbulence.
- Difficulties in reconciling these predictions against the observed Nastrom-Gage spectrum.
- The Tung-Orlando theory of double cascades.
- The superposition principle conjecture.
- The essential difference between 2D and QG turbulence.

Governing equations for 2D

- In 2D turbulence, the scalar vorticity $\zeta(x, y, t)$ is governed by

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = -[\nu(-\Delta)^{\kappa} + \nu_1(-\Delta)^{-m}]\zeta + F, \tag{1}$$

  where $\psi(x, y, t)$ is the streamfunction and $\zeta(x, y, t) = -\nabla^2 \psi(x, y, t)$.
- The Jacobian term $J(\psi, \zeta)$ describes the advection of ζ by ψ, and is defined as

$$J(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial y}. \tag{2}$$

- Two conserved quadratic invariants: energy E and enstrophy G defined as

$$E(t) = -\frac{1}{2}\int \psi(x, y, t)\zeta(x, y, t)\ dxdy \quad G(t) = \frac{1}{2}\int \zeta^2(x, y, t)\ dxdy. \tag{3}$$

KLB theory I

Kraichnan, Leith, and Batchelor (KLB) proposed that in two-dimensional turbulence there is an upscale energy cascade and a downscale enstrophy cascade. The energy spectrum in the upscale energy range is

$$E(k) = C_{ir}\varepsilon^{2/3}k^{-5/3}, \tag{4}$$

and in the downscale enstrophy range is

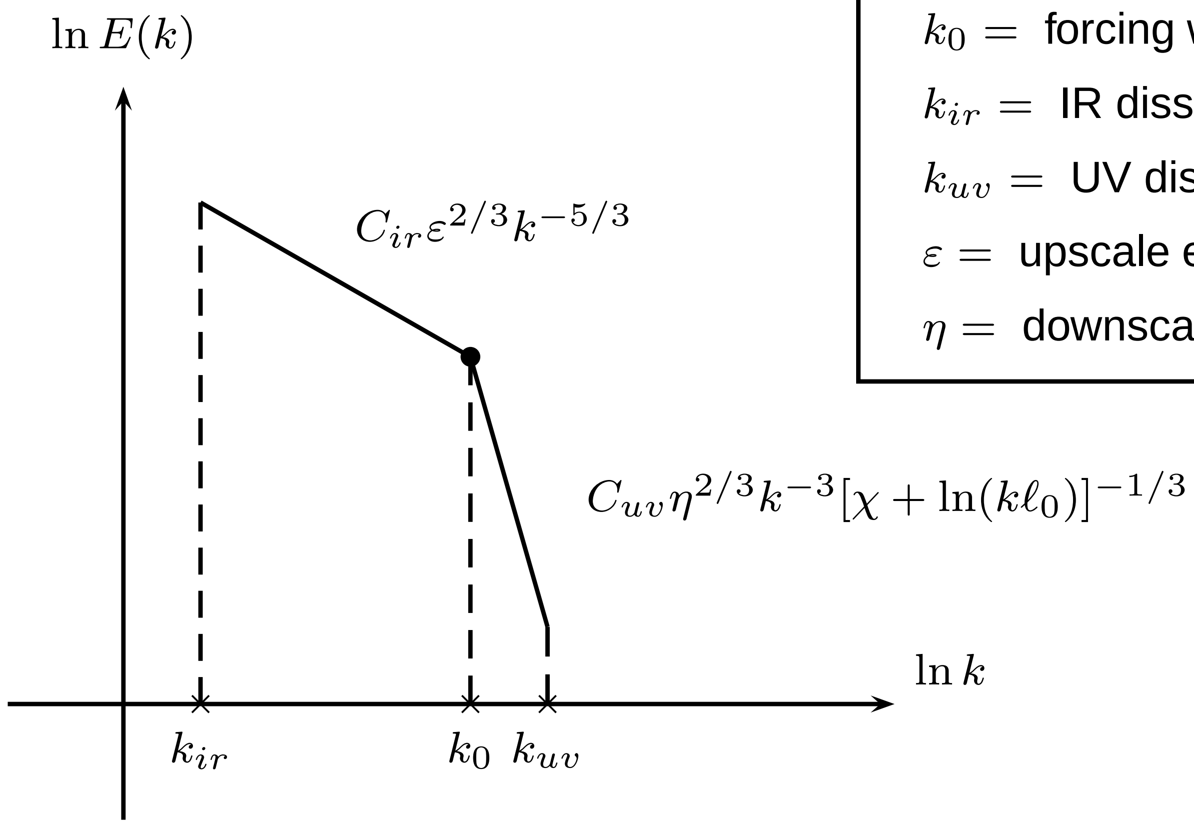
$$E(k) = C_{uv}\eta^{2/3}k^{-3}[\chi + \ln(k\ell_0)]^{-1/3}. \tag{5}$$

Falkovich and Lebedev (1994) predict that the vorticity ζ structure functions have logarithmic scaling given by

$$\langle[\zeta(\mathbf{r}_1) - \zeta(\mathbf{r}_2)]^n\rangle \sim [\eta \ln(\ell_0/r_{12})]^{2n/3}. \tag{6}$$

Confirmed using spectral reduction by Bowman, Shadwick and Morrison (1999).

