Using restrictions to accept or reject solutions of radical equations

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Introduction

- We define a radical equation as an equation where the unknown variable appears at least once inside a square root.
- Standard procedure:
  1. Square both sides of equation once or several times to eliminate radicals
  2. This results in false solutions *(extraneous solutions)*
  3. Substitute to original equation and check if they work.
- **Two challenges:**
  - Can we identify extraneous solutions without substituting to the original equation?
  - What is the appropriate definition of a solution?
- Details are given in my two papers:
  1. E. Gkioulekas (2018): "Using restrictions to accept or reject solutions of radical equations", *International Journal of Mathematical Education in Science and Technology* **49**, 1278-1292
  2. E. Gkioulekas (2020): "Solving parametric radical equations with depth 2 rigorously using the restriction set method", *International Journal of Mathematical Education in Science and Technology* **51**, 1255-1277
- Disadvantages of the standard approach
  - From a theoretical standpoint, it is fair to say that a correct procedure should not result in "extraneous" solutions that don't work.
  - From a practical standpoint, unless the candidate solutions are integers, it can be quite tedious to verify them directly on the original equation.
  - With parametric radical equations, a solution that is rejected for some values of the parameter, may be a valid solution for other values of the parameter, and working that out by direct verification is also impractical

Definition of solution

- Consider the equation $\sqrt{1-3x} = \sqrt{x-7}$.
- Solution $x = 2$ with $\sqrt{1-3x} = \sqrt{x-7} = i\sqrt{5}$
- Do we want to accept or reject this solution?
- *Formalist viewpoint:* solution should be accepted, if our goal is to find the elements of the set $S = \{x \in \mathbb{R} \mid \sqrt{1-3x} = \sqrt{x-7}\}$
- *Geometric viewpoint:* if we define real-valued functions $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ with $f(x) = \sqrt{1-3x}$ and $g(x) = \sqrt{x-7}$ and with the widest possible implied domains $A = (-\infty, 1/3]$ and $B = [7, +\infty)$, and are interested in the set S of all points where the graphs of f and g intersect, then the formal definition of S reads $S = \{x \in A \cap B \mid f(x) = g(x)\}$ and the solution $x = 2$ should be rejected, because $A \cap B = \varnothing$
- A *strong solution* is defined to be a real-valued solution that verifies the original equation without encountering any negative numbers under any radical sign.
- A *formal solution* is defined to be a real-valued solution that verifies the original equation, where, in doing so, we allow radicals to evaluate to imaginary numbers.
- To the best of my knowledge, this distinction was not previously discussed in the literature.

Pedagogical citicism of standard approach by Hegeman

- A.S. Hegeman (1922): "Certain cases of extraneous roots", *The Mathematics Teacher* **15** (2), 110-118
- Quote: *"The student naturally wants to be shown why a root obtained by a process which he has been taught to consider correct is not a root at all. He is usually told that it is an extraneous root. This explains nothing and he is just as puzzled as before".*
- Recommendation:
  1. Move all terms of the equation to the left-hand side.
  2. Multiply both sides with rationalizing factors to progressively eliminate the radicals.
  3. One still needs to verify all solutions against the original equation.
  4. The extraneous solutions can be easily explained as zeroes of the rationalizing factors introduced in the process
- Good idea for explaining the origin of extraneous solutions. The downside is:
  - You have to do more writing
  - You still have no better way to eliminate the extraneous solutions
- During the 20th century, a few additional teacher-scholars highlighted the need for a more rigorous approach to the teaching of radical equations:
  - J.M. Taylor (1910): "Equations", *The Mathematics Teacher* **2**, 135-146
  - R.E. Bruce (1931): "Equivalence of Equations in One Unknown", *The Mathematics Teacher* **24**, 238-244
  - C.B. Allendoerfer (1966): "The Method of Equivalence or How to Cure a Cold", *The Mathematics Teacher* **59**, 531-535

Once upon a time...

