The effect of asymmetric large-scale dissipation on energy and potential enstrophy injection in two-layer quasi-geostrophic turbulence

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Publications

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2. K.K. Tung and W.W. Orlando (2003a), *J. Atmos. Sci.* **60**,, 824-835.
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8. E. Gkioulekas and K.K. Tung (2007), *J. Fluid Mech.*, **576**, 173-189.
9. E. Gkioulekas and K.K. Tung (2007), *Discrete Contin. Dyn. Syst. Ser. B*, **7**, 293-314
10. E. Gkioulekas (2010), *J. Fluid Mech.*, submitted. [arXiv:1011.3163v1 [nlin.CD]]

Outline

- Review of 2D turbulence.
- The Nastrom-Gage spectrum
- The Tung-Orlando theory of double cascade.
- The two-layer quasi-geostrophic model.

2D Navier-Stokes equations

- In 2D turbulence, the scalar vorticity $\zeta(x,y,t)$ is governed by

$$\frac{\partial \zeta}{\partial t} + J(\psi,\zeta) = d + f, \tag{1}$$

where $\psi(x,y,t)$ is the streamfunction, and $\zeta(x,y,t) = -\nabla^2\psi(x,y,t)$, and

$$d = -[\nu(-\Delta)^{\kappa} + \nu_1(-\Delta)^{-m}]\zeta \tag{2}$$

- The Jacobian term $J(\psi,\zeta)$ describes the advection of ζ by ψ, and is defined as

$$J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial y}. \tag{3}$$

Energy and enstrophy spectrum. I

- Two conserved quadratic invariants: energy E and enstrophy G defined as

$$E(t) = -\frac{1}{2}\int \psi(x,y,t)\zeta(x,y,t)\,dxdy \quad G(t) = \frac{1}{2}\int \zeta^2(x,y,t)\,dxdy. \tag{4}$$

- Let $a^{<k}(\mathbf{x})$ be the field obtained from $a(\mathbf{x})$ by setting to zero, in Fourier space, the components corresponding to wavenumbers with norm greater than k:

$$a^{<k}(\mathbf{x}) = \int d\mathbf{y}\, P(k|\mathbf{x}-\mathbf{y})a(\mathbf{y}) \tag{5}$$

$$= \int_{\mathbb{R}^2} d\mathbf{x}_0 \int_{\mathbb{R}^2} d\mathbf{k}_0\, \frac{H(k-\|\mathbf{k}_0\|)}{4\pi^2}\exp(i\mathbf{k}_0\cdot(\mathbf{x}-\mathbf{x}_0))a(\mathbf{x}_0) \tag{6}$$

- Filtered inner product:

$$\langle a,b\rangle_k = \frac{d}{dk}\int_{\mathbb{R}^2} d\mathbf{x}\; a^{<k}(\mathbf{x})b^{<k}(\mathbf{x}) \tag{7}$$

Energy and enstrophy spectrum. II

- Energy spectrum: $E(k) = \langle \psi, \zeta \rangle_k$
- Enstrophy spectrum $G(k) = \langle \zeta, \zeta \rangle_k$
- Consider the conservation laws for $E(k)$ and $G(k)$:

$$\frac{\partial E(k)}{\partial t} + \frac{\partial \Pi_E(k)}{\partial k} = D_E(k) + F_E(k) \tag{8}$$