KLB theory II



$k_0 =$ forcing wavenumber

$k_{ir} =$ IR dissipation wavenumber

$k_{uv} =$ UV dissipation wavenumber

$\varepsilon =$ upscale energy flux

$\eta =$ downscale ensrropht flux

KLB theory III

- Let $T(k, p, q)$ be the rate of energy transfered to wavenumber k from wavenumbers p and q.
- Suppose that the triad interactioms are self-similar in the sense

$$T(\lambda k, \lambda p, \lambda q) = \lambda^{\zeta} T(k, p, q) \tag{7}$$

Then, Kraichnan (1967) showed that

$$\zeta = -3 \Longleftrightarrow [\Pi_E(k) = \varepsilon] \wedge [\Pi_G(k) = 0] \tag{8}$$

$$\zeta = -5 \Longleftrightarrow [\Pi_E(k) = 0] \wedge [\Pi_G(k) = \eta]. \tag{9}$$

- Therefore, Kraichnan's argument can the leveraged to show that the interactions responsible for the enstrophy transfer do not interfere with the energy transfer and vice versa.

KLB theory IV

- Assume that the forcing spectrum $F_E(k)$ is confined to a narrow interval of wavenumbers $[k_1, k_2]$

$$F_E(k) = 0 \text{ and } F_G(k) = 0, \forall k \in (0, k_1) \cup (k_2, +\infty), \tag{10}$$

- Then, under stationarity the fluxes $\Pi_E(k)$ and $\Pi_G(k)$ will satisfy the inequalities

$$\int_0^k q\Pi_E(q)\, dq < 0, \ \forall k > k_2 \tag{11}$$

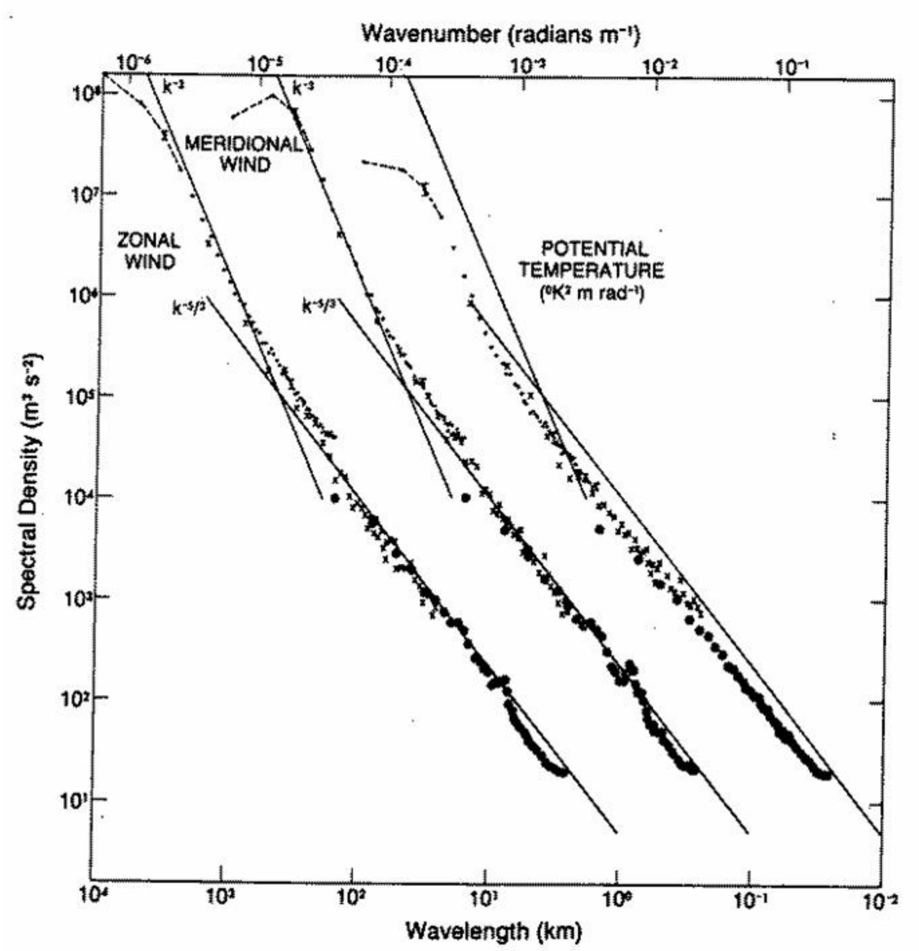
$$\int_k^{+\infty} q^{-3}\Pi_G(q) > 0, \ \forall k < k_1. \tag{12}$$

