The effect of asymmetric large-scale dissipation on energy and potential enstrophy injection in two-layer quasi-geostrophic turbulence

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Outline

- Review of 2D turbulence.
- The Nastrom-Gage spectrum
- The Tung-Orlando theory of double cascade.
- The two-layer quasi-geostrophic model.
In 2D turbulence, the scalar vorticity $\zeta(x, y, t)$ is governed by

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = d + f,$$

(1)

where $\psi(x, y, t)$ is the streamfunction, and $\zeta(x, y, t) = -\nabla^2 \psi(x, y, t)$, and

$$d = -[\nu(-\Delta)^\kappa + \nu_1(-\Delta)^{-m}]\zeta$$

(2)

The Jacobian term $J(\psi, \zeta)$ describes the advection of $\zeta$ by $\psi$, and is defined as

$$J(a, b) = \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial b}{\partial x} \frac{\partial a}{\partial y}.$$
Two conserved quadratic invariants: energy $E$ and enstrophy $G$ defined as

$$E(t) = -\frac{1}{2} \int \psi(x, y, t) \zeta(x, y, t) \, dx \, dy \quad G(t) = \frac{1}{2} \int \zeta^2(x, y, t) \, dx \, dy. \quad (4)$$

Let $a^{<k}(x)$ be the field obtained from $a(x)$ by setting to zero, in Fourier space, the components corresponding to wavenumbers with norm greater than $k$:

$$a^{<k}(x) = \int dy \, P(k| x - y) a(y) \quad (5)$$

$$= \int dx_0 \int d\mathbf{k}_0 \frac{H(k - \||\mathbf{k}_0||)}{4\pi^2} \exp(i\mathbf{k}_0 \cdot (x - x_0)) a(x_0) \quad (6)$$

Filtered inner product:

$$\langle a, b \rangle_k = \frac{d}{dk} \int d\mathbf{x} \, a^{<k}(x) b^{<k}(x) \quad (7)$$
Energy spectrum: $E(k) = \langle \psi, \zeta \rangle_k$

Enstrophy spectrum $G(k) = \langle \zeta, \zeta \rangle_k$

Consider the conservation laws for $E(k)$ and $G(k)$:

$$\frac{\partial E(k)}{\partial t} + \frac{\partial \Pi_E(k)}{\partial k} = D_E(k) + F_E(k) \quad (8)$$

$$\frac{\partial G(k)}{\partial t} + \frac{\partial \Pi_G(k)}{\partial k} = D_G(k) + F_G(k) \quad (9)$$

In two-dimensional turbulence, the energy flux $\Pi_E(k)$ and the enstrophy flux $\Pi_G(k)$ are constrained by

$$k^2 \Pi_E(k) - \Pi_G(k) < 0 \quad (10)$$

for all $k$ not in the forcing range.
ln $E(k)$

$C_{ir}^{2/3} \varepsilon_{ir}^{-5/3}$

$C_{uv}^{2/3} \eta_{uv}^{-3}[\chi + \ln(k\ell_0)]^{-1/3}$

$k_f$ = forcing wavenumber

$k_{ir}$ = IR dissipation wavenumber

$k_{uv}$ = UV dissipation wavenumber

$\varepsilon_{ir}$ = upscale energy flux

$\eta_{uv}$ = downscale entropy flux
Nastrom-Gage spectrum schematic

\[ \ln E(k) \]

\[ \ln k \]

\[ k^{-3} \rightarrow 3000\text{km} - 800\text{km} \]
\[ k^{-5/3} \rightarrow 600\text{km} - \ll 1\text{km} \]
\[ k_t \approx 700\text{km} \approx k_R \]
Nastrom-Gage spectrum
Interpretation of NG spectrum.

The $k^{-3}$ range is interpreted as downscale enstrophy cascade.

The $k^{-5/3}$ range used to be interpreted as an 2D inverse energy cascade forced at small scales by thunderstorms.

This tortured interpretation followed from the perceived need to explain the Nastrom-Gage spectrum in terms of 2D turbulence.