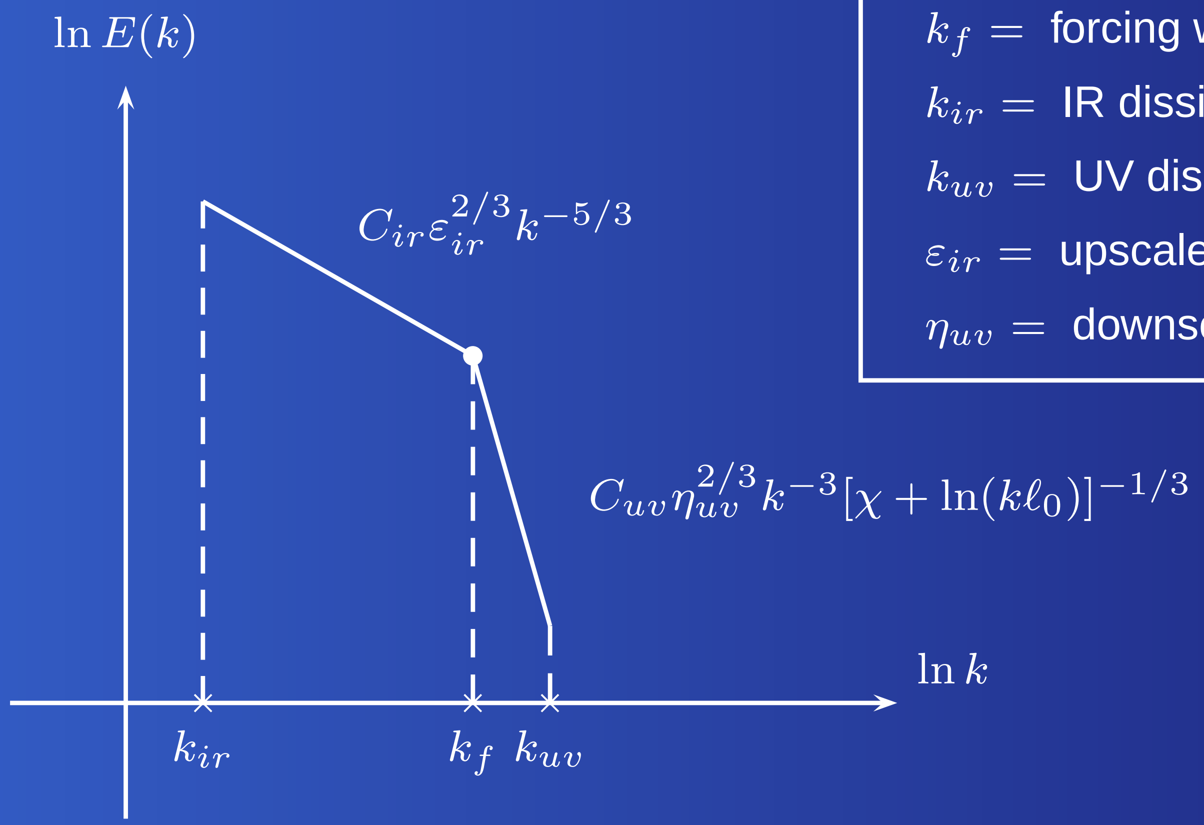
$$\frac{\partial G(k)}{\partial t} + \frac{\partial \Pi_E(k)}{\partial k} = D_G(k) + F_G(k) \tag{9}$$

- In two-dimensional turbulence, the energy flux $\Pi_E(k)$ and the enstrophy flux $\Pi_G(k)$ are constrained by

$$k^2 \Pi_E(k) - \Pi_G(k) < 0 \tag{10}$$

for all k not in the forcing range.

KLB theory.