- Radical equations first appeared in mathematics textbooks around 1860.
- J.E. Oliver and L.A. Wait and G.W. Jones (1887):, *"A Treatise on Algebra"*, J.S. Cushing and Co. Printers, Boston MA
  - The notation $\sqrt{a}$ had a multivalued interpretation where it could be equal to either zero of the polynomial $p(x) = x^2 - a$
  - Introduced the notation $\sqrt[-]{a}$ and $\sqrt[+]{a}$ to distinguish between the negative and positive zero of $p(x) = x^2 - a$
  - Under this multivalued definition $\sqrt{4x+1} = x - 5$ was viewed as equivalent to $\sqrt{4x+1} = x - 5 \vee -\sqrt{4x+1} = x + 5$, under the modern single-valued definition of the radical sign
  - As a result, any solution that is an extraneous solution, for one of the two equations in the disjunction above, will satisfy the other equation and vice versa.
  - Expended a substantial amount of effort to present a very rigorous and interesting theory of radicals, from the bottom up, under the multivalued definition
- G.E. Fisher and I.J. Schwatt (1898): *"Text-Book of Algebra with Exercises for Secondary Schools and Colleges. Part 1"*, Norwood Press, Norwood MA
  - Introduced the term *principal root* for the positive root, and used the notation $\sqrt{a}$ to represent the positive root.
  - Revealed the problem of extraneous solutions in radical equations.
  - Tried using simple contradiction arguments to eliminate extraneous solutions.
  - No systematic methodology for these arguments

Literature on radical equations

- Overview of the history of extraneous solutions in rational and radical equations
  - K.R. Manning (1970): "A history of extraneous solutions", *The Mathematics Teacher* **63** (2), 165-175
- Inverse problem: Constructing radical equations given the desired solution set
  - J.W. Beach (1952): "Equations involving Radicals", *School Science and Mathematics* **52**, 473-474.
  - S. Schwartz, C.E. Moulton and J. O'Hara (1997): "Constructing Radical Equations with Two Roots–a Student-Generated Algorithm", *The Mathematics Teacher* **90 (9)**, 742-744
  - W. Hildebrand (1998): "Radical equations with two solutions", *The Mathematics Teacher* **91 (7)**, 620-622
- Papers on solution techniques for radical equations
  - B. Bompart (1982): "An Alternate Method for Solving Radical Equations", *The Two-Year College Mathematics Journal* **13 (3)**, 198-199
  - J.V. Roberti (1984): "Radical solutions", *The Mathematics Teacher* 77 (3), 166
  - V.J. Gurevich (2003): "A Reasonable Restriction Set for Solving Radical Equations", *The Mathematics Teacher* **96 (9)**, 662-664
- Papers on systematic theories for specific types of radical equations
  - G.B. Huff and D.F. Barrow (1952): "A Minute Theory of Radical Equations", *The American Mathematical Monthly* **59 (5)**, 320-323
  - G. Nagase (1987): "Existence of Real Roots of a Radical Equation", *The Mathematics Teacher* **80**, 369-370
- My papers on radical equations

Classification of radical equations

- We define *depth* as the number of equivalence steps needed to eliminate all radicals.
- Radical equations of depth 1

$$\sqrt{f(x)} = \sqrt{g(x)},$$
$$\sqrt{f(x)} = g(x),$$
$$\sqrt{f_1(x)} + \sqrt{f_2(x)} + \cdots + \sqrt{f_n(x)} = 0. \quad (1)$$

- Radical equations of depth 2

$$\sqrt{f(x)} + \sqrt{g(x)} = h(x), \quad (2)$$
$$\sqrt{f(x)} + \sqrt{g(x)} = \sqrt{h(x)}, \quad (3)$$
$$\sqrt{f(x)} - \sqrt{g(x)} = h(x). \quad (4)$$

- $\sqrt{f(x)} - \sqrt{g(x)} = \sqrt{h(x)}$ is equivalent to $\sqrt{g(x)} + \sqrt{h(x)} = \sqrt{f(x)}$
- $\sqrt{f(x)} - \sqrt{g(x)} = -\sqrt{h(x)}$ reduces to $\sqrt{f(x)} + \sqrt{h(x)} = \sqrt{g(x)}$
- $\sqrt{f(x)} + \sqrt{g(x)} = -\sqrt{h(x)}$ reduces to $\sqrt{f(x)} + \sqrt{g(x)} + \sqrt{h(x)} = 0$
- There are additional forms with higher depth that I haven't considered.
- The total number of forms that are solvable is finite.