Tung and Orlando (formerly: Welch) challenged the Charney QG-2D equivalence:


New interpretation: A double downscale cascade of enstrophy and energy with enstrophy flux $\eta_{uv}$ and energy flux $\varepsilon_{uv}$ and transition from $k^{-3}$ scaling to $k^{-5/3}$ scaling at the transition wavenumber $k_t \sim \sqrt{\eta_{uv}/\varepsilon_{uv}}$. 
Tung and Orlando spectrum

lower axis: zonal wavelength (km); upper axis: zonal wavenumber
More on Tung-Orlando theory

Cho-Lindborg: Confirm downscale energy cascade.

Smith-Tung debate: Transition not possible in 2D turbulence

Gkioulekas-Tung superposition principle

Transition may be possible in 2-layer QG with asymmetric dissipation – still open question.
The two-layer model. I

The governing equations for the two-layer quasi-geostrophic model are

\[
\frac{\partial \zeta_1}{\partial t} + J(\psi_1, \zeta_1 + f) = -\frac{2f}{h}\omega + d_1 \tag{11}
\]

\[
\frac{\partial \zeta_2}{\partial t} + J(\psi_2, \zeta_2 + f) = +\frac{2f}{h}\omega + d_2 + 2e_2 \tag{12}
\]

\[
\frac{\partial T}{\partial t} + \frac{1}{2}[J(\psi_1, T) + J(\psi_2, T)] = -\frac{N^2}{f}\omega + Q_0 \tag{13}
\]

where \( \zeta_1 = \nabla^2 \psi_1; \ zeta_2 = \nabla^2 \psi_2; \ T = (2/h)(\psi_1 - \psi_2) \). \( f \) is the Coriolis term; \( N \) the Brunt-Väisälä frequency; \( Q_0 \) is the thermal forcing on the temperature equation; \( d_1, d_2, e_2 \) the dissipation terms:

\[
d_1 = (-1)^{\kappa + 1}\nu \nabla^{2\kappa} \zeta_1 \tag{14}
\]

\[
d_2 = (-1)^{\kappa + 1}\nu \nabla^{2\kappa} \zeta_2 \tag{15}
\]

\[
e_2 = -\nu E \zeta_2 \tag{16}
\]
The two-layer model. II

The potential vorticity is defined as

\[ q_1 = \nabla^2 \psi_1 + f + \frac{k_R^2}{2} (\psi_2 - \psi_1) \tag{17} \]

\[ q_2 = \nabla^2 \psi_2 + f - \frac{k_R^2}{2} (\psi_2 - \psi_1) \tag{18} \]

with \( k_R \equiv 2\sqrt{2} f / (hN) \) and it satisfies

\[ \frac{\partial q_1}{\partial t} + J(\psi_1, q_1) = f_1 + d_1 \tag{19} \]

\[ \frac{\partial q_2}{\partial t} + J(\psi_2, q_2) = f_2 + d_2 + e_2 \tag{20} \]

with \( f_1 = -(k_R^2 Q)/(2f) \) and \( f_2 = (k_R^2 Q)/(2f) \) where \( Q = (1/4)k_R^2 hQ_0 \).
The two-layer model conserves total energy $E$ and potential layer enstrophies $G_1$ and $G_2$:

$$E = \int \left[ -\psi_1(x, y, t)q_1(x, y, t) - \psi_2(x, y, t)q_2(x, y, t) \right] \, dx \, dy \quad (21)$$

$$G_1 = \int q_1^2(x, y, t) \, dx \, dy \quad (22)$$

$$G_2 = \int q_2^2(x, y, t) \, dx \, dy \quad (23)$$

We define:

- The energy spectrum $E(k) = \langle \psi_1, q_1 \rangle_k + \langle \psi_2, q_2 \rangle_k$
- Top-layer potential enstrophy spectrum $G_1(k) = \langle q_1, q_1 \rangle_k$
- Bottom-layer potential enstrophy spectrum $G_2(k) = \langle q_2, q_2 \rangle_k$
- Total potential enstrophy spectrum $G(k) = G_1(k) + G_2(k)$
The two-layer model. IV