- Further discussion in
  - E. Gkioulekas and K.K. Tung (2007), *J. Fluid Mech.*, **576**, 173-189.

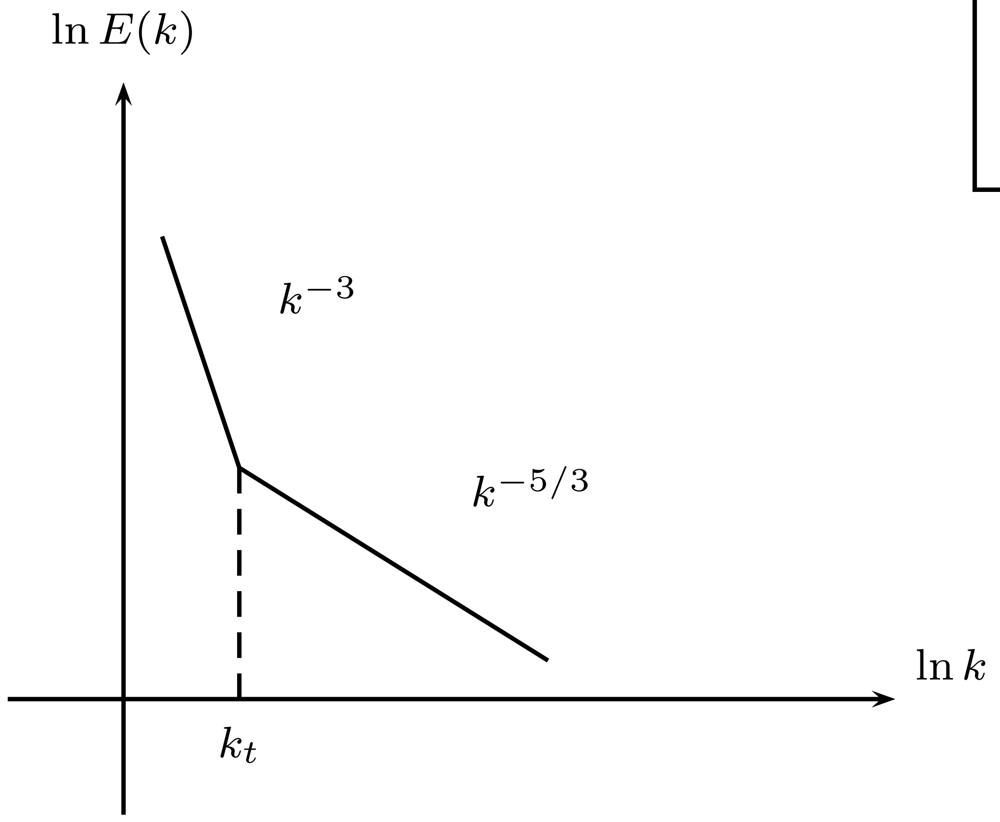
Motivation

- The study of two-dimensional turbulence was originally motivated by the hope that it would prove a useful model for atmospheric turbulence.
- This idea was later encouraged by Charney (1971) who claimed that quasi-geostrophic turbulence is isomorphic to two-dimensional turbulence.
- Early observations suggested that the energy spectrum of the atmosphere follows a k^{-3} power law behavior (see Tung and Orlando (2003) for review).
- Analysis of GASP measurements by Nastrom and Gage (1984) shows a transition to $k^{-5/3}$ scaling.

Nastrom-Gage spectrum



Nastrom-Gage spectrum schematic



$k^{-3} \longrightarrow 3000\text{km} - 800\text{km}$

$k^{-5/3} \longrightarrow 600\text{km} - \ll 1\text{km}$

$k_t \approx 700\text{km}$

The $k^{-5/3}$ part of NG spectrum

- The Nastrom-Gage energy spectrum was confirmed recently with MOSAIC program and GCM simulation
- The k^{-3} is interpretated as downscale enstrophy cascade.
- An explanation of the $k^{-5/3}$ in terms of internal gravity waves was ruled out by Gage and Nastrom (1986).
- Gage and Nastrom (1986) suggest a large-scale source of enstrophy (baroclinic instability) and a small scale source of energy that sends some energy upscale and downscale.
- Lilly (1989) theorized that energy source at small scales can be attributed to thunderstorms.
- The $k^{-5/3}$ portion of the spectrum, appears to be approximately the same whether it is in winter or summer, and whether the airplane flew over storms or not.