Equations with two equal radicals $\sqrt{f(x)} = \sqrt{g(x)}$

Proposition 1

Consider the equation $\sqrt{f(x)} = \sqrt{g(x)}$ with $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ polynomial or rational functions with $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. The set S_1 of all strong solutions is given by

$$S_1 = S_0 \cap A_1, \quad (5)$$
$$S_0 = \{x \in A \cap B \mid f(x) = g(x)\}, \quad (6)$$
$$A_1 = \{x \in A \cap B \mid f(x) \geq 0 \wedge g(x) \geq 0\}, \quad (7)$$

and the set S_2 of all formal solutions is given by $S_2 = S_0$.

- Methodology
  1. We find the domain A_1 of the equation by requiring that $f(x) \geq 0$ and $g(x) \geq 0$:
  $$\begin{cases} f(x) \geq 0 \\ g(x) \geq 0 \end{cases} \iff \cdots \iff x \in A_1. \quad (8)$$
  2. We solve the equation by squaring both sides:
  $$\sqrt{f(x)} = \sqrt{g(x)} \iff f(x) = g(x) \iff \cdots \iff x \in S_0. \quad (9)$$
  3. We accept the solutions in S_0 that also belong to A. Consequently, the solution set for all strong solutions is given by $S = S_0 \cap A_1$.

Equations with two equal radicals – Example

Example 2
Find all strong solutions of the equation $\sqrt{2x+3} = \sqrt{3x+5}$

Solution.
We require that

$$\begin{cases} 2x+3 \geq 0 \\ 3x+5 \geq 0 \end{cases} \iff \begin{cases} 2x \geq -3 \\ 3x \geq -5 \end{cases} \iff \begin{cases} x \geq -3/2 \\ x \geq -5/3 \end{cases} \quad (10)$$

$$\iff x \in [-3/2, +\infty) \cap [-5/3, +\infty) \iff x \in [-3/2, +\infty), \quad (11)$$

and therefore the domain of the equation is $A = [-3/2, +\infty)$. Solving the equation gives:

$$\sqrt{2x+3} = \sqrt{3x+5} \iff 2x+3 = 3x+5 \iff 3x - 2x = 3 - 5 \iff x = -2. \quad (12)$$

This solution is rejected, because $-2 \notin A$, consequently the equation has no strong solutions □

- It should be emphasized that the rejected solution is, in fact, a *formal solution* of the radical equation, and as such it would have been accepted if the problem was to find the set of all formal solutions.

Equations with one radical $\sqrt{f(x)} = g(x)$

Proposition 3
Consider the equation $\sqrt{f(x)} = g(x)$ with $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ polynomial or rational functions with $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. The set S_1 of all strong solutions and the set S_2 of all formal solutions to the equation are given by

$$S_1 = S_2 = S_0 \cap A_1, \quad (13)$$

$$S_0 = \{x \in A \cap B \mid f(x) = [g(x)]^2\}, \quad (14)$$

$$A_1 = \{x \in A \cap B \mid g(x) \geq 0\}. \quad (15)$$

- Methodology:
  1. The domain of the equation is determined by requiring that the left-hand-side of the equation be greater or equal to zero:
  $$g(x) \geq 0 \iff \cdots \iff x \in A. \quad (16)$$
  2. We solve the equation by squaring both sides:
  $$\sqrt{f(x)} = g(x) \iff f(x) = [g(x)]^2 \iff \cdots \iff x \in S_0. \quad (17)$$
  3. We accept only those solutions of S_0 that belong also to the domain A of the equation. Consequently, the solution set is given by $S = S_0 \cap A$.