Consider the generalized form of an n-layer model:

$$\frac{\partial q_\alpha}{\partial t} + J(\psi_\alpha, q_\alpha) = \sum_\beta D_{\alpha\beta} \psi_\beta + f_\alpha$$  \hspace{1cm} (24)

$$\hat{q}_\alpha(k, t) = \sum_\beta L_{\alpha\beta} (\|k\|) \hat{\psi}_\beta(k, t)$$  \hspace{1cm} (25)

The energy spectrum $E(k)$ and the potential enstrophy spectrum $G(k)$ are given by:

$$E(k) = \sum_\alpha \langle q_\alpha, q_\alpha \rangle_k = \sum_\alpha L_{\alpha\beta}(k) C_{\alpha\beta}(k)$$  \hspace{1cm} (26)

$$G(k) = \sum_\alpha \langle q_\alpha, q_\alpha \rangle_k = \sum_{\alpha\beta\gamma} L_{\alpha\beta}(k) L_{\alpha\gamma}(k) C_{\beta\gamma}(k)$$  \hspace{1cm} (27)

with $C_{\alpha\beta}(k) = \langle \psi_\alpha, \psi_\beta \rangle_k$. 
The two-layer model. V

Let $\phi_{\alpha\beta}(k) = \langle f_\alpha, \psi_\beta \rangle_k$, and $D_{\alpha\beta}(k)$ be the spectrum of the operator $D_{\alpha\beta}$.

The energy forcing spectrum $F_E(k)$ and the potential enstrophy forcing spectrum $F_G(k)$ are:

$$F_E(k) = 2 \sum_{\alpha} \phi_{\alpha\alpha}(k) \quad (28)$$

$$F_G(k) = 2 \sum_{\alpha\beta} L_{\alpha\beta}(k) \phi_{\alpha\beta}(k) \quad (29)$$

The energy dissipation spectrum $D_E(k)$ and the potential dissipation enstrophy spectrum $D_G(k)$ are related with $C_{\alpha\beta}(k) = \langle \psi_\alpha, \psi_\beta \rangle_k$ according to:

$$D_E(k) = 2 \sum_{\alpha\beta} D_{\alpha\beta}(k) C_{\alpha\beta}(k) \quad (30)$$

$$D_G(k) = 2 \sum_{\alpha\beta\gamma} L_{\alpha\beta}(k) D_{\alpha\gamma}(k) C_{\beta\gamma}(k) \quad (31)$$
Forcing spectrum. I.

For antisymmetric forcing $f_1 = f$ and $f_2 = -f$:

$$F_E(k) = 2(\Phi_1(k) - \Phi_2(k))$$  \hspace{1cm} (32)

$$F_G(k) = (k^2 + k_R^2)F_E(k)$$  \hspace{1cm} (33)

with $\Phi_1(k) = \langle f, \psi_1 \rangle_k$ and $\Phi_2(k) = \langle f, \psi_2 \rangle_k$.

It follows that for $k_f \ll k_R$: $(\eta/\varepsilon) \sim k_R^2$.

If both are dissipated at small scales, then $k_t \sim k_R$.

In 2D turbulence Ekman damping dissipates most of injected energy and some of injected enstrophy.

Not true in two-layer QG model because the Ekman term appears only on bottom layer.
Forcing spectrum. II.

For Ekman-damped forcing: $f_1 = f$ and $f_2 = -f - \nu_E \Delta \psi_2$, incorporating the damping effect to the forcing spectrum gives:

$$F_E(k) = 2(\Phi_1(k) - \Phi_2(k)) + 2\nu_E k^2 U_2(k)$$
$$F_G(k) = (k^2 + k^2_R) F_E(k) - \nu_E k^2 k^2_R (U_2(k) + C_{12}(k))$$

with $U_1(k) = \langle \psi_1, \psi_1 \rangle_k$, $U_2(k) = \langle \psi_2, \psi_2 \rangle_k$, and $C_{12}(k) = \langle \psi_1, \psi_2 \rangle_k$.

