The effect of asymmetric Ekman damping on energy and enstrophy injection in two-layer quasi-geostrophic turbulence

Eleftherios Gkioulekas [1] and Ka-Kit Tung [2]

[1] Department of Mathematics, University of Texas-Pan American

[2] Department of Applied Mathematics, University of Washington

Overview

- Review of 2D turbulence.
- The Nastrom-Gage spectrum of atmospheric turbulence.
- The Tung-Orlando theory of double cascade.
- The two-layer quasi-geostrophic model.

2D Navier-Stokes equations

- In 2D turbulence, the scalar vorticity $\zeta(x, y, t)$ is governed by

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = d + f, \tag{1}$$

where $\psi(x, y, t)$ is the streamfunction, and $\zeta(x, y, t) = -\nabla^2 \psi(x, y, t)$, and

$$d = -[\nu(-\Delta)^{\kappa} + \nu_1(-\Delta)^{-m}]\zeta \tag{2}$$

- The Jacobian term $J(\psi, \zeta)$ describes the advection of ζ by ψ, and is defined as

$$J(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial y}. \tag{3}$$

Energy and enstrophy spectrum. I

- Two conserved quadratic invariants: energy E and enstrophy G defined as

$$E(t) = -\frac{1}{2}\int \psi(x,y,t)\zeta(x,y,t)\;dxdy \quad G(t) = \frac{1}{2}\int \zeta^2(x,y,t)\;dxdy. \tag{4}$$

- Let $a^{<k}(\mathbf{x})$ be the field obtained from $a(\mathbf{x})$ by setting to zero, in Fourier space, the components corresponding to wavenumbers with norm greater than k:

$$a^{<k}(\mathbf{x}) = \int d\mathbf{y}\, P(k|\mathbf{x}-\mathbf{y})a(\mathbf{y}) \tag{5}$$

$$= \int_{\mathbb{R}^2} d\mathbf{x}_0 \int_{\mathbb{R}^2} d\mathbf{k}_0\, \frac{H(k-\|\mathbf{k}_0\|)}{4\pi^2}\exp(i\mathbf{k}_0\cdot(\mathbf{x}-\mathbf{x}_0))a(\mathbf{x}_0) \tag{6}$$

- Filtered inner product:

$$\langle a,b\rangle_k = \frac{d}{dk}\int_{\mathbb{R}^2} d\mathbf{x}\; a^{<k}(\mathbf{x})b^{<k}(\mathbf{x}) \tag{7}$$

Energy and enstrophy spectrum. II

- Energy spectrum: $E(k) = \langle \psi, \zeta \rangle_k$
- Enstrophy spectrum $G(k) = \langle \zeta, \zeta \rangle_k$
- Consider the conservation laws for $E(k)$ and $G(k)$:

$$\frac{\partial E(k)}{\partial t} + \frac{\partial \Pi_E(k)}{\partial k} = D_E(k) + F_E(k) \tag{8}$$