The double cascade theory. I

- Tung and Orlando (2003) conjectured that the observed atmospheric energy spectrum results from the downscale cascade of enstrophy and energy injected at the large scales by baroclinic instability and dissipated at the smallest length scales.
- If η_{uv} is the downscale enstrophy flux and ε_{uv} is the downscale energy flux, the transition from -3 slope to $-5/3$ slope occurs at the transition wavenumber k_t with order of magnitude estimated by

$$k_t \approx \sqrt{\eta_{uv}/\varepsilon_{uv}}. \tag{13}$$

The double cascade theory. II

- Recent measurements and data analysis by Cho and Lindborg (2001) have confirmed the existence of a *downscale* energy flux and estimate

$$\eta_{uv} \approx 2 \times 10^{-15}\mathrm{s}^{-3} \tag{14}$$

$$\varepsilon_{uv} \approx 6 \times 10^{-11}\mathrm{km}^2\mathrm{s}^{-3} \tag{15}$$

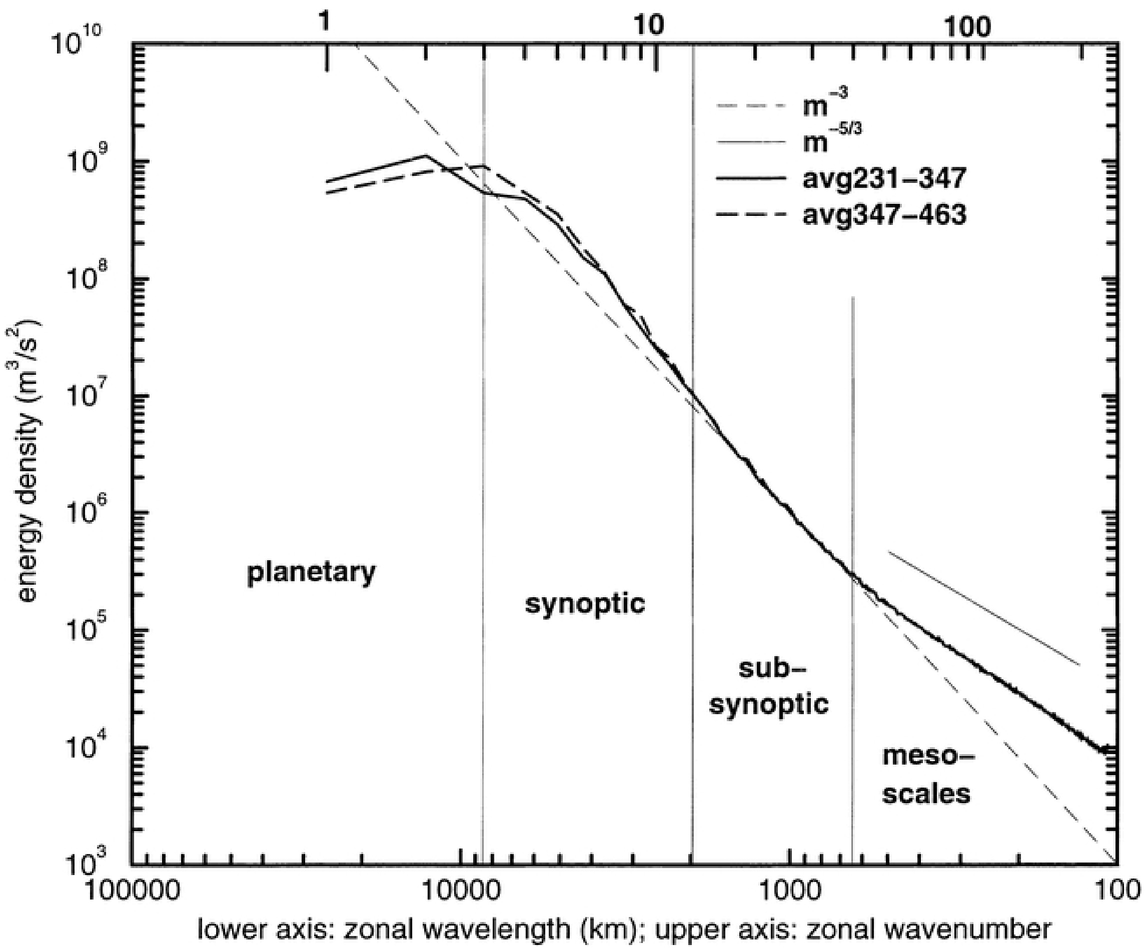
- From these estimates we find the mean value of the transition scale

$$k_t = \sqrt{\eta_{uv}/\varepsilon_{uv}} \approx 0.57 \times 10^{-2}\mathrm{km}^{-1} \Longrightarrow \lambda_t = 2\pi/k_t \approx 1 \times 10^3\mathrm{km} \tag{16}$$

  which has the correct order of magnitude.