Note the $F_E(k)$ increases, because $U_2(k) > 0$.

If $E_K(k) \geq E_P(k) \implies C_{12}(k) \geq 0 \implies F_G(k)$ decreases.

Thus, the tendency is to decrease $k_t$.

Layer interaction makes it non-obvious whether $F_G(k)$ increases or decreases.

In SQG: $E_K(k)/E_P(k) = 1$.

In stratified 3D: $E_K(k)/E_P(k) \approx 3$. (Lindborg (2009))

In full QG: $E_K(k)/E_P(k) \approx 2$. (Charney theory – Vallgren and Lindborg (2010))
Claim: Suppressing bottom-layer forcing tends to decrease $k_t$.

Let $f_1 = \varphi$ and $f_2 = -\mu \varphi$ with $\mu \in (0, 1)$ (suppression factor).

Assume $\varphi$ is random Gaussian with

$$\langle \varphi(x_1, t_1) \varphi(x_2, t_2) \rangle = 2Q(x_1, x_2)\delta(t_1 - t_2)$$  \hspace{1cm} (36)$$

$$\langle f_\alpha(x_1, t_1)f_\beta(x_2, t_2) \rangle = 2Q_{\alpha\beta}(x_1, x_2)\delta(t_1 - t_2)$$  \hspace{1cm} (37)$$

Define the forcing spectrum:

$$Q(k) = \frac{d}{dk} \int dx dy dz \ P(k|\mathbf{x} - \mathbf{y})P(k|\mathbf{x} - \mathbf{z})Q(\mathbf{y}, \mathbf{z})$$  \hspace{1cm} (38)$$

$$Q_{\alpha\beta}(k) = \frac{d}{dk} \int dx dy dz \ P(k|\mathbf{x} - \mathbf{y})P(k|\mathbf{x} - \mathbf{z})Q_{\alpha\beta}(\mathbf{y}, \mathbf{z})$$  \hspace{1cm} (39)$$
It follows that the streamfunction-forcing spectrum reads:

$$\varphi_{\alpha\beta}(k) = \langle f_\alpha, \psi_\beta \rangle_k = \sum_\gamma L_{\beta\gamma}^{-1}(k) Q_{\alpha\gamma}(k)$$  \hspace{1cm} (40)

Energy and enstrophy forcing spectrum:

$$F_E(k) = 2 \sum_\alpha \varphi_{\alpha\alpha}(k) = \frac{2Q(k)[2(1 + \mu^2)k^2 + (1 - \mu)^2k_R^2]}{2k^2(k^2 + k_R^2)}$$  \hspace{1cm} (41)

$$F_G(k) = 2 \sum_\alpha L_{\alpha\beta}(k) \varphi_{\alpha\beta}(k) = 2(1 + \mu^2)Q(k)$$  \hspace{1cm} (42)

Consider the forcing range limit: \( k \ll k_R \).

For \( \mu = 1 \): \( F_G(k) \sim k_R^2 F_E(k) \implies k_t \sim k_R \)

For \( \mu = 0 \): \( F_G(k) \sim 2k^2 F_E(k) \implies k_t \sim 2k_f \)

Thus: Baroclinically damped forcing contributes to inertial range transition.
Conclusion

The energy flux constraint prevents a transition from $k^{-3}$ scaling to $k^{-5/3}$ scaling in 2D turbulence.

The energy flux constraint can be broken in two-layer QG turbulence under asymmetric dissipation.

In the two-layer QG model, the rates of enstrophy over energy injection satisfy:

$$(\eta/\varepsilon) = k^2 R$$

Asymmetric Ekman dissipation increases $\varepsilon$ and may decrease $\eta$.

Suppressing the bottom layer forcing directly always decreases the ratio $(\eta/\varepsilon)$.

Open question: Can the injected energy and enstrophy be dissipated?