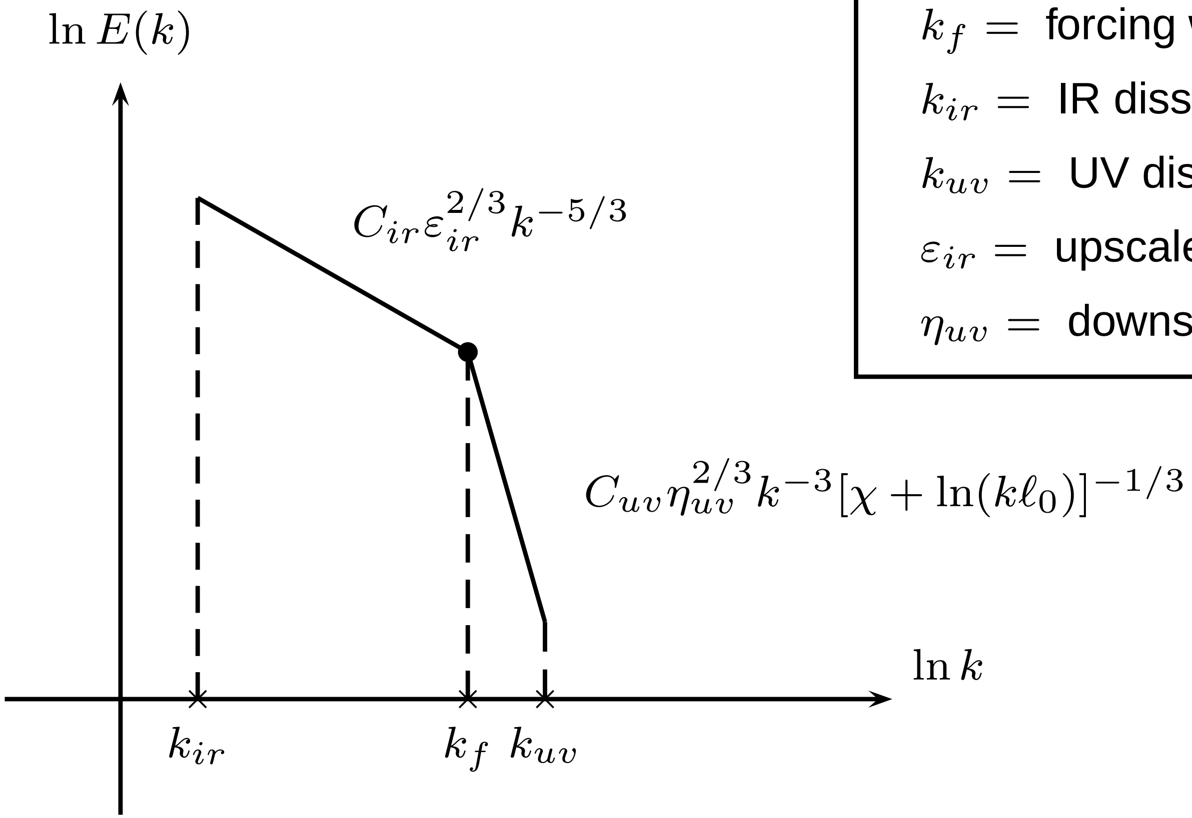
$$\frac{\partial G(k)}{\partial t} + \frac{\partial \Pi_E(k)}{\partial k} = D_G(k) + F_G(k) \tag{9}$$

- In two-dimensional turbulence, the energy flux $\Pi_E(k)$ and the enstrophy flux $\Pi_G(k)$ are constrained by

$$k^2 \Pi_E(k) - \Pi_G(k) < 0 \tag{10}$$

for all k not in the forcing range.

KLB theory.



k_f = forcing wavenumber

k_{ir} = IR dissipation wavenumber

k_{uv} = UV dissipation wavenumber

ε_{ir} = upscale energy flux

η_{uv} = downscale ensrropht flux

The need for a large-scale sink

- It can also be shown that: $F_E(k) = k^{-2} F_G(k)$. It follows that
  - At the forcing range: $(\eta/\varepsilon) \sim k_f^2$
  - At the dissipation range: $(\eta_{uv}/\varepsilon_{uv}) \sim k_d^2 \gg k_f^2$
- It follows that most of the energy injected at the forcing range along with the enstrophy cannot be dissipated at small scales.
- A large scale sink that can reduce predominantly ε_{uv} is needed.
- In the papers:
  - E. Gkioulekas and K.K. Tung (2005), *DCDS B*, **5**, 79-102
  - E. Gkioulekas and K.K. Tung (2005), *DCDS B*, **5**, 103-124.

  we have theorized that if it were possible to have $(\eta_{uv}/\varepsilon_{uv}) \gg k_d^2$, then within the enstrophy cascade we would see a transition from k^{-3} scaling to $k^{-5/3}$ scaling at the transition wavenumber $k_t \sim (\eta_{uv}/\varepsilon_{uv})$.

Superposition principle. I

Specifically, Gkioulekas and Tung (2005) have argued that in 2D turbulence a leading downscale enstrophy cascade *and a subleading downscale energy cascade* contribute linearly to the total energy spectrum:

$$E(k) = E_{uv}^{(\varepsilon)}(k) + E_{uv}^{(\eta)}(k) + E_{uv}^{(p)}(k), \; \forall k\ell_0 \gg 1, \tag{11}$$

where $E_{uv}^{(\varepsilon)}(k)$, $E_{uv}^{(\eta)}(k)$ are the contributions of the downscale energy and enstrophy cascade, given by

$$\begin{aligned} E_{uv}^{(\varepsilon)}(k) &= a_{uv}\varepsilon_{uv}^{2/3}k^{-5/3}\mathcal{D}_{uv}^{(\varepsilon)}(k\ell_{uv}^{(\varepsilon)}) \\ E_{uv}^{(\eta)}(k) &= b_{uv}\eta_{uv}^{2/3}k^{-3}[\chi + \ln(k\ell_0)]^{-1/3}\mathcal{D}_{uv}^{(\eta)}(k\ell_{uv}^{(\eta)}), \end{aligned} \tag{12}$$