k_f = forcing wavenumber

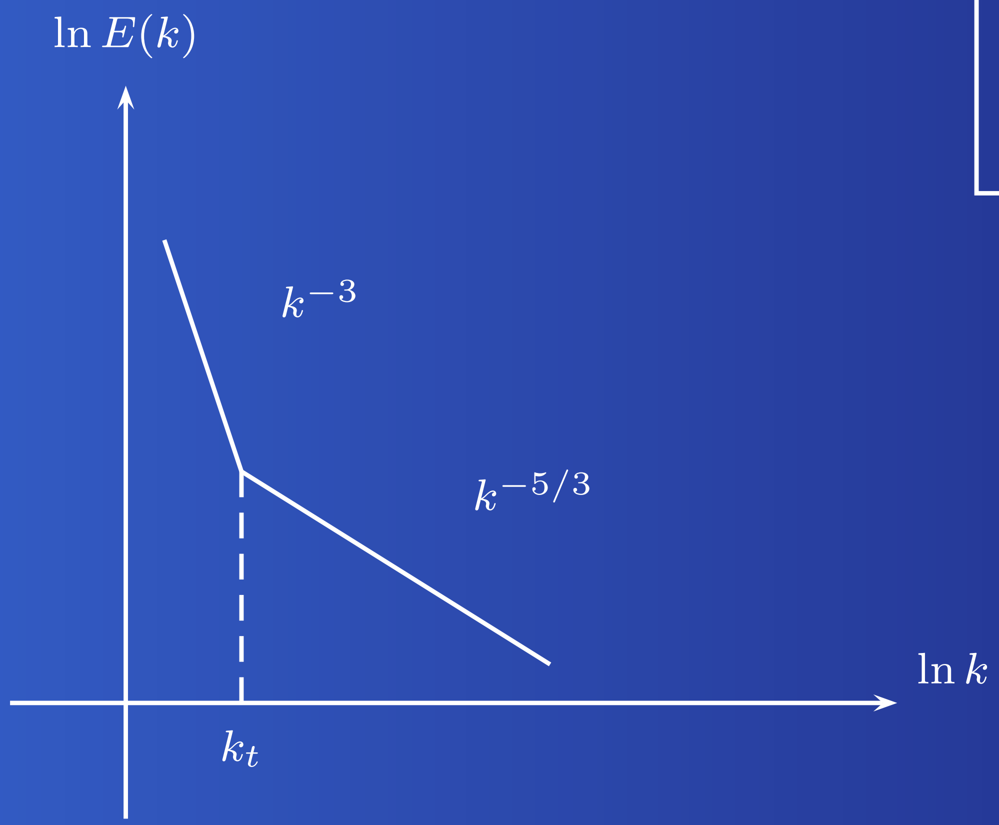
k_{ir} = IR dissipation wavenumber

k_{uv} = UV dissipation wavenumber

ε_{ir} = upscale energy flux

η_{uv} = downscale ensrropht flux

Nastrom-Gage spectrum schematic

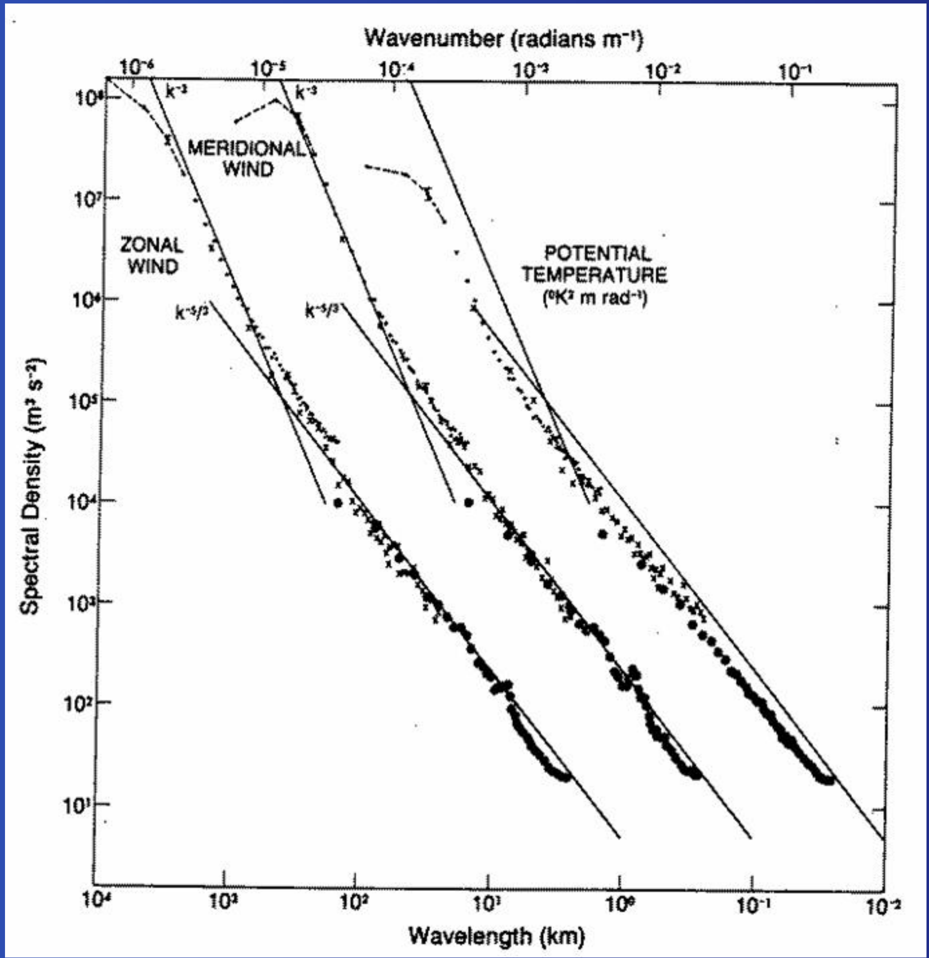


$k^{-3} \longrightarrow$ 3000km – 800km

$k^{-5/3} \longrightarrow$ 600km– $\ll$ 1km

$k_t \approx$ 700km $\approx k_R$

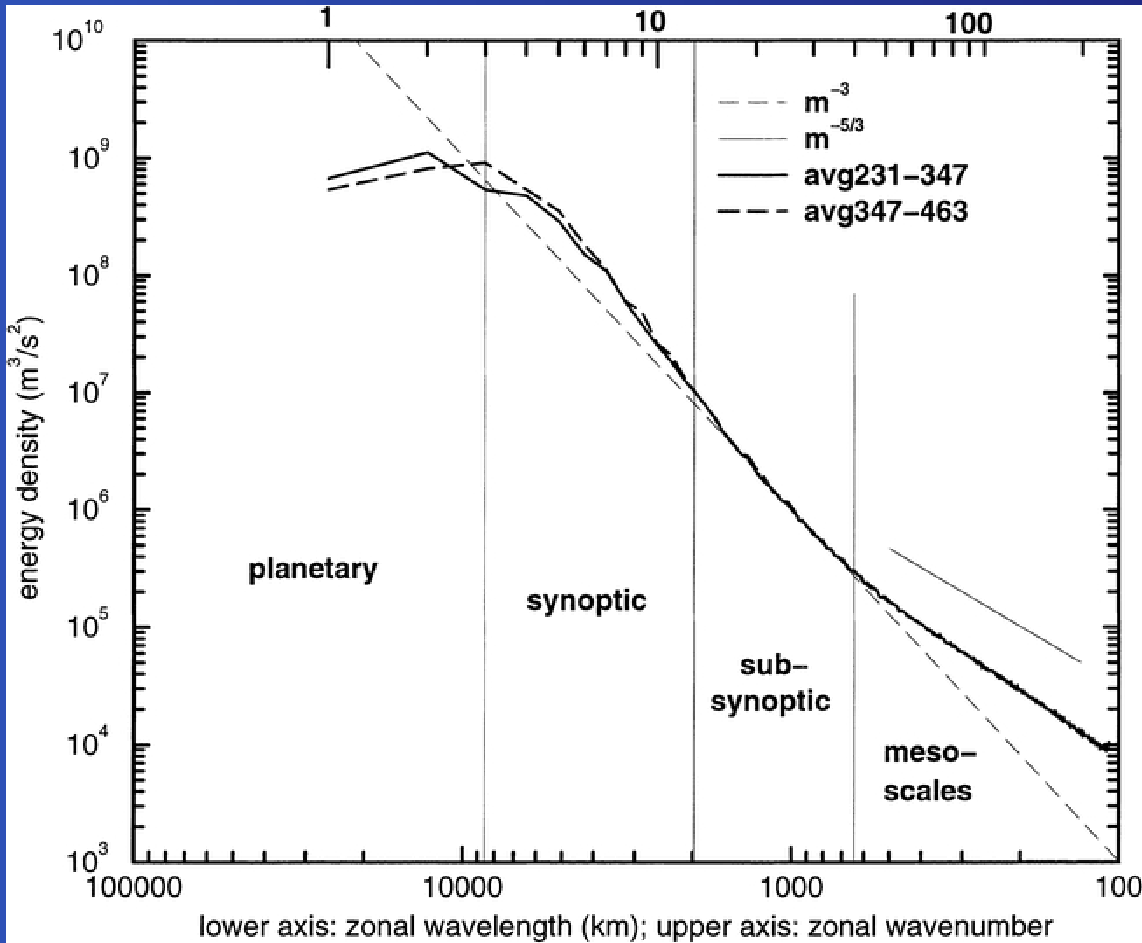
Nastrom-Gage spectrum



Interpretation of NG spectrum.

- The k^{-3} range is interpretated as downscale enstrophy cascade.
- The $k^{-5/3}$ range used to be interpretated as an 2D inverse energy cascade forced at small scales by thunderstorms.
- This tortured interpretation followed from the perceived need to explain the Nastrom-Gage spectrum in terms of 2D turbulence.
- Tung and Orlando (formerly: Welch) challenged the Charney QG-2D equivalence:
  - K.K. Tung and W.T. Welch (2001), *J. Atmos. Sci.* **58**, 2009-2012.
  - K.K. Tung and W.W. Orlando (2003a), *J. Atmos. Sci.* **60**,, 824-835.
  - K.K. Tung and W.W. Orlando (2003b), *Discrete Contin. Dyn. Syst. Ser. B*, **3**, 145-162.
- New interpretation: A double downscale cascade of enstrophy and energy with enstrophy flux η_{uv} and energy flux ε_{uv} and transition from k^{-3} scaling to $k^{-5/3}$ scaling at the transition wavenumber $k_t \sim \sqrt{\eta_{uv}/\varepsilon_{uv}}$.