- Tung and Orlando (2003) have also demonstrated numerically that a two-layer quasi-geostrophic channel model with thermal forcing, Ekman damping, and hyperdiffusion can reproduce the atmospheric energy spectrum.

Tung and Orlando spectrum



The two-layer model. I

The governing equations for the two-layer quasi-geostrophic model are

$$\frac{\partial \zeta_1}{\partial t} + J(\psi_1, \zeta_1 + f) = -\frac{2f}{h}\omega + d_1 \tag{17}$$

$$\frac{\partial \zeta_2}{\partial t} + J(\psi_2, \zeta_2 + f) = +\frac{2f}{h}\omega + d_2 + 2e_2 \tag{18}$$

$$\frac{\partial T}{\partial t} + \frac{1}{2}[J(\psi_1, T) + J(\psi_2, T)] = -\frac{N^2}{f}\omega + Q_0 \tag{19}$$

where $\zeta_1 = \nabla^2\psi_1$; $\zeta_2 = \nabla^2\psi_2$; $T = (2/h)(\psi_1 - \psi_2)$. f is the Coriolis term; N the Brunt-Väisälä frequency; Q_0 is the thermal forcing on the temperature equation; d_1, d_2, e_2 the dissipation terms:

$$d_1 = (-1)^{\kappa+1}\nu\nabla^{2\kappa}\zeta_1 \tag{20}$$

$$d_2 = (-1)^{\kappa+1}\nu\nabla^{2\kappa}\zeta_2 \tag{21}$$

$$e_2 = -\nu_E\zeta_2 \tag{22}$$

The two-layer model. II

- The potential vorticity is defined as

$$q_1 = \nabla^2 \psi_1 + f + \frac{k_R^2}{2}(\psi_2 - \psi_1) \quad (23)$$

$$q_2 = \nabla^2 \psi_2 + f - \frac{k_R^2}{2}(\psi_2 - \psi_1) \quad (24)$$

with $k_R \equiv 2\sqrt{2}f/(hN)$ and it satisfies

$$\frac{\partial q_1}{\partial t} + J(\psi_1, q_1) = f_1 + d_1 \quad (25)$$

$$\frac{\partial q_2}{\partial t} + J(\psi_2, q_2) = f_2 + d_2 + e_2 \quad (26)$$

with $f_1 = -(k_R^2 Q)/(2f)$ and $f_2 = (k_R^2 Q)/(2f)$ where $Q = (1/4)k_R^2 h Q_0$.

- The two-layer model conserves total energy $E = -\psi_1 q_1 - \psi_2 q_2$ and potential layer enstrophies $G_1 = q_1^2$ and $G_2 = q_2^2$.

Debate

- Smith (2004) debated the theory of Tung and Orlando (2003) by arguing that the downscale energy cascade can never have enough flux to move the transition wavenumber k_t into the inertial range.
- Smith (2004) uses two-dimensional Navier-Stokes for his argument. Tung (2004) replies that the two-layer model is a different dynamical system than the two-dimensional Navier-Stokes equations
- Debate clarified further in a series of papers by Gkioulekas and Tung:
  1. K.K. Tung (2004), *J. Atmos. Sci.*, **61**, 943-948.
  2. E. Gkioulekas and K.K. Tung (2005), *Discrete and Continuous Dynamical Systems B*, **5**, 79-102
  3. E. Gkioulekas and K.K. Tung (2005), *Discrete and Continuous Dynamical Systems B*, **5**, 103-124.
  4. E. Gkioulekas and K.K. Tung (2007), *Discrete Contin. Dyn. Syst. Ser. B*, **7**, 293-314