Thus, in the inertial range where the effect of forcing and dissipation can be ignored, the energy spectrum will take the simple form

$$E(k) \approx a_{uv}\varepsilon_{uv}^{2/3}k^{-5/3} + b_{uv}\eta_{uv}^{2/3}k^{-3}[\chi + \ln(k\ell_0)]^{-1/3}. \tag{13}$$

Superposition principle. II

- To recap: the downscale enstrophy cascade contributes a dominant k^{-3} term to $E(k)$ and the downscale energy cascade a subdominant $k^{-5/3}$ term.
- Why is the $k^{-5/3}$ term hidden in two-dimensional turbulence?
- In two-dimensional turbulence, the energy flux $\Pi_E(k)$ and the enstrophy flux $\Pi_G(k)$ are constrained by

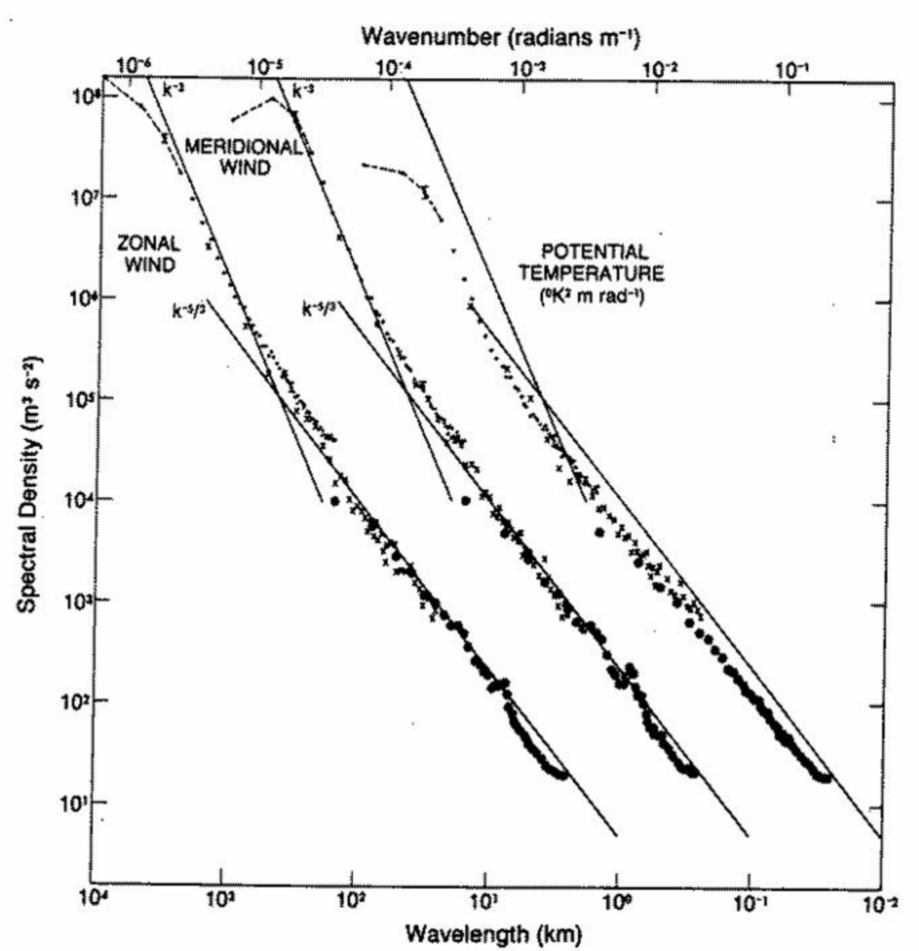
$$k^2\Pi_E(k) - \Pi_G(k) < 0, \tag{14}$$

  for all wavenumbers k outside of the forcing range.
- The transition wavenumber k_t where the break from k^{-3} to $k^{-5/3}$ should occur, approaches the dissipation scale from the dissipation range, thus a transition cannot be seen visually in two-dimensional turbulence.
- Thus, the contribution of the downscale energy cascade to the energy spectrum is overwhelmed by the contribution of the downscale enstrophy cascade.