Tung and Orlando spectrum



More on Tung-Orlando theory

- Cho-Lindborg: Confirm downscale energy cascade.
  - J.Y.N. Cho and E. Lindborg (2001), *J. Geophys. Res.* **106 D10**, 10,223-10,232.
  - J.Y.N. Cho and E. Lindborg (2001), *J. Geophys. Res.* **106 D10**, 10,232-10,241.
- Smith-Tung debate: Transition not possible in 2D turbulence
  - K.S. Smith (2004), *J. Atmos. Sci.* **61** , 937-942
  - K.K. Tung (2004), *J. Atmos. Sci.*, **61**, 943-948.
- Gkioulekas-Tung superposition principle
  - E. Gkioulekas and K.K. Tung (2005), *Discr. Cont. Dyn. Syst. Ser. B*, **5**, 79-102
  - E. Gkioulekas and K.K. Tung (2005), *Discr. Cont. Dyn. Syst. Ser. B* , **5**, 103-124.
- Transition may be possible in 2-layer QG with asymmetric dissipation – stil open question.
  - E. Gkioulekas and K.K. Tung (2007), *Discr. Cont. Dyn. Syst. Ser. B*, **7**, 293-314

The two-layer model. I

The governing equations for the two-layer quasi-geostrophic model are

$$\frac{\partial \zeta_1}{\partial t} + J(\psi_1, \zeta_1 + f) = -\frac{2f}{h}\omega + d_1 \tag{11}$$

$$\frac{\partial \zeta_2}{\partial t} + J(\psi_2, \zeta_2 + f) = +\frac{2f}{h}\omega + d_2 + 2e_2 \tag{12}$$

$$\frac{\partial T}{\partial t} + \frac{1}{2}[J(\psi_1, T) + J(\psi_2, T)] = -\frac{N^2}{f}\omega + Q_0 \tag{13}$$

where $\zeta_1 = \nabla^2\psi_1$; $\zeta_2 = \nabla^2\psi_2$; $T = (2/h)(\psi_1 - \psi_2)$. f is the Coriolis term; N the Brunt-Väisälä frequency; Q_0 is the thermal forcing on the temperature equation; d_1, d_2, e_2 the dissipation terms:

$$d_1 = (-1)^{\kappa+1}\nu\nabla^{2\kappa}\zeta_1 \tag{14}$$

$$d_2 = (-1)^{\kappa+1}\nu\nabla^{2\kappa}\zeta_2 \tag{15}$$

$$e_2 = -\nu_E\zeta_2 \tag{16}$$

The two-layer model. II

- The potential vorticity is defined as

$$q_1 = \nabla^2\psi_1 + f + \frac{k_R^2}{2}(\psi_2 - \psi_1) \quad (17)$$

$$q_2 = \nabla^2\psi_2 + f - \frac{k_R^2}{2}(\psi_2 - \psi_1) \quad (18)$$

with $k_R \equiv 2\sqrt{2}f/(hN)$ and it satisfies

$$\frac{\partial q_1}{\partial t} + J(\psi_1, q_1) = f_1 + d_1 \quad (19)$$

$$\frac{\partial q_2}{\partial t} + J(\psi_2, q_2) = f_2 + d_2 + e_2 \quad (20)$$

with $f_1 = -(k_R^2 Q)/(2f)$ and $f_2 = (k_R^2 Q)/(2f)$ where $Q = (1/4)k_R^2 h Q_0$.