Superposition principle. I

Gkioulekas and Tung (2005) have argued that in 2D turbulence a leading downscale enstrophy cascade *and a subleading downscale energy cascade* contribute linearly to the total energy spectrum:

$$E(k) = E_{uv}^{(\varepsilon)}(k) + E_{uv}^{(\eta)}(k) + E_{uv}^{(p)}(k), \ \forall k\ell_0 \gg 1, \tag{27}$$

where $E_{uv}^{(\varepsilon)}(k)$, $E_{uv}^{(\eta)}(k)$ are the contributions of the downscale energy and enstrophy cascade, given by

$$\begin{aligned} E_{uv}^{(\varepsilon)}(k) &= a_{uv}\varepsilon_{uv}^{2/3}k^{-5/3}\mathcal{D}_{uv}^{(\varepsilon)}(k\ell_{uv}^{(\varepsilon)}) \\ E_{uv}^{(\eta)}(k) &= b_{uv}\eta_{uv}^{2/3}k^{-3}[\chi + \ln(k\ell_0)]^{-1/3}\mathcal{D}_{uv}^{(\eta)}(k\ell_{uv}^{(\eta)}), \end{aligned} \tag{28}$$

Thus, in the inertial range where the effect of forcing and dissipation can be ignored, the energy spectrum will take the simple form

$$E(k) \approx a_{uv}\varepsilon_{uv}^{2/3}k^{-5/3} + b_{uv}\eta_{uv}^{2/3}k^{-3}[\chi + \ln(k\ell_0)]^{-1/3}. \tag{29}$$

Superposition principle. II

- Justification based on work by L'vov and Procaccia as follows:
- Define the fully unfused correlation tensors for velocity u_α and vorticity ζ:

$$F_n^{\alpha_1\alpha_2\ldots\alpha_n}(\{\mathbf{x}_k,\mathbf{x}'_k\}_{k=1}^n,t) = \left\langle \prod_{k=1}^n \left[u_{\alpha_k}(\mathbf{x}_k,t) - u_{\alpha_k}(\mathbf{x}'_k,t)\right] \right\rangle, \tag{30}$$

$$V_n(\{\mathbf{x}_k,\mathbf{x}'_k\}_{k=1}^n,t) = \left\langle \prod_{k=1}^n \left[\zeta(\mathbf{x}_k,t) - \zeta(\mathbf{x}'_k,t)\right] \right\rangle \tag{31}$$

- The relation between F_n and V_n is $V_n = \mathcal{T}_n F_n$ or:

$$V_n(\{\mathbf{x}_k,\mathbf{x}'_k\}_{k=1}^n,t) = \prod_{k=1}^n [\varepsilon_{\alpha_k\beta_k}(\partial_{\alpha_k,\mathbf{x}_k} + \partial_{\alpha_k,\mathbf{x}'_k})] F_n^{\alpha_1\cdots\alpha_n}(\{\mathbf{x}_k,\mathbf{x}'_k\}_{k=1}^n,t) \tag{32}$$

Superposition principle. III

- F_n and V_n satisfy the balance equations:

$$\frac{\partial F_n}{\partial t} + \mathcal{O}_n F_{n+1} + I_n = \mathcal{D}_n F_n + Q_n \tag{33}$$

$$\frac{\partial V_n}{\partial t} + \mathcal{T}_n \mathcal{O}_n F_{n+1} + \mathcal{I}_n = \mathcal{D}_n V_n + \mathcal{Q}_n \tag{34}$$

Here $Q_n, \mathcal{Q}_n$ are forcing terms and $I_n, \mathcal{I}_n$ are sweeping terms, $\mathcal{O}_n$ local interactions, and $\mathcal{D}_n$ the dissipation operator.

- Belinicher, L'vov, Pomyalov and Procaccia (1998) argue that in 3D turbulence, the scaling of the downscale energy cascade originates from the solvability condition on the homogeneous equation

$$\mathcal{O}_n F_{n+1} = 0 \tag{35}$$

This argument leads to a scheme for computing the scaling exponents ζ_n of F_n.

Superposition principle. IV

- This argument requires the elimination of the sweeping interactions represented by the terms $I_n, \mathcal{I}_n$ given by

$$I_n(\{\mathbf{x},\mathbf{x}'\}_{k=1}^n, t) = \sum_{k=1}^n (\partial_{\gamma,\mathbf{x}_k} + \partial_{\gamma,\mathbf{x}'_k}) \left\langle \mathcal{U}_\gamma(\{\mathbf{x}_k,\mathbf{x}'_k\}_{k=1}^n, t) \left[\prod_{l=1}^n w_{\alpha_l}(\mathbf{x}_l,\mathbf{x}'_l,t)\right]\right\rangle$$

$$\mathcal{I}_n(\{\mathbf{x},\mathbf{x}'\}_{k=1}^n, t) = \sum_{k=1}^n (\partial_{\gamma,\mathbf{x}_k} + \partial_{\gamma,\mathbf{x}'_k}) \left\langle \mathcal{U}_\gamma(\{\mathbf{x}_k,\mathbf{x}'_k\}_{k=1}^n, t) \left[\prod_{l=1}^n \xi(\mathbf{x}_l,\mathbf{x}'_l,t)\right]\right\rangle$$

  where $\mathcal{U}_\alpha(\{\mathbf{x}_k,\mathbf{x}'_k\}_{k=1}^n, t)$ is the mean velocity:

$$\mathcal{U}_\alpha(\{\mathbf{x}_k,\mathbf{x}'_k\}_{k=1}^n, t) = \frac{1}{2n}\sum_{k=1}^n (u_\alpha(\mathbf{x}_k,t) + u_\alpha(\mathbf{x}'_k,t)) \qquad (36)$$