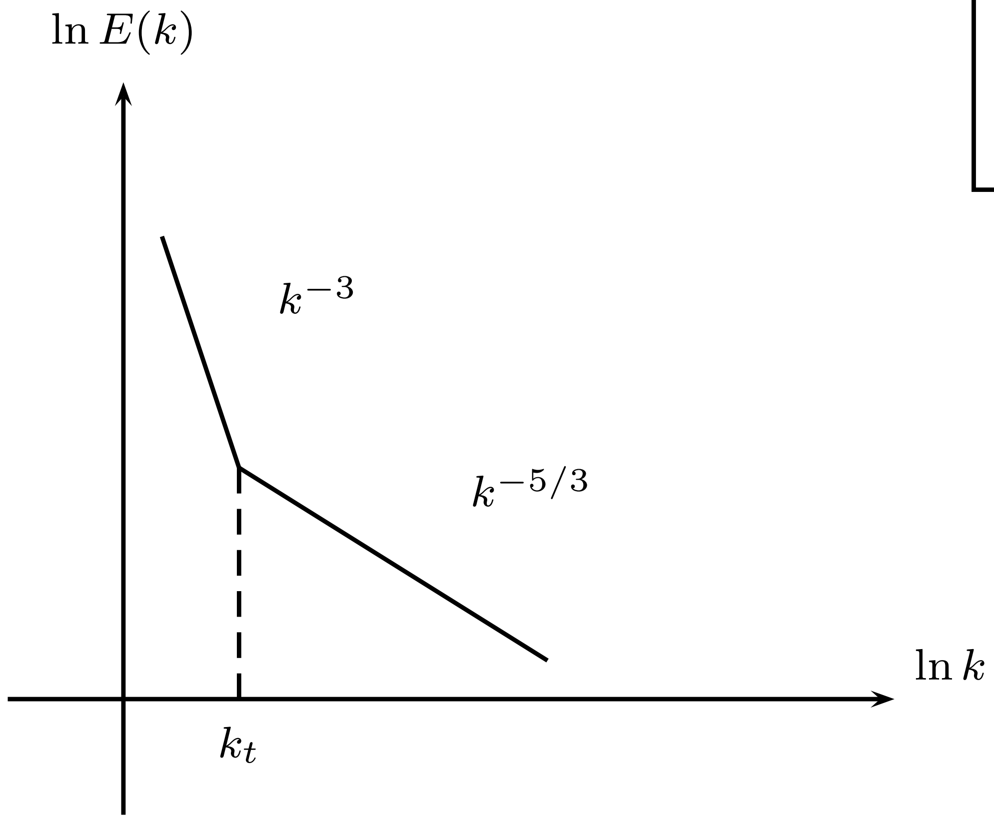
The Nastrom-Gage spectrum. I

- A transition from k^{-3} scaling to $k^{-5/3}$ scaling has been observed in the energy spectrum of large-scale atmospheric turbulence.
  - G.D. Nastrom and K.S. Gage (1984), *J. Atmos. Sci.* **42**, 950–960.
  - K.S. Gage and G.D. Nastrom (1986), *J. Atmos. Sci.* **43**, 729–740.
- This energy spectrum is known as the **Nastrom-Gage spectrum**.
- The QG model (quasi-geostrophic) describes large-scale atmospheric turbulence down to a scale of 100km.
- Charney claimed that the QG model is isomorphic to two-dimensional turbulence:
  - J.G. Charney (1971), *J. Atmos. Sci.* **28**, 1087–1095.
- This result by Charney has caused considerable confusion when efforts were made to explain the Nastrom-Gage spectrum in terms of 2D turbulence.