Equations with one radical – Example

Example 4
Solve the equation $\sqrt{x^2 - 2x + 6} + 3 = 2x$.

Solution.
We note that

$$\sqrt{x^2 - 2x + 6} + 3 = 2x \iff \sqrt{x^2 - 2x + 6} = 2x - 3. \quad (18)$$

To determine the domain of the equation we require that

$$2x - 3 \geq 0 \iff 2x \geq 3 \iff x \geq 3/2 \iff x \in [3/2, +\infty) \equiv A. \quad (19)$$

Solving the equation for all $x \in A$ gives

$$\text{Eq. (18)} \iff x^2 - 2x + 6 = (2x-3)^2 \iff x^2 - 2x + 6 = 4x^2 - 12x + 9 \quad (20)$$

$$\iff (4-1)x^2 + (-12+2)x + (9-6) = 0 \iff 3x^2 - 10x + 3 = 0 \quad (21)$$

$$\iff \cdots \iff x = 3 \lor x = 1/3. \quad (22)$$

The solution $x = 3 \in A$ is accepted and the solution $x = 1/3 \notin A$ is rejected. It follows that the solution set of all strong solutions or all formal solutions is given by $S = \{3\}$. □

Equations with a sum of roots equal to zero

Lemma 5
Let $n \in \mathbb{N} - \{0\}$ be a natural number. Then, it follows that

$$\forall a_1, \ldots, a_n \in \mathbb{R} : \left(\sum_{k=1}^{n} \sqrt{a_k} = 0 \iff \forall k \in [n] : a_k = 0 \right). \quad (23)$$

Example 6
Solve the equation $\sqrt{x^2 - 9} + \sqrt{x^2 + 5x + 6} = 0$.

Solution.
Since,

$$\sqrt{x^2 - 9} + \sqrt{x^2 + 5x + 6} = 0 \iff \begin{cases} x^2 - 9 = 0 \\ x^2 + 5x + 6 = 0 \end{cases} \iff \begin{cases} (x-3)(x+3) = 0 \\ (x+2)(x+3) = 0 \end{cases} \quad (24)$$

$$\iff \begin{cases} x - 3 = 0 \lor x + 3 = 0 \\ x + 2 = 0 \lor x + 3 = 0 \end{cases} \iff \begin{cases} x = 3 \lor x = -3 \\ x = -2 \lor x = -3 \end{cases} \quad (25)$$

$$\iff x \in \{3, -3\} \cap \{-2, -3\} \iff x = -3, \quad (26)$$

it follows that the set of all formal or strong solutions is given by $S = \{-3\}$. □

Sum of two radicals equal to a function: $\sqrt{f(x)} + \sqrt{g(x)} = h(x)$

Proposition 7

Consider the equation $\sqrt{f(x)} + \sqrt{g(x)} = h(x)$ with $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ and $h : C \to \mathbb{R}$ polynomial or rational functions with $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ and $C \subseteq \mathbb{R}$. The set of all strong solutions S_1 and the set of all formal solutions S_2 are both given by

$$\begin{aligned} S_1 &= S_2 = S_0 \cap A_1 \cap A_2, \\ S_0 &= \{x \in A \cap B \cap C \mid 4f(x)g(x) = [(h(x))^2 - f(x) - g(x)]^2\}, \\ A_1 &= \{x \in A \cap B \cap C \mid h(x) \geq 0\}, \\ A_2 &= \{x \in A \cap B \cap C \mid (h(x))^2 - f(x) - g(x) \geq 0\}. \end{aligned}$$

Example 8

The equation $\sqrt{x^2 - a^2} + \sqrt{x^2 + a^2} = bx$ with $a \neq 0$ has two candidate solutions:

$$x_1 = -\left[\frac{4a^4}{(2-b)(2+b)b^2}\right]^{1/4} \quad \text{and } x_2 = +\left[\frac{4a^4}{(2-b)(2+b)b^2}\right]^{1/4}$$

which are accepted or rejected as follows:

1. If $b \in (-\infty, -2] \cup (-\sqrt{2}, \sqrt{2}) \cup [2, +\infty)$, then both x_1 and x_2 are rejected
2. If $b \in (-2, -\sqrt{2}]$, then x_1 is accepted as a strong solution and x_2 is rejected.
3. If $b \in [\sqrt{2}, 2)$, then x_2 is accepted as a strong solution and x_1 is rejected

Sum of two radicals equal to a function – Methodology

- Intuitive methodology for $\sqrt{f(x)} + \sqrt{g(x)} = h(x)$
  1. We require that $h(x) \geq 0 \iff \cdots \iff x \in A_1$.
  2. We raise both sides to power 2 and obtain:
  $$\begin{aligned} \sqrt{f(x)} + \sqrt{g(x)} = h(x) &\iff (\sqrt{f(x)} + \sqrt{g(x)})^2 = [h(x)]^2 \\ &\iff f(x) + 2\sqrt{f(x)g(x)} + g(x) = [h(x)]^2 \\ &\iff 2\sqrt{f(x)g(x)} = [h(x)]^2 - f(x) - g(x). \end{aligned}$$
  3. Before raising both sides to power 2 again, we introduce the requirement
  $$[h(x)]^2 - f(x) - g(x) \geq 0 \iff \cdots \iff x \in A_2.$$
  4. We raise to power 2 again and obtain the set S_0 of all candidate solutions:
  $$\begin{aligned} 2\sqrt{f(x)g(x)} &= [h(x)]^2 - f(x) - g(x) \\ &\iff 4f(x)g(x) = ([h(x)]^2 - f(x) - g(x))^2 \\ &\iff \cdots \iff x \in S_0. \end{aligned}$$
  5. We accept all solutions in S_0 that belong to both A_1 and A_2. The solution set S for all strong solutions is given by $S = S_0 \cap A_1 \cap A_2$. This is also the set of all formal solutions.

Sum of two square roots equal to another square root $\sqrt{f(x)} + \sqrt{g(x)} = \sqrt{h(x)}$

Proposition 9

Consider the equation $\sqrt{f(x)} + \sqrt{g(x)} = \sqrt{h(x)}$ with $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ and $h : C \to \mathbb{R}$ polynomial or rational functions with $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ and $C \subseteq \mathbb{R}$. Then the set of all strong solutions S_1 is given by

$$\begin{aligned} S_1 &= S_0 \cap A_1 \cap A_2, \\ S_0 &= \{x \in A \cap B \cap C \mid 4f(x)g(x) = [h(x) - f(x) - g(x)]^2\}, \\ A_1 &= \{x \in A \cap B \cap C \mid f(x) \geq 0 \wedge g(x) \geq 0 \wedge h(x) \geq 0\}, \\ A_2 &= \{x \in A \cap B \cap C \mid h(x) - f(x) - g(x) \geq 0\}, \end{aligned}$$

and the set S_2 of all formal solutions is given by

$$\begin{aligned} S_2 &= (S_0 \cap A_1 \cap A_2) \cup (S_0 \cap A_3 \cap A_4), \\ A_3 &= \{x \in A \cap B \cap C \mid f(x) \leq 0 \wedge g(x) \leq 0 \wedge h(x) \leq 0\}, \\ A_4 &= \{x \in A \cap B \cap C \mid h(x) - f(x) - g(x) \leq 0\}. \end{aligned}$$