The two-layer model. III

- The two-layer model conserves total energy E and potential layer enstrophies G_1 and G_2:

$$E = \int \left[-\psi_1(x,y,t) q_1(x,y,t) - \psi_2(x,y,t) q_2(x,y,t) \right] dxdy \tag{21}$$

$$G_1 = \int q_1^2(x,y,t)\, dxdy \tag{22}$$

$$G_2 = \int q_2^2(x,y,t)\, dxdy \tag{23}$$

- We define:
  - The energy spectrum $E(k) = \langle \psi_1, q_1 \rangle_k + \langle \psi_2, q_2 \rangle_k$
  - Top-layer potential enstrophy spectrum $G_1(k) = \langle q_1, q_1 \rangle_k$
  - Bottom-layer potential enstrophy spectrum $G_2(k) = \langle q_2, q_2 \rangle_k$
  - Total potential enstrophy spectrum $G(k) = G_1(k) + G_2(k)$

The two-layer model. IV

- Consider the generalized form of an n-layer model:

$$\frac{\partial q_\alpha}{\partial t} + J(\psi_\alpha, q_\alpha) = \sum_\beta \mathcal{D}_{\alpha\beta}\psi_\beta + f_\alpha \tag{24}$$

$$\hat{q}_\alpha(\mathbf{k}, t) = \sum_\beta L_{\alpha\beta}(\|\mathbf{k}\|)\hat{\psi}_\beta(\mathbf{k}, t) \tag{25}$$

- The energy spectrum $E(k)$ and the potential enstrophy spectrum $G(k)$ are given by:

$$E(k) = \sum_\alpha \langle \psi_\alpha, q_\alpha \rangle_k = \sum_{\alpha\beta} L_{\alpha\beta}(k) C_{\alpha\beta}(k) \tag{26}$$

$$G(k) = \sum_\alpha \langle q_\alpha, q_\alpha \rangle_k = \sum_{\alpha\beta\gamma} L_{\alpha\beta}(k) L_{\alpha\gamma}(k) C_{\beta\gamma}(k) \tag{27}$$

with $C_{\alpha\beta}(k) = \langle \psi_\alpha, \psi_\beta \rangle_k$.

The two-layer model. V

- Let $\phi_{\alpha\beta}(k) = \langle f_\alpha, \psi_\beta \rangle_k$, and $D_{\alpha\beta}(k)$ be the spectrum of the operator $\mathcal{D}_{\alpha\beta}$
- The energy forcing spectrum $F_E(k)$ and the potential enstrophy forcing spectrum $F_G(k)$ are:

$$F_E(k) = 2 \sum_{\alpha} \phi_{\alpha\alpha}(k) \tag{28}$$

$$F_G(k) = 2 \sum_{\alpha\beta} L_{\alpha\beta}(k) \phi_{\alpha\beta}(k) \tag{29}$$

- The energy dissipation spectrum $D_E(k)$ and the potential dissipation enstrophy spectrum $D_G(k)$ are related with $C_{\alpha\beta}(k) = \langle \psi_\alpha, \psi_\beta \rangle_k$ according to:

$$D_E(k) = 2 \sum_{\alpha\beta} D_{\alpha\beta}(k) C_{\alpha\beta}(k) \tag{30}$$

$$D_G(k) = 2 \sum_{\alpha\beta\gamma} L_{\alpha\beta}(k) D_{\alpha\gamma}(k) C_{\beta\gamma}(k) \tag{31}$$

Forcing spectrum. I.

- For antisymmetric forcing $f_1 = f$ and $f_2 = -f$:

$$F_E(k) = 2(\Phi_1(k) - \Phi_2(k)) \tag{32}$$

$$F_G(k) = (k^2 + k_R^2)F_E(k) \tag{33}$$

  with $\Phi_1(k) = \langle f, \psi_1 \rangle_k$ and $\Phi_2(k) = \langle f, \psi_2 \rangle_k$.
- It follows that for $k_f \ll k_R$: $(\eta/\varepsilon) \sim k_R^2$.
- If both are dissipated at small scales, then $k_t \sim k_R$.
- In 2D turbulence Ekman damping dissipates most of injected energy and some of injected enstrophy.
- Not true in two-layer QG model because the Ekman term appears only on bottom layer.

Forcing spectrum. II.

- For Ekman-damped forcing: $f_1 = f$ and $f_2 = -f - \nu_E \Delta\psi_2$, incorporating the damping effect to the forcing spectrum gives:

$$F_E(k) = 2(\Phi_1(k) - \Phi_2(k)) + 2\nu_E k^2 U_2(k) \quad (34)$$

$$F_G(k) = (k^2 + k_R^2)F_E(k) - \nu_E k^2 k_R^2 (U_2(k) + C_{12}(k)) \quad (35)$$

  with $U_1(k) = \langle\psi_1, \psi_1\rangle_k$, $U_2(k) = \langle\psi_2, \psi_2\rangle_k$, and $C_{12}(k) = \langle\psi_1, \psi_2\rangle_k$.
- Note the $F_E(k)$ increases, because $U_2(k) > 0$.
- If $E_K(k) \geq E_P(k) \Longrightarrow C_{12}(k) \geq 0 \Longrightarrow F_G(k)$ decreases.
- Thus, the tendency is to decrease k_t.
- Layer interaction makes it non-obvious whether $F_G(k)$ increases or decreases.
- In SQG: $E_K(k)/E_P(k) = 1$.
- In stratified 3D: $E_K(k)/E_P(k) \approx 3$. (Lindborg (2009))
- In full QG: $E_K(k)/E_P(k) \approx 2$. (Charney theory – Vallgren and Lindborg (2010))

Forcing spectrum. IV.

- Claim: Suppressing bottom-layer forcing tends to decrease k_t.
- Let $f_1 = \varphi$ and $f_2 = -\mu\varphi$ with $\mu \in (0,1)$ (suppression factor).
- Assume φ is random Gaussian with

$$\langle \varphi(\mathbf{x}_1, t_1)\varphi(\mathbf{x}_2, t_2)\rangle = 2Q(\mathbf{x}_1, \mathbf{x}_2)\delta(t_1 - t_2) \tag{36}$$

$$\left\langle f_\alpha(\mathbf{x}_1, t_1) f_\beta(\mathbf{x}_2, t_2)\right\rangle = 2Q_{\alpha\beta}(\mathbf{x}_1, \mathbf{x}_2)\delta(t_1 - t_2) \tag{37}$$

- Define the forcing spectrum:

$$\mathcal{Q}(k) = \frac{d}{dk}\int d\mathbf{x}d\mathbf{y}d\mathbf{z}\; P(k|\mathbf{x}-\mathbf{y})P(k|\mathbf{x}-\mathbf{z})Q(\mathbf{y},\mathbf{z}) \tag{38}$$

$$Q_{\alpha\beta}(k) = \frac{d}{dk}\int d\mathbf{x}d\mathbf{y}d\mathbf{z}\; P(k|\mathbf{x}-\mathbf{y})P(k|\mathbf{x}-\mathbf{z})Q_{\alpha\beta}(\mathbf{y},\mathbf{z}) \tag{39}$$

Forcing spectrum. V.

- It follows that the streamfunction-forcing spectrum reads:

$$\varphi_{\alpha\beta}(k) = \langle f_\alpha, \psi_\beta \rangle_k = \sum_\gamma L^{-1}_{\beta\gamma}(k) \mathcal{Q}_{\alpha\gamma}(k) \tag{40}$$

- Energy and enstrophy forcing spectrum:

$$F_E(k) = 2 \sum_\alpha \varphi_{\alpha\alpha}(k) = \frac{2\mathcal{Q}(k)[2(1+\mu^2)k^2 + (1-\mu)^2 k_R^2]}{2k^2(k^2 + k_R^2)} \tag{41}$$

$$F_G(k) = 2 \sum_\alpha L_{\alpha\beta}(k) \varphi_{\alpha\beta}(k) = 2(1+\mu^2)\mathcal{Q}(k) \tag{42}$$

- Consider the forcing range limit: $k \ll k_R$.
- For $\mu = 1$: $F_G(k) \sim k_R^2 F_E(k) \Longrightarrow k_t \sim k_R$
- For $\mu = 0$: $F_G(k) \sim 2k^2 F_E(k) \Longrightarrow k_t \sim 2k_f$
- Thus: Baroclinically damped forcing contributes to inertial range transition.

Conclusion

- The energy flux constraint prevents a transition from k^{-3} scaling to $k^{-5/3}$ scaling in 2D turbulence.
- The energy flux constraint can be broken in two-layer QG turbulence under asymmetric dissipation.
- In the two-layer QG model, the rates of enstrophy over energy injection satisfy: $(\eta/\varepsilon) = k_R^2$
- Asymmetric Ekman dissipation increases ε and may decrease η,
- Suppressing the bottom layer forcing directly always decreases the ratio (η/ε).
- Open question: Can the injected energy and enstrophy be dissipated?