The Nastrom-Gage spectrum. II



Nastrom-Gage spectrum schematic



$k^{-3} \longrightarrow$ 3000km $-$ 800km

$k^{-5/3} \longrightarrow$ 600km$- \ll$ 1km

$k_t \approx$ 700km $\approx k_R$

Interpretation of NG spectrum. I

- The k^{-3} range is interpretated as downscale enstrophy cascade.
- The $k^{-5/3}$ range used to be interpretated as an 2D inverse energy cascade forced at small scales by thunderstorms.
- This tortured interpretation followed from the perceived need to explain the Nastrom-Gage spectrum in terms of 2D turbulence.
- Tung and Orlando (formerly: Welch) challenged the Charney QG-2D equivalence:
  - K.K. Tung and W.T. Welch (2001), *J. Atmos. Sci.* **58**, 2009-2012.
  - K.K. Tung and W.W. Orlando (2003a), *J. Atmos. Sci.* **60**,, 824-835.
  - K.K. Tung and W.W. Orlando (2003b), *DCDS B*, **3**, 145-162.
- New interpretation: A double downscale cascade of enstrophy and energy with enstrophy flux η_{uv} and energy flux ε_{uv} and transition from k^{-3} scaling to $k^{-5/3}$ scaling at the transition wavenumber $k_t \sim \sqrt{\eta_{uv}/\varepsilon_{uv}}$.

Interpretation of NG spectrum. II

- Recent measurements and data analysis by Cho and Lindborg (2001) have confirmed the existence of a *downscale* energy flux and estimate

$$\eta_{uv} \approx 2 \times 10^{-15}\text{s}^{-3} \tag{15}$$

$$\varepsilon_{uv} \approx 6 \times 10^{-11}\text{km}^2\text{s}^{-3} \tag{16}$$

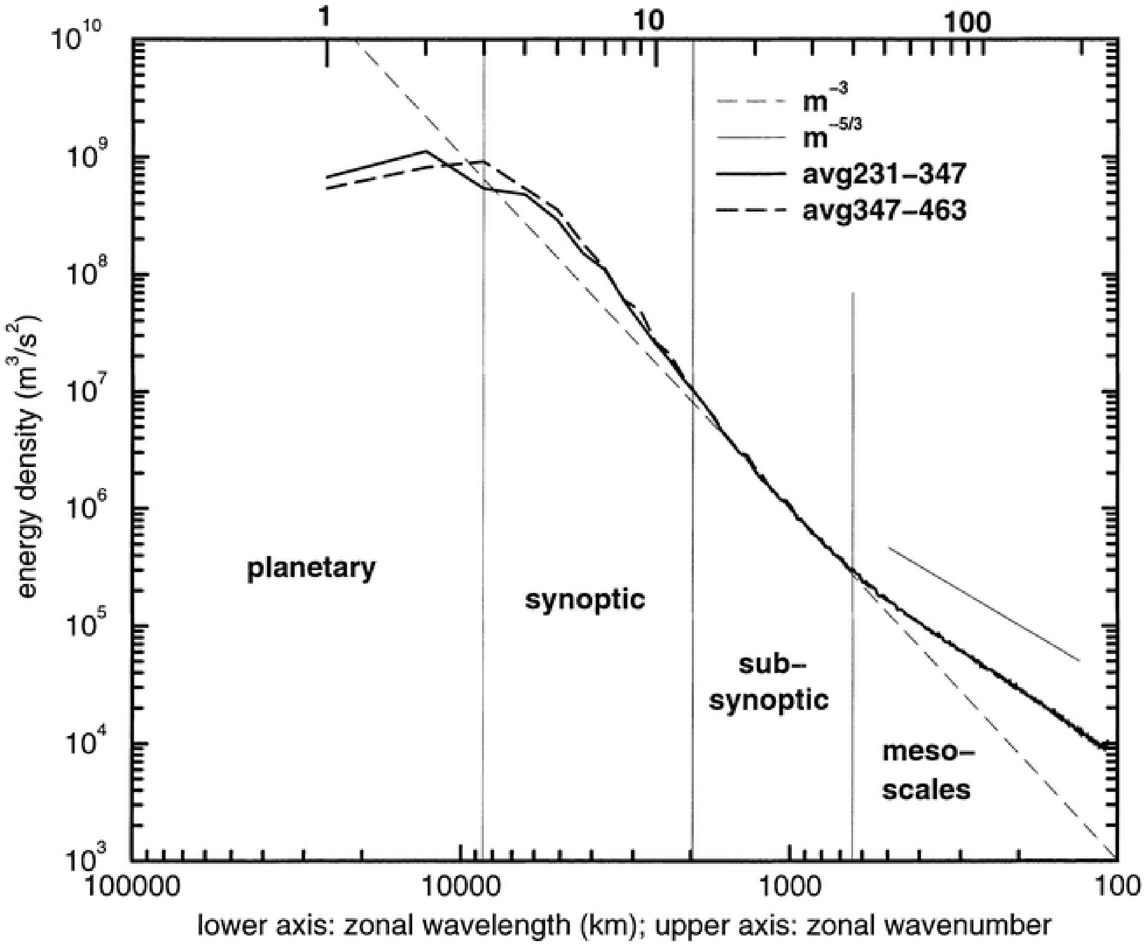
- From these estimates we find the mean value of the transition scale

$$k_t = \sqrt{\eta_{uv}/\varepsilon_{uv}} \approx 0.57 \times 10^{-2}\text{km}^{-1} \Longrightarrow \lambda_t = 2\pi/k_t \approx 1 \times 10^3\text{km} \tag{17}$$

  which has the correct order of magnitude.