Sum of two square roots equal to another square root – Methodology

- Intuitive methodology for $\sqrt{f(x)} + \sqrt{g(x)} = \sqrt{h(x)}$
  1. First, we require that all expressions under a square root be positive or zero:
  $$f(x) \geq 0 \wedge g(x) \geq 0 \wedge h(x) \geq 0 \iff \cdots \iff x \in A_1.$$
  2. Then we raise both sides of the equation to the power 2, which reads:
  $$\begin{aligned} \sqrt{f(x)} + \sqrt{g(x)} = \sqrt{h(x)} &\iff (\sqrt{f(x)} + \sqrt{g(x)})^2 = h(x) \\ &\iff f(x) + 2\sqrt{f(x)g(x)} + g(x) = h(x) \\ &\iff 2\sqrt{f(x)g(x)} = h(x) - f(x) - g(x). \qquad (27) \end{aligned}$$
  3. Now, we introduce the additional requirement that
  $$h(x) - f(x) - g(x) \geq 0 \iff \cdots \iff x \in A_2.$$
  4. Finally we raise both sides to power 2 again to eliminate the remaining root:
  $$\begin{aligned} \text{Eq. (27)} &\iff 4f(x)g(x) = (h(x) - f(x) - g(x))^2 \\ &\iff \cdots \iff x \in S_0. \end{aligned}$$
  5. We accept all solutions of S_0 that also belong to A_1 and A_2. Thus, the set of all *strong solutions* is given by $S = S_0 \cap A_1 \cap A_2$.
  6. If we want to find *all formal solutions*, it is necessary and sufficient to also accept all solutions that satisfy the restriction
  $$\begin{cases} f(x) \leq 0 \wedge g(x) \leq 0 \wedge h(x) \leq 0 \\ h(x) - f(x) - g(x) \leq 0. \end{cases} \qquad (28)$$

Sum of two square roots equal to another square root – Example

Example 10

The equation $\sqrt{x+a}+\sqrt{x-a}=\sqrt{x+b}$ with $a \in (0,+\infty)$ and $b \in \mathbb{R}$ has two candidate solutions

$$x_1 = \frac{-b-2\sqrt{b^2+3a^2}}{3} \text{ and } x_2 = \frac{-b+2\sqrt{b^2+3a^2}}{3},$$

which are accepted or rejected as follows:

1. If $b \in (-\infty,-a]$, then x_1 is a formal but not a strong solution and x_2 is rejected.
2. If $b \in (-a,a)$, then x_1 and x_2 are both rejected.
3. If $b \in [a,+\infty)$, then x_2 is a strong solution and x_1 is rejected.

- With a simple change of variables, the result of Example 10 can be used to handle the more general form $\sqrt{x+a}+\sqrt{x+b}=\sqrt{x+c}$.

Difference of square roots equal to a function

Proposition 11

Consider the equation $\sqrt{f(x)}-\sqrt{g(x)}=h(x)$ with $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and $h: C \to \mathbb{R}$ polynomial or rational functions with $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ and $C \subseteq \mathbb{R}$. Then the set S_1 of all strong solutions is given by

$$\begin{aligned}
S_1 &= S_0 \cap A_1 \cap A_2 \cap A_3, \\
S_0 &= \{x \in A \cap B \cap C \mid 4[h(x)]^2 f(x) = [f(x)+[h(x)]^2 - g(x)]^2\}, \\
A_1 &= \{x \in A \cap B \cap C \mid f(x) \geq 0\}, \\
A_2 &= \{x \in A \cap B \cap C \mid \sqrt{f(x)} - h(x) \geq 0\}, \\
A_3 &= \{x \in A \cap B \cap C \mid h(x)[f(x)+[h(x)]^2 - g(x)] \geq 0\},
\end{aligned}$$

and the set S_2 of all formal solutions is given by

$$\begin{aligned}
S_2 &= (S_0 \cap A_1 \cap A_2 \cap A_3) \cup B_1, \\
B_1 &= \{x \in A \cap B \cap C \mid f(x) = g(x) < 0 \land h(x) = 0\}.
\end{aligned}$$

- Underlying methodology is counterintuitive, so it is better to just apply the theorem.

Example 12

The equation $\sqrt{2a-x}-\sqrt{x-2b}=x-(a+b)$ with $a<b$, has no strong solutions and has $x=a+b$ as a formal, but not strong, solution.

Thank you!