- The problem of the elimination of the sweeping interactions is discussed in
  1. E. Gkioulekas (2007), *Physica D*, **226**, 151-172

Superposition principle. V

- The homogeneous equations $\mathcal{O}_n F_{n+1} = 0$ are invariant with respect to the following group of transformations

$$\mathbf{r} \mapsto \lambda \mathbf{r}, \quad F_n \mapsto \lambda^{nh+\mathcal{Z}(h)} F_n. \tag{37}$$

- This means that in an inertial range solutions $F_{n,h}$ that satisfy the self-similarity property

$$F_{n,h}(\{\lambda \mathbf{x}_k, \lambda \mathbf{x}'_k\}_{k=1}^{n}, t) = \lambda^{nh+\mathcal{Z}(h)} F_{n,h}(\mathbf{x}_k, \mathbf{x}'_k\}_{k=1}^{n}, t), \tag{38}$$

are admissible.

- The correct solution is the linear combination of these solutions, given by

$$F_n = \int d\mu(h) F_{n,h}. \tag{39}$$

Superposition principle. VI

- In two-dimensional turbulence, homogeneous solutions originate from

$$\mathcal{O}_n F_{n+1} = 0 \Longrightarrow \text{1 solution: energy cascade} \tag{40}$$

$$\mathcal{T}_n \mathcal{O}_n F_{n+1} = 0 \Longrightarrow \text{2 solutions: energy and enstrophy cascade} \tag{41}$$

- The superposition principle is also valid in 2D turbulence. However the solvability argument cannot be trivially extended to 2D; for the enstrophy cascade the reducible contributions (unlinked Feynman diagrams) are not negligible. The sufficient condition to neglect the reducible contributions is $\mathcal{Z}(h) > 0$ for all h.
- The enstrophy solution contributes at $h = 1$ exclusively, and the energy solution at $0 < h < 1$. Thus our ansatz is $\mathcal{Z}(1) = 0$ and $\mathcal{Z}(h) > 0$ for $0 < h < 1$.
- We conjecture that the balance equations essentially have two homogeneous solutions (energy/enstrophy cascade) from $\mathcal{T}_n \mathcal{O}_n F_{n+1} = 0$, and a particular solution (coherent structures) which is caused by Q_n and I_n.

Superposition principle. VII

- The realistic solutions for each cascade include a dissipation range. These solutions originate from the modified equation

$$\mathcal{T}_n \mathcal{O}_n F_{n+1} - \mathcal{T}_n \mathcal{D}_n F_n = 0. \tag{42}$$

- The dissipative terms $\mathcal{D}_n$ modify the linear operator $\mathcal{O}_n$ and in doing so *modify the homogeneous solutions* responsible both for the leading and subleading cascades both downscale and upscale. The modification amounts to truncating the inertial range with the dissipation range.
- The location of the dissipation scale corresponding to one of the homogeneous solutions present is independent of the energy or enstrophy flux corresponding to the other homogeneous solutions.
- Thus, the dissipation scales can be estimated with dimensional analysis.

Superposition principle. VIII

- The dissipation scale $\ell_{uv}^{(\eta)}$ of the downscale enstrophy cascade and the dissipation scale $\ell_{uv}^{(\varepsilon)}$ of the downscale energy cascade are given by

$$\ell_{uv}^{(\eta)} = \ell_0 \left[\frac{\mathcal{R}_{uv}^{(\eta)}}{\mathcal{R}_{0,uv}^{(\eta)}}\right]^{-1/(2\kappa)} = \left[\frac{1}{\mathcal{R}_{0,uv}^{(\eta)}} \frac{\eta_{uv}^{1/3}}{\nu}\right]^{-1/(2\kappa)} \tag{43}$$

$$\ell_{uv}^{(\varepsilon)} = \ell_0 \left[\frac{\mathcal{R}_{uv}^{(\varepsilon)}}{\mathcal{R}_{0,uv}^{(\varepsilon)}}\right]^{3/(2-6\kappa)} = \left[\frac{1}{\mathcal{R}_{0,uv}^{(\varepsilon)}} \frac{\varepsilon_{uv}^{1/3}}{\nu}\right]^{3/(2-6\kappa)} \tag{44}$$