- Tung and Orlando (2003) have also demonstrated numerically that a two-layer quasi-geostrophic channel model with thermal forcing, Ekman damping, and hyperdiffusion can reproduce the atmospheric energy spectrum.

Tung and Orlando spectrum



More on Tung-Orlando theory

- Smith-Tung debate: Transition not possible in 2D turbulence
  - K.S. Smith (2004), *J. Atmos. Sci.* **61** , 937-942
  - K.K. Tung (2004), *J. Atmos. Sci.* **61**, 943-948.
  - E. Gkioulekas and K.K. Tung (2007), *DCDS B* **7**, 293-314
- Smith argues: In 2D turbulence, transition from k^{-3} to $k^{-5/3}$ cannot occur in the inertial range.
- Objections:
  - 2D turbulence is not QG turbulence.
  - In 2D turbulence we can rule out transition in $S_3(r)$ but not in $E(k)$.
  - Very high-order hyperdiffusion can amplify ratio of Kolmogorov constants in 2D turbulence thereby allowing a transition in $E(k)$.
  - In QG turbulence, asymmetric dissipation can violate the constraint of vanishing downscale energy flux.

The two-layer model. I

The governing equations for the two-layer quasi-geostrophic model are

$$\frac{\partial \zeta_1}{\partial t} + J(\psi_1, \zeta_1 + f) = -\frac{2f}{h}\omega + d_1 \tag{18}$$

$$\frac{\partial \zeta_2}{\partial t} + J(\psi_2, \zeta_2 + f) = +\frac{2f}{h}\omega + d_2 + 2e_2 \tag{19}$$

$$\frac{\partial T}{\partial t} + \frac{1}{2}[J(\psi_1, T) + J(\psi_2, T)] = -\frac{N^2}{f}\omega + Q_0 \tag{20}$$

where $\zeta_1 = \nabla^2\psi_1$; $\zeta_2 = \nabla^2\psi_2$; $T = (2/h)(\psi_1 - \psi_2)$. f is the Coriolis term; N the Brunt-Väisälä frequency; Q_0 is the thermal forcing on the temperature equation; d_1, d_2, e_2 the dissipation terms:

$$d_1 = (-1)^{\kappa+1}\nu\nabla^{2\kappa}\zeta_1 \tag{21}$$

$$d_2 = (-1)^{\kappa+1}\nu\nabla^{2\kappa}\zeta_2 \tag{22}$$

$$e_2 = -\nu_E\zeta_2 \tag{23}$$

The two-layer model. II

- The potential vorticity is defined as

$$q_1 = \nabla^2\psi_1 + f + \frac{k_R^2}{2}(\psi_2 - \psi_1) \quad (24)$$

$$q_2 = \nabla^2\psi_2 + f - \frac{k_R^2}{2}(\psi_2 - \psi_1) \quad (25)$$

with $k_R \equiv 2\sqrt{2}f/(hN)$ and it satisfies

$$\frac{\partial q_1}{\partial t} + J(\psi_1, q_1) = f_1 + d_1 \quad (26)$$

$$\frac{\partial q_2}{\partial t} + J(\psi_2, q_2) = f_2 + d_2 + e_2 \quad (27)$$

with $f_1 = -(k_R^2 Q)/(2f)$ and $f_2 = (k_R^2 Q)/(2f)$ where $Q = (1/4)k_R^2 h Q_0$.

The two-layer model. III

- The two-layer model conserves total energy E and potential layer enstrophies G_1 and G_2:

$$E = \int \left[-\psi_1(x,y,t)q_1(x,y,t) - \psi_2(x,y,t)q_2(x,y,t)\right] dxdy \tag{28}$$

$$G_1 = \int q_1^2(x,y,t)\; dxdy \tag{29}$$

$$G_2 = \int q_2^2(x,y,t)\; dxdy \tag{30}$$

- We define:
  - The energy spectrum $E(k) = \langle \psi_1, q_1 \rangle_k + \langle \psi_2, q_2 \rangle_k$
  - Top-layer potential enstrophy spectrum $G_1(k) = \langle q_1, q_1 \rangle_k$
  - Bottom-layer potential enstrophy spectrum $G_2(k) = \langle q_2, q_2 \rangle_k$
  - Total potential enstrophy spectrum $G(k) = G_1(k) + G_2(k)$

The two-layer model. IV

- Consider the generalized form of an n-layer model:

$$\frac{\partial q_\alpha}{\partial t} + J(\psi_\alpha, q_\alpha) = \sum_\beta \mathcal{D}_{\alpha\beta}\psi_\beta + f_\alpha \tag{31}$$

$$\hat{q}_\alpha(\mathbf{k}, t) = \sum_\beta L_{\alpha\beta}(\|\mathbf{k}\|)\hat{\psi}_\beta(\mathbf{k}, t) \tag{32}$$

- The energy spectrum $E(k)$ and the potential enstrophy spectrum $G(k)$ are given by:

$$E(k) = \sum_\alpha \langle \psi_\alpha, q_\alpha \rangle_k = \sum_{\alpha\beta} L_{\alpha\beta}(k) C_{\alpha\beta}(k) \tag{33}$$

$$G(k) = \sum_\alpha \langle q_\alpha, q_\alpha \rangle_k = \sum_{\alpha\beta\gamma} L_{\alpha\beta}(k) L_{\alpha\gamma}(k) C_{\beta\gamma}(k) \tag{34}$$

with $C_{\alpha\beta}(k) = \langle \psi_\alpha, \psi_\beta \rangle_k$.

The two-layer model. V

- Let $\phi_{\alpha\beta}(k) = \left\langle f_\alpha, \psi_\beta \right\rangle_k$, and $D_{\alpha\beta}(k)$ be the spectrum of the operator $\mathcal{D}_{\alpha\beta}$
- The energy forcing spectrum $F_E(k)$ and the potential enstrophy forcing spectrum $F_G(k)$ are given by:

$$F_E(k) = 2\sum_{\alpha} \phi_{\alpha\alpha}(k) \tag{35}$$

$$F_G(k) = 2\sum_{\alpha\beta} L_{\alpha\beta}(k)\phi_{\alpha\beta}(k) \tag{36}$$

- The energy dissipation spectrum $D_E(k)$ and the potential dissipation enstrophy spectrum $D_G(k)$ are given by:

$$D_E(k) = 2\sum_{\alpha\beta} D_{\alpha\beta}(k)C_{\alpha\beta}(k) \tag{37}$$

$$D_G(k) = 2\sum_{\alpha\beta\gamma} L_{\alpha\beta}(k)D_{\alpha\gamma}(k)C_{\beta\gamma}(k) \tag{38}$$

The energy flux constraint

- Can the 2-layer QG model violate the energy flux constraint $k^2\Pi_E(k) - \Pi_G(k) < 0$?
  - E. Gkioulekas and K.K. Tung (2007), *Discr. Contin. Dyn. Syst. Ser. B*, **7**, 293-314
- Recall that in the two-layer model, the dissipation terms read:

$$d_1 = \nu(-\Delta)^{\kappa+1}\psi_1, \tag{39}$$

$$d_2 = \nu(-\Delta)^{\kappa+1}\psi_2 - \nu_E\Delta\psi_2 \tag{40}$$

- Only the **asymmetric** presense of $\nu_E\Delta\psi_2$ can break the energy flux constraint.
- Then the transition wavenumber k_t can occur in the inertial range.
- Necessary condition:

$$\nu_E > 4\nu k_{\max}^{2p}\left(\frac{k_{\max}}{k_R}\right)^2 \tag{41}$$

Forcing spectrum. I.

- For antisymmetric forcing $f_1 = f$ and $f_2 = -f$:

$$F_E(k) = 2(\Phi_1(k) - \Phi_2(k)) \tag{42}$$

$$F_G(k) = (k^2 + k_R^2) F_E(k) \tag{43}$$

  with $\Phi_1(k) = \langle f, \psi_1 \rangle_k$ and $\Phi_2(k) = \langle f, \psi_2 \rangle_k$.
- It follows that for $k_f \ll k_R$: $(\eta/\varepsilon) \sim k_R^2$.
- If both are dissipated at small scales, then $k_t \sim k_R$.
- In 2D turbulence Ekman damping dissipates most of injected energy and some of injected enstrophy.
- Not true in two-layer QG model because the Ekman term appears only on bottom layer.

Forcing spectrum. II.

- For Ekman-damped forcing: $f_1 = f$ and $f_2 = -f - \nu_E \Delta \psi_2$, incorporating the damping effect to the forcing spectrum gives:

$$F_E(k) = 2(\Phi_1(k) - \Phi_2(k)) + 2\nu_E k^2 U_2(k) \tag{44}$$

$$F_G(k) = (k^2 + k_R^2) F_E(k) - \nu_E k^2 k_R^2 (U_2(k) + C_{12}(k)) \tag{45}$$

  with $U_1(k) = \langle \psi_1, \psi_1 \rangle_k$, $U_2(k) = \langle \psi_2, \psi_2 \rangle_k$, and $C_{12}(k) = \langle \psi_1, \psi_2 \rangle_k$.
- Note the $F_E(k)$ increases, because $U_2(k) > 0$.
- If $E_K(k) \geq E_P(k) \Longrightarrow C_{12}(k) \geq 0 \Longrightarrow F_G(k)$ decreases.
- Thus, the tendency is to decrease k_t.
- Layer interaction makes it non-obvious whether $F_G(k)$ increases or decreases.

Forcing spectrum. IV.

- Claim: Suppressing bottom-layer forcing tends to decrease k_t.
- Let $f_1 = f$ and $f_2 = -\mu f$ with $\mu \in (0,1)$ (suppression factor).
- Assume f is random Gaussian with

$$\langle f_\alpha(\mathbf{x}_1, t_1) f_\beta(\mathbf{x}_2, t_2)\rangle = 2Q_{\alpha\beta}(\mathbf{x}_1, \mathbf{x}_2)\delta(t_1 - t_2) \tag{46}$$

- Define the forcing spectrum:

$$\mathcal{Q}_{\alpha\beta}(k) = \frac{d}{dk}\int d\mathbf{x}d\mathbf{y}d\mathbf{z}\; P(k|\mathbf{x}-\mathbf{y})P(k|\mathbf{x}-\mathbf{z})Q_{\alpha\beta}(\mathbf{y}, \mathbf{z}) \tag{47}$$

- It follows that the streamfunction-forcing spectrum reads:

$$\varphi_{\alpha\beta}(k) = \langle f_\alpha, \psi_\beta\rangle_k = \sum_\gamma L^{-1}_{\beta\gamma}(k)\mathcal{Q}_{\alpha\gamma}(k) \tag{48}$$

Forcing spectrum. V.

- Energy forcing spectrum:

$$F_E(k) = 2\sum_\alpha \varphi_{\alpha\alpha}(k) = \frac{2\Omega(k)[2(1+\mu^2)k^2 + (1-\mu)^2 k_R^2]}{2k^2(k^2+k_R^2)} \tag{49}$$

- Enstrophy forcing spectrum:

$$F_G(k) = 2\sum_\alpha L_{\alpha\beta}(k)\varphi_{\alpha\beta}(k) = 2(1+\mu^2)\Omega(k) \tag{50}$$

- Consider the forcing range limit: $k \ll k_R$.
- For $\mu = 1$: $F_G(k) \sim k_R^2 F_E(k) \Longrightarrow k_t \sim k_R$
- Thus: Antisymmetric forcing indicates transition at k_R.
- For $\mu = 0$: $F_G(k) \sim 2k^2 F_E(k) \Longrightarrow k_t \sim 2k_f$
- Thus: Baroclinically damped forcing contributes to inertial range transition.

Conclusion

- The energy flux constraint prevents a transition from k^{-3} scaling to $k^{-5/3}$ scaling in 2D turbulence.
- The energy flux constraint can be broken in two-layer QG turbulence under asymmetric dissipation.
- In the two-layer QG model, the rates of enstrophy over energy injection satisfy: $(\eta/\varepsilon) = k_R^2$
- Asymmetric Ekman dissipation increases ε and may decrease η,
- Suppressing the bottom layer forcing directly always decreases the ratio (η/ε).
- Open question: Can the injected energy and enstrophy be dissipated?