- In the KLB limit $\mathcal{R}_{uv}^{(\eta)} \to +\infty$ the dissipation scale of the subleading downscale energy cascade is given asymptotically by

$$\ell_{uv}^{(\varepsilon)} \approx \ell_{uv}^{(\eta)} \left(\frac{\mathcal{R}_{0,uv}^{(\varepsilon)}}{\mathcal{R}_{0,uv}^{(\eta)}}\right)^{3/(6\kappa-2)}, \quad \text{and } k_t \lambda_{uv} \to 1 \tag{45}$$

Danilov Inequality. I

- To recap: the downscale enstrophy cascade contributes a dominant k^{-3} term to $E(k)$ and the downscale energy cascade a subdominant $k^{-5/3}$ term.
- Why is the $k^{-5/3}$ term hidden in two-dimensional turbulence?
- In two-dimensional turbulence, the energy flux $\Pi_E(k)$ and the enstrophy flux $\Pi_G(k)$ are constrained by

$$k^2\Pi_E(k) - \Pi_G(k) < 0, \tag{46}$$

  for all wavenumbers k outside of the forcing range.
- The transition wavenumber k_t where the break from k^{-3} to $k^{-5/3}$ should occur, approaches the dissipation scale from the dissipation range, thus a transition cannot be seen visually in two-dimensional turbulence.
- Thus, the contribution of the downscale energy cascade to the energy spectrum is overwhelmed by the contribution of the downscale enstrophy cascade.

Danilov Inequality. II

- Recall that in the two-layer model, the dissipation terms read:

$$d_1 = \nu(-\Delta)^{\kappa+1}\psi_1, \tag{47}$$

$$d_2 = \nu(-\Delta)^{\kappa+1}\psi_2 - \nu_E\Delta\psi_2 \tag{48}$$

- We have shown that it is the asymmetric presence of Ekman damping on the bottom layer but not the top layer which causes the violation of the Danilov inequality in the two-layer model and moves the transition wavenumber k_t into the inertial range.
- A necessary (but not sufficient) condition to *violate* Danilov's inequality is

$$\nu_E > 4\nu k_{\max}^{2p}\left(\frac{k_{\max}}{k_R}\right)^2 \tag{49}$$

Danilov Inequality. II

- We have derived a *necessary and sufficient condition* for violating the Danilov inequality.
- The condition has the form

$$\nu_E k_R^2 \geq \Lambda \nu k_{\max}^{2p+2}, \tag{50}$$

  but it is not possible to find a universal value for Λ that will *always* work.
- The necessary requirement needed to *have* a sufficient condition for violating the Danilov inequality at wavenumber k is

$$G_1(q) - (1 + 4(q/k_R)^2)G_2(q) > 2q^2 E_K(q), \tag{51}$$

  for all q such that $k < q < k_{max}$. Here $G_1(k)$ and $G_2(k)$, are the enstrophy spectra for each layer and $E_K(k)$ is the kinetic energy spectrum.
- Many open questions remain.

Open question: Helicity cascade

- It has been claimed that in the Nastrom-Gage spectrum we have a transition from $k^{-7/3}$ (downscale helicity cascade), instead of k^{-3}, to $k^{-5/3}$. References advocating this position include:
  1. A.Bershadskii, E.Kit, and A. Tsinober, *Proc. R. Soc. Lond. A* **441** (1993), 147–155.
  2. S.S. Moiseev and O.G. Chkhetiani, *JETP* **83** (1996), 192–198.
  3. H. Branover, A. Eidelman, E. Golbraikh, and S. Moiseev, *Turbulence and structures: chaos, fluctuations, and helical self-organization in nature and the laboratory*, Academic Press, San Diego, 1999.
- Cho and Lindborg (2001) showed that $S_3(r) \sim r^3$ (diagonal components) in the polar stratosphere data, which supports an enstrophy cascade. There is also an unexplained robust r^2 contribution to the off-diagonal components in the stratosphere from 10 km to 1,000 km in scale.
- Open question. Need transition scale calculation to confirm or deny.

Conclusion

- The $k^{-5/3}$ portion of the Nastrom-Gage spectrum is a downscale energy cascade.
- The k^{-3} interpretation for small wavenumbers could be wrong.
- The $k^{-3} \rightarrow k^{-5/3}$ interpretation can be accounted for with a two-layer model
- The dynamics of the two-layer model are interesting and not well